Bipolar fuzzy soft sets and its applications in decision making problem

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Abstract. In this article, we combine the concept of a bipolar fuzzy set and a soft set. We introduce the notion of bipolar fuzzy soft set and study fundamental properties. We study basic operations on bipolar fuzzy soft set. We define extended union, intersection of two bipolar fuzzy soft set. We also give an application of bipolar fuzzy soft set into decision making problem. We give a general algorithm to solve decision making problems by using bipolar fuzzy soft set.

Keywords: Soft set, bipolar fuzzy set, fuzzy soft set and bipolar fuzzy soft set

1. Introduction

Complicated problems in different field like engineering, economics, environmental science, medicine and social sciences, which arising due to classical mathematical modelling and manipulating of various type of uncertainty. While some of traditional mathematical tool fail to solve these complicated problems. We used some mathematical modelling like fuzzy set theory[1], rough set theory[2], interval mathematics[12] and probability theory are well-known and operative tools for handling with vagueness and uncertainty, each of them has its own inherent limitations; one major fault shared by these mathematical methodologies may be due to the inadequacy of parametrization tools[4]. Molodtsov, [4] adopted the notion of soft set. Soft set theory is powerful tool to describe uncertainties. Recently, researcher are engaged in soft set theory. Maji et al. [6] defined new notions on soft sets. Ali et al.[21] studied some new concepts of a soft set. Seyin and Atanin [22] studied some new theoretical soft set operations. Majumdar and Samanta, worked on soft mappings [24]. Choudure et al. defined the concept of soft relation and fuzzy soft relation and then applied them to solve a number of decision-making problems. Feng et al. studied and applied softness to semirings[8]. Recently, Acar studied soft rings [9]. Jun et. al, applied the concept of soft set to BCK/BCI-algebras [10–12]. Sezgin and Atagin initiated the concept of normalistic soft groups [13]. Zhan et al. worked on soft ideal of BL-algebras [15]. In [16], Kazancı et. al, used the concept of soft set to BCH-algebras. Sezgin et al. studied soft nearrings [17]. Çagman et al. considered two types of notions of a soft set with group, which is called group Soft intersection group soft union groups of a group [20], see [14]. Fuzzy set originally proposed by Zadeh in [1] of 1965. After semblance of the concept of fuzzy set, researcher given much attention to developed fuzzy set theory. Maji et al. [36] introduced the concept of fuzzy soft sets. Afterwards, many researchers have worked on...
this concept. Roy and Maji [28] provided some results on an application of fuzzy soft sets in decision making problems. F. Feng et al. give application in decision making problem [31, 32].

Fuzzy set is a type of important mathematical structure to represent a collection of objects whose boundary is vague. There are several types of fuzzy set extensions in the fuzzy set theory, for example, intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, etc. bipolar-valued fuzzy set is another an extension of fuzzy set whose membership degree is different from the above extensions. In 2000, Lee [35] initiated an extension of fuzzy set named bi-polar valued fuzzy set. He gave two kinds of representations of the notion of bipolar-valued fuzzy sets. In case of bi-polar valued fuzzy sets membership degree range is enlarged from the interval [0, 1] to [−1, 1]. In a bi-polar valued fuzzy set, the membership degree 0 indicate that elements somewhat satisfy the above conditions: (i) \( C \subseteq A \cup B \) (ii) for all \( c \in C \),

\[ H(c) = F(c) \text{ if } c \in A \setminus B \]

\[ = G(c) \text{ if } c \notin A \setminus B \]

\[ = F(c) \cup G(c) \text{ if } c \in A \cap B \]

This relation is denoted by \((H, C) = (F, A) \cup(G, B)\).

Definition 2.5. [21] Let \((F, A)\) and \((G, B)\) be two soft sets over a common universe \(U\). The union of \((F, A)\) and \((G, B)\) is defined to be the soft set \((H, C)\) satisfying the following conditions: (i) \( C = A \cup B \) (ii) \( c \in A \cap B \)

\[ H(c) = F(c) \text{ if } c \in A \setminus B \]

\[ = G(c) \text{ if } c \notin A \setminus B \]

\[ = F(c) \cup G(c) \text{ if } c \in A \cap B \]

This relation is denoted by \((H, C) = (F, A) \cup(G, B)\).

3. Bipolar fuzzy soft sets

In this section we introduce the concept of bipolar fuzzy soft set, absolute bipolar fuzzy soft set, null
bipolar fuzzy soft set and complement of bipolar fuzzy soft set

Definition 3.1. Let $U$ be a universe, $E$ a set of parameters and $A \subseteq E$. Define $F : A \rightarrow BF\mathcal{F}$, where $BF\mathcal{F}$ is the collection of all bipolar fuzzy subsets of $U$. Then $(F, A)$ is said to be a bipolar fuzzy soft set over a universe $U$. It is defined by

$$(F, A) = F(a) = \{ (c_i, \mu^+(c_i), \mu^-(c_i)) : \forall c_i \in A, \forall a \in A \}$$

Example 3.2. Let $U = \{ e_1, e_2, e_3, e_4 \}$ be the set of four cars under consideration and $E = \{ e_1 = \text{Costly}, e_2 = \text{Beautiful}, e_3 = \text{Fuel Efficient}, e_4 = \text{Modern Technology} \}$ be the set of parameters and $A = \{ e_1, e_2 \}$ be subset of $E$. Then

$$(F, A) = \{ (e_1, 0.5, 0), (e_2, 0.3, 0.6), (e_3, 0.4, 0.2), (e_4, 0.7, 0.2) \}$$

Definition 3.3. Let $U$ be a universe and $E$ a set of attributes. Then $(U, E)$ is the collection of all bipolar fuzzy soft sets on $U$ with attributes from $E$ and is said to be bipolar fuzzy soft class.

Definition 3.4. A bipolar fuzzy soft set $(F, A)$ is said to be null bipolar fuzzy soft set denoted by empty set $\emptyset$, if for all $e \in A$, $F(e) = \emptyset$.

Definition 3.5. A bipolar fuzzy soft set $(F, A)$ is said to be an absolute bipolar fuzzy soft set if for all $e \in A$, $F(e) = BF\mathcal{F}$

Definition 3.6. The complement of a bipolar fuzzy soft set $(F, A)$ is denoted $(F, A^c)$ and is defined by

$$(F, A^c) = \{ (x, 1 - \mu^+_A(x), 1 - \mu^-_A(x)) : x \in U \}.$$

Definition 3.7. Let $U = \{ b_1, b_2, b_3, b_4 \}$ be the set of four bikes under consideration and $E = \{ e_1 = \text{Style}, e_2 = \text{Heavy Duty}, e_3 = \text{Light}, e_4 = \text{Steel} \}$ be the set of parameters and $A = \{ e_1, e_2 \}$ be subset of $E$. Then

$$(F, A) = \{ (b_1, 0.5, 0), (b_2, 0.7, 0.6), (b_3, 0.3, 0.2), (b_4, 0.4, 0.2) \}$$

The complement of the bipolar fuzzy soft set $(F, A)$ is

$$(F, A^c) = \{ (b_1, 0.9, -0.5), (b_2, 0.7, -0.4), (b_3, 0.3, -0.8), (b_4, 0.6, -0.8) \}$$

4. Bipolar fuzzy soft subsets

Definition 4.1. Let $(F, A)$ and $(G, B)$ be two bipolar fuzzy soft sets over a common universe $U$. We say that $(F, A)$ is a bipolar fuzzy soft subset of $(G, B)$, if (1) $A \subseteq B$ and (2) $\forall e \in A, F(e)$ is a bipolar fuzzy subset of $G(e)$. We write $(F, A) \subseteq (G, B)$.

Definition 4.2. Every element of $(F, A)$ is presented in $(G, B)$ and do not depend on its membership or non-membership.

Example 4.3. Let $U = \{ m_1, m_2, m_3, m_4 \}$ be the set of four men under consideration and $E = \{ e_1 = \text{Educated}, e_2 = \text{Government employee}, e_3 = \text{Businessman}, e_4 = \text{Smart} \}$ be the set of parameters and $A = \{ e_1, e_2 \}$, $B = \{ e_1, e_2, e_3 \}$ be subsets of $E$. Then
### Definition 5.1

An intersection of two bipolar fuzzy soft sets over a common universe \( U \) and \( (A, B) \) is a bipolar fuzzy soft set \((H, C)\), where \( C = A \cap B \neq \emptyset \) and \( H : C \rightarrow BF^U \) is defined by \( H(e) = F(e) \cap G(e) \forall e \in \emptyset \) and denoted by \( (H, C) = (F, A) \cap (G, B) \).

### Example 5.2

Let \( U = \{b_1, b_2, b_3, b_4\} \) be the set of four bikes under consideration and \( E = \{e_1 = \text{Light, e}_2 = \text{Beautiful, e}_3 = \text{Good milage, e}_4 = \text{Modern Technology}\} \) be the set of parameters and \( A = \{e_1, e_2\} \subseteq E, B = \{e_1, e_2, e_3\} \subseteq E \). Then,

### Definition 5.3

Union of two bipolar fuzzy soft sets over a common universe \( U \) is a bipolar fuzzy soft set \((H, C)\), where \( C = A \cup B \) and \( H : C \rightarrow BF^U \) is defined by

\[
H(e) = F(e) \quad \text{if} \quad e \in A \setminus B
\]
\[ F(e) = \begin{cases} \{ (c_1, 0.1, -0.5), \\ (c_2, 0.3, -0.6), \\ (c_3, 0.4, -0.2), \\ (c_4, 0.7, -0.2) \} & \text{if } e \in B \setminus A \\ \{ (c_1, 0.3, -0.5), \\ (c_2, 0.4, -0.2), \\ (c_3, 0.5, -0.2), \\ (c_4, 0.4, -0.2) \} & \text{if } e \in A \cap B \end{cases} \]

Then \((H, C) = (F, A) \cup (G, B)\), where \(C = A \cup B = \{ e_1, e_2, e_3, e_4 \}\).

**Example 5.4.** Let \(U = \{ e_1, e_2, e_3, e_4 \}\) be the set of four cars under consideration and \(E = \{ e_1 = \text{Costly}, e_2 = \text{Modern Technology} \} \) be the set of parameters and \(A = \{ e_1, e_2, e_3 \} \subseteq E\). Let \(B = \{ e_1, e_2, e_3, e_4 \} \subseteq E\). Then

\[
(F, A) = \begin{cases} F(e_1) = \{ (c_1, 0.2, -0.5), \\ (c_2, 0.2, -0.6), \\ (c_3, 0.2, -0.3), \\ (c_4, 0.7, -0.1) \} \\ F(e_2) = \{ (c_1, 0.3, -0.6), \\ (c_2, 0.2, -0.5), \\ (c_3, 0.5, -0.3), \\ (c_4, 0.5, -0.2) \} \\ F(e_3) = \{ (c_1, 0.8, -0.1), \\ (c_2, 0.3, -0.6), \\ (c_3, 0.4, -0.3), \\ (c_4, 0.6, -0.2) \} \end{cases}
\]

and

\[
(G, B) = \begin{cases} G(e_1) = \{ (c_1, 0.2, -0.5), \\ (c_2, 0.2, -0.6), \\ (c_3, 0.2, -0.3), \\ (c_4, 0.7, -0.1) \} \\ G(e_2) = \{ (c_1, 0.3, -0.6), \\ (c_2, 0.2, -0.5), \\ (c_3, 0.5, -0.3), \\ (c_4, 0.5, -0.2) \} \\ G(e_3) = \{ (c_1, 0.8, -0.1), \\ (c_2, 0.4, -0.6), \\ (c_3, 0.2, -0.3), \\ (c_4, 0.7, -0.2) \} \\ G(e_4) = \{ (c_1, 0.1, -0.6), \\ (c_2, 0.3, -0.4), \\ (c_3, 0.1, -0.6), \\ (c_4, 0.0, -0.2) \} \end{cases}
\]

**Definition 5.5.** Let \(T = \{ (F_i, A_i) : i \in I \} \) be a family of bipolar fuzzy soft sets in a bipolar fuzzy soft class \((U, E)\). Then the intersection of bipolar fuzzy soft sets in \(T\) is a bipolar fuzzy soft set \((H, C)\), where \(C = \bigcap A_i \neq \emptyset\) for all \(i \in I\), \(H(e) = \bigcap F_i(e)\) for all \(e \in C\).

**Definition 5.6.** Let \(T = \{ (F_i, A_i) : i \in I \} \) be a family of bipolar fuzzy soft sets in a bipolar fuzzy soft class \((U, E)\). Then the union of bipolar fuzzy soft sets in \(T\) is a bipolar fuzzy soft set \((H, C)\), where \(C = \bigcup A_i\) for all \(i \in I\).

\[
H(e) = \begin{cases} F_i(e) & \text{if } e \in A_i \\ \emptyset & \text{if } e \notin A_i \end{cases}
\]

**Definition 5.7.** Let \((F, A)\) and \((G, B)\) be two bipolar fuzzy soft sets over a common universe \(U\). The extended intersection of \((F, A)\) and \((G, B)\) is defined as the bipolar fuzzy soft set \((H, C)\), where \(C = A \cup B\) and for all \(e \in C\),

\[
H(e) = \begin{cases} F(e) & \text{if } e \in A \setminus B \\ G(e) & \text{if } e \in B \setminus A \\ F(e) \cap G(e) & \text{if } e \in A \cap B \end{cases}
\]

This intersection is denoted by \((H, C) = (F, A) \cap (G, B)\).
Definition 5.8. Let \((F, A)\) and \((G, B)\) be two bipolar fuzzy soft sets over a common universe \(U\). The restricted union of \((F, A)\) and \((G, B)\) is defined to be the bipolar fuzzy soft set \((H, C)\), where \(C = A \cap B \neq \emptyset\) and for all \(e \in C\)
\[
H(e) = F(e) \cup G(e)
\]
This union is denoted by \((H, C) = (F, A) \cup_R (G, B)\).

Proposition 5.9. Let \((F, A)\) be bipolar fuzzy soft set over a common universe \(U\). Then,

1. \((F, A) \cup (F, A) = (F, A)\)
2. \((F, A) \cap (F, A) = (F, A)\)
3. \((F, A) \cup \emptyset = (F, A)\) where \(\emptyset\) is a null bipolar fuzzy soft set.
4. \((F, A) \cap \emptyset = \emptyset\), where \(\emptyset\) is a null bipolar fuzzy soft set.

Proof. (1)
\[
(F, A) \cup (F, A) = (F, A).
\]
A bipolar fuzzy soft set \((H, C)\) is union of two bipolar fuzzy soft sets \((F, A)\) and \((F, A)\) which is
\[
(H, C) = (F, A) \cup (F, A) \quad (1)
\]
Define by
\[
H(e) = F(e) \text{ if } e \in A \setminus A
\]
\[
= F(e) \text{ if } e \in A \setminus A
\]
\[
= F(e) \cup F(e) \text{ if } e \in A \cap A
\]
L.H.S. There are three cases.

Case (1) If \(a \in A \setminus A\).
\[
H(a) = F(a) \text{ if } a \in A \setminus A = \emptyset
\]
Case (2) If \(a \in A \setminus A\).
\[
H(a) = F(a) \text{ if } a \in A \setminus A = \emptyset
\]
Case (3) If \(a \in A \cap A\).
\[
H(a) = F(a) \cup F(a) \text{ if } a \in A \cap A = A
\]
\[
= F(a) \text{ if } a \in A
\]
\[
H(a) = F(a) \text{ if } a \in A
\]
\[
= (H, C) = (F, A) \text{ from Eq } 1
\]
\[
(F, A) \cup (F, A) = (F, A) \text{ from Eq } 1
\]
It is satisfied in all three cases. Hence
\[
(F, A) \cup (F, A) = (F, A)
\]
A bipolar fuzzy soft set \((H, C)\) is intersection of two bipolar fuzzy soft sets \((F, A)\) and \((F, A)\) which is
\[
(H, C) = (F, A) \cap (F, A) \text{ where } C = A \cap A \quad (2)
\]
Define by
\[
H(e) = F(e) \cap F(e) \text{ if } e \in C = A \cap A
\]
L.H.S. Let \(a \in C = A \cap A\).
\[
H(e) = F(e) \cap F(e) \text{ if } e \in C = A \cap A
\]
Proof. (1)
\[
(F, A) \cup (G, B) = (F, A)
\]
Let bipolar fuzzy soft set \((H, C)\) is an intersection of two bipolar fuzzy soft sets \((F, A)\) and \((G, B)\), where
\[
C = A \cap B
\]
\[
(H, C) = (F, A) \cap (G, B) \quad (3)
\]
where \(C = A \cap B\) Define if \(e \in C = A \cap B\)
\[
H(e) = F(e) \cap G(e)
\]
Let bipolar fuzzy soft set \((K, M)\) is union of two bipolar fuzzy soft sets \((F, A)\) and \((H, C)\) which is
\[
(K, M) = (F, A) \cup (H, C) \quad (4)
\]
Define by
\[
K(e) = F(e) \text{ if } e \in A \cap C
\]
\[
= H(e) \text{ if } e \in C \setminus A
\]
\[
= F(e) \cup H(e) \text{ if } e \in A \cap C
\]
L.H.S. There are three cases.
Proof. (a) If \( e \in A \cap C \),
\[
K(e) = F(e) \quad \text{if} \quad e \in A \cap C
\]
\[
= F(e) \quad \text{if} \quad e \in A
\]
\[
K(e) = F(e)
\]
\[
(K, M) = (F, A) \quad \text{from Eq} \ 4
\]

Cases (2) If \( e \in C \setminus A = A \cap B - A = 0 \).
\[
K(e) = H(e) \quad \text{if} \quad e \in C \cap A = 0
\]
\[
= 0 \quad \text{if} \quad e \in 0
\]
\[
K(e) = 0
\]
\[
(K, M) = 0 \quad \text{from Eq} \ 4
\]

Cases (3) If \( e \in A \cap C \).
\[
K(e) = (F(e) \cup H(e)) \quad \text{if} \quad e \in A \cap C
\]
\[
C = A \cap B
\]
\[
= F(e) \cup (F(e) \cap G(e)) \quad \text{from Eq} \ 3
\]
\[
= F(e) \quad \text{since} \quad (F(e) \cap G(e)) \subseteq F(e)
\]
\[
= F(e)
\]
\[
K(e) = F(e)
\]
\[
(K, M) = (F, A) \quad \text{from Eq} \ 4
\]

It is satisfied in three cases. Hence
\[
(F, A) \cap ((F, A) \cap (G, B)) = (F, A)
\]

(2) same as above. \( \square \)

Theorem 5.11. Commutative property of bipolar fuzzy soft sets \( (F, A) \) and \( (G, B) \).

(a) \((F, A) \cap (G, B) = (G, B) \cap (F, A)\)

(b) \((F, A) \cup (G, B) = (G, B) \cup (F, A)\)

Proof. (1) To show that
\[
(F, A) \cap (G, B) = (G, B) \cap (F, A)
\]
A bipolar fuzzy soft set \((H, C)\) is an intersection of two bipolar fuzzy soft sets \((F, A)\) and \((G, B)\), where \( C = A \cap B \)
\[
H(e) = F(e) \cap G(e) \quad \text{if} \quad e \in C = A \cap B
\]

A bipolar fuzzy soft set \((K, D)\) is an intersection of two bipolar fuzzy soft sets \((G, B)\) and \((F, A)\), where \( D = B \cap A \)
\[
K(e) = G(e) \cap F(e) \quad \text{if} \quad e \in D = B \cap A
\]

To show that \((H, C) = (K, D)\)
\[
\text{L.H.S.}
\]
\[
H(e) = F(e) \cap G(e) = G(e) \cap F(e) = G(e) \cap F(e)
\]
\[
= K(e) \quad \text{for all} \quad e \in B \cap A = D
\]
\[
\text{R.H.S.}
\]

(2) To show that \((F, A) \cup (G, B) = (G, B) \cup (F, A)\)

L.H.S.
\[
(F, A) \cup (G, B) = (G, B) \cup (F, A)
\]

Hence \((F, A) \cup (G, B) = (G, B) \cup (F, A)\)

There are three cases.

Case (1) If \( e \in A \cap B \)
\[
H(e) = F(e) \quad \text{if} \quad e \in A \cap B \quad \text{from Eq} \ 8
\]

Case (2) If \( e \in B \cap A \)
\[
H(e) = G(e) \quad \text{if} \quad e \in B \cap A \quad \text{from Eq} \ 9
\]

Case (3) If \( e \in A \cap B \)
\[
H(e) = F(e) \cup G(e) \quad \text{if} \quad e \in A \cap B \quad \text{from Eq} \ 12
\]

Combine Eq 11, Eq 12 and Eq 13. We get
\[
H(e) = G(e) \quad \text{if} \quad e \in B \cap A
\]

\[= G(e) \cup F(e) \quad \text{if} \quad e \in B \cap A
\]

\[= (H, C) \quad \text{becomes}
\]

\[
(H, C) = (G, B) \cup (F, A) \quad \text{where} \quad C = B \cap A
\]

\[= R. H. S.
\]
Theorem 5.12.  Associative law of bipolar fuzzy soft sets \((F, A), (G, B)\) and \((H, C)\).

1. \( (F, A) \cap ((G, B) \cap (H, C)) = (F, A) \cap (G, B) \cap (H, C) \)

2. \( (F, A) \cup ((G, B) \cap (H, C)) = (F, A) \cup (G, B) \cap (H, C) \)

Proof. \((1)\) A bipolar fuzzy soft set \((L, D)\) is an intersection of two bipolar fuzzy soft sets \((G, B)\) and \((H, C)\) which is \((G, B) \cap (H, C) = (L, D)\) where \(D = B \cap C\) Define by \(L(x) = G(x) \cap H(x)\) \((14)\)

A bipolar fuzzy soft set \((M, X)\) is an intersection of two bipolar fuzzy soft sets \((F, A)\) and \((L, D)\) which is \((F, A) \cap (L, D) = (M, X)\) \((15)\)

Define by \(M(e) = F(e) \cap L(e) \) if \(e \in X = A \cap D\) \((16)\)

L.H.S:

\[
M(e) = F(e) \cap L(e) = F(e) \cap (G(e) \cap H(e)) = (F(e) \cap G(e)) \cap H(e) \\
M(e) = (F(e) \cap G(e)) \cap H(e) \\
(M, X) = (F, A) \cap (G, B) \cap (H, C) \\
(F, A) \cap (L, D) = (F, A) \cap (G, B) \cap (H, C) \\
(F, A) \cap ((G, B) \cap (H, C)) = ((F, A) \cap (G, B)) \cap (H, C) \\
\]

Hence \( (F, A) \cap ((G, B) \cap (H, C)) = (F, A) \cap (G, B) \cap (H, C) \)

(2) Same as above.

Hence \( (F, A) \cup ((G, B) \cap (H, C)) = (F, A) \cup (G, B) \cap (H, C) \)

Theorem 5.13.  Distributive law of bipolar fuzzy soft sets \((F, A), (G, B)\) and \((H, C)\).

1. \( (F, A) \cap ((G, B) \cup (H, C)) = (F, A) \cap (G, B) \cup (H, C) \)

2. \( (F, A) \cup ((G, B) \cap (H, C)) = (F, A) \cup (G, B) \cap (H, C) \)

Proof. \((1)\) A bipolar fuzzy soft set \((L, D)\) is union of two bipolar fuzzy soft sets \((G, B)\) and \((H, C)\) over a common universe \(U\).

\((G, B) \cup (H, C) = (L, D)\) where \(D = B \cup C\) \((17)\)

Define by \(L(x) = G(x) \cup H(x) \) if \(e \in B \cap C\)

\[= H(x) \) if \(e \in C \setminus B\]

\[= G(x) \) if \(e \in B \cap C\]

A bipolar fuzzy soft set \((M, V)\) is an intersection of two bipolar fuzzy soft sets \((F, A)\) and \((L, D)\).

\((F, A) \cap (L, D) = (M, V)\) \((18)\)

Define by \(M(e) = F(e) \cap L(e) \) if \(e \in V = A \cap D\) \((19)\)

L.H.S

\[M(e) = F(e) \cap L(e) = F(e) \cap (G(e) \cup H(e)) = (F(e) \cup G(e)) \cup H(e) \\
M(e) = (F(e) \cup G(e)) \cup H(e) \\
(M, V) = (F, A) \cap (G, B) \cup (H, C) \\
(F, A) \cap ((G, B) \cup (H, C)) = ((F, A) \cup (G, B)) \cup (H, C) \\
\]

Hence \( (F, A) \cap ((G, B) \cup (H, C)) = (F, A) \cup (G, B) \cup (H, C) \)

(2) Same as above.
Put Eq 21, Eq 22 and Eq 23 in Eq 20
\[ M(e) = F(e) \cap G(e) \]
for all \( e \in A, e \in B \cap C \)
\[ = F(e) \cap H(e) \]
if \( e \in A, e \in C \setminus B \)
\[ = F(e) \cap (G(e) \cup H(e)) \]
if \( e \in A, e \in B \setminus C \)
\[ = (F(e) \cap G(e)) \cup (F(e) \cap H(e)) \]
\[ M(e) = (F(e) \cap G(e)) \cup (F(e) \cap H(e)) \]
\[ (M, V) = \left( (F, A) \cap (G, B), (F, A) \sqcap (H, C) \right) \]
using Eq 18
\[ (F, A) \cap (G, B) \]
\[ (F, A) \cap (H, C) \]
\[ = (F, A) \cap (G, B) \cap (H, C) \]
using Eq 17

Hence proof.

(2). Same as above.

Lemma 5.14. \((F, A)\) and \((G, B)\) are two bipolar fuzzy soft sets.

1. \((F, A) \subseteq (G, B)\) \(\Rightarrow (F, A) \cap (G, B) = (F, A)\)
2. \((F, A) \subseteq (G, B)\) \(\Rightarrow (F, A) \cap (G, B) = (G, B)\).

6. De Morgan’s law of bipolar fuzzy soft sets

Theorem 6.1. De Morgan’s law of bipolar fuzzy soft sets \((F, A)\) and \((G, B)\).

1. \(\big((F, A) \cup (G, B)\big)^c = (F, A)^c \cap (G, B)^c\)
2. \(\big((F, A) \cap (G, B)\big)^c = (F, A)^c \cup (G, B)^c\)

Proof. (1) Let \((F, A)\) and \((G, B)\) be a two bipolar fuzzy soft sets over a common universe \(U\). Then the union of two bipolar fuzzy soft sets \((F, A)\) and \((G, B)\) is a bipolar fuzzy soft set \((H, C)\) where \(A = A \cup B\) and \((H, C) = (F, A) \cup (G, B)\) is defined by
\[ H(e) = F(e) \text{ if } e \in A \setminus B \]
\[ = G(e) \text{ if } e \in B \setminus A \]
\[ = F(e) \cup G(e) \text{ if } e \in A \cap B \]

Putting Eq 30, Eq 31 in Eq 29. We get
\[ (H(e))^c = (F(e))^c \cap (G(e))^c \]
if \( e \in A \cap B \).

\[ (H(e))^c = (F(e))^c \cap (G(e))^c \]
if \( e \in A \cap B \) and \( C = A \cap B \)
From Eq. 27, Eq 28 and Eq 32. We get
\[(H(e))' = (F(e))' \quad \text{if } e \in A \cup B\]
\[= (G(e))' \quad \text{if } e \in B \setminus A\]
\[= (F(e))' \cap \Gamma(G(e))'\]
if \(e \in A \cap B \) and \(C = A \cap B\)

Then
\[(H(e))' = (F(e))' \cap \Gamma(G(e))'\]

Thus
\[(F, A) \cup (G, B) = (F, A) \cap (G, B)'\]

Hence it is proved.
(2). Same as above.

7. OR and AND operations on bipolar fuzzy soft sets

Definition 7.1. Let \((F, A)\) and \((G, B)\) be two bipolar fuzzy soft sets over a common universe \(U\). Then,

1. \((F, A) \lor (G, B)\) is a bipolar fuzzy soft set defined by \((F, A) \lor (G, B) = (H, A \times B)\) where \(H(a, b) = F(a) \cap G(b)\) for all \((a, b) \in C = A \times B\), where \(\cap\) is the intersection operation of sets.

2. \((F, A) \land (G, B)\) is a bipolar fuzzy soft set defined by \((F, A) \land (G, B) = (H, A \times B)\) where \(H(a, b) = F(a) \lor G(b)\) for all \((a, b) \in C = A \times B\), where \(\lor\) is the intersection operation of sets.

Definition 7.2. Let \(T = \{(F_i, A_i) : i \in I\}\) be a family of bipolar fuzzy soft sets in a bipolar fuzzy soft class \((U, E)\). Then the union of bipolar fuzzy soft sets in \(T\) is bipolar fuzzy soft set \((H, C)\), where \(C = A_i\) for all \(i \in I\) and \(H(e) = \bigcup_{i \in I} H_i(e)\) for all \(e \in C, (H, C) = (\bigcup_{i \in I} (F_i, A_i))\).

Definition 7.3. Let \(T = \{(F_i, A_i) : i \in I\}\) be a family of bipolar fuzzy soft sets in a bipolar fuzzy soft class \((U, E)\). Then the union of bipolar fuzzy soft sets in \(T\) is bipolar fuzzy soft set \((H, C)\), where \(C = A_i\) for all \(i \in I\) and \(H(e) = \bigcup_{i \in I} H_i(e)\) for all \(e \in C, (H, C) = (\bigcup_{i \in I} (F_i, A_i))\) for all \(i \in I\).

Example 7.4. Let \(U = \{m_1, m_2, m_3, m_4\}\) be the set of four men under consideration and \(E = \{e_1 = \text{Educated}, e_2 = \text{Government employee}, e_3 = \text{Businessman}, e_4 = \text{Smart, } e_5 = \text{Weak}\}\) be the set of parameters and \(A = \{e_1, e_2\} \subseteq E, B = \{e_3, e_4\} \subseteq E\) Then

\[
F(e_1) = \{(m_1, 0.1, -0.5), (m_2, 0.3, -0.6), (m_3, 0.4, -0.2), (m_4, 0.7, -0.2)\},
\]

\[
F(e_2) = \{(m_1, 0.3, -0.5), (m_2, 0.4, -0.2), (m_3, 0.5, -0.2), (m_4, 0.4, -0.2)\}.
\]

\[
G(e_3) = \{(m_1, 0.1, -0.6), (m_2, 0.3, -0.4), (m_3, 0.1, -0.6), (m_4, 0.0, -0.2)\},
\]

\[
G(e_4) = \{(m_1, 0.4, -0.1), (m_2, 0.2, -0.4), (m_3, 0.6, -0.4), (m_4, 0.7, -0.0)\}.
\]

Then \((F, A) \lor (G, B) = (H, A \times B)\) and \(C = A \times B = \{e_1, e_2\} \times \{e_3, e_4\}\) define by \(H(a, b) = F(a) \cap G(b)\) for all \((a, b) \in C\).
\[ e_2 = \text{Beautiful}, \quad e_3 = \text{Wooden}, \quad e_4 = \text{In the green surrounding} \]

\[ A = \{e_1, e_2\} \subseteq E, \quad B = \{e_3, e_4\} \subseteq E. \]

For \( F(A) \) defined by

\[
\begin{align*}
F(e_1) = & \begin{cases} 
(h_1, 0.1, -0.5), \\
(h_2, 0.3, -0.6), \\
(h_3, 0.4, -0.2), \\
(h_4, 0.7, -0.2)
\end{cases}, \\
F(e_2) = & \begin{cases} 
(h_1, 0.3, -0.5), \\
(h_2, 0.4, -0.2), \\
(h_3, 0.5, -0.2), \\
(h_4, 0.4, -0.2)
\end{cases}, \\
F(e_3) = & \begin{cases} 
(h_1, 0.1, -0.6), \\
(h_2, 0.3, -0.4), \\
(h_3, 0.1, -0.6), \\
(h_4, 0.0, -0.2)
\end{cases}, \\
F(e_4) = & \begin{cases} 
(h_1, 0.4, -0.1), \\
(h_2, 0.2, -0.4), \\
(h_3, 0.6, -0.4), \\
(h_4, 0.7, -0.0)
\end{cases}
\end{align*}
\]

Then \( (F, A) \lor (G, B) = (H, A \times B) \) and \( C = A \times B \).

The set \( C = A \times B \) is defined by \( H(a) = F(a) \cup G(a) \), for all \( a \in C \).

\[
(H, C) = \begin{cases} 
(h_1, 0.1, -0.6), \\
(h_2, 0.3, -0.6), \\
(h_3, 0.4, -0.6), \\
(h_4, 0.7, -0.2)
\end{cases}, \\
H(e_1, e_2) = \begin{cases} 
(h_1, 0.3, -0.6), \\
(h_2, 0.4, -0.6), \\
(h_3, 0.6, -0.4), \\
(h_4, 0.7, -0.2)
\end{cases}, \\
H(e_1, e_3) = \begin{cases} 
(h_1, 0.3, -0.6), \\
(h_2, 0.4, -0.4), \\
(h_3, 0.5, -0.6), \\
(h_4, 0.4, -0.2)
\end{cases}, \\
H(e_1, e_4) = \begin{cases} 
(h_1, 0.3, -0.5), \\
(h_2, 0.4, -0.4), \\
(h_3, 0.6, -0.4), \\
(h_4, 0.7, -0.2)
\end{cases}, \\
H(e_2, e_3) = \begin{cases} 
(h_1, 0.2, -0.5), \\
(h_2, 0.3, -0.6), \\
(h_3, 0.4, -0.3), \\
(h_4, 0.7, -0.2)
\end{cases}, \\
H(e_2, e_4) = \begin{cases} 
(h_1, 0.1, -0.6), \\
(h_2, 0.3, -0.5), \\
(h_3, 0.6, -0.1), \\
(h_4, 0.4, -0.4)
\end{cases}
\]

Example 7.7. Let \( U = \{e_1, e_2, e_3, e_4\} \) be the set of four cars under consideration, \( E = \{e_1 = \text{Costly}, e_2 = \text{Beautiful}, e_3 = \text{Fuel efficient}, e_4 = \text{Luxurious}\} \) be the set of parameters and \( A = \{e_1, e_2, e_3\} \subseteq E \) then \( F : A \rightarrow BFU \) defined by

\[
\begin{align*}
F(e_1) = & \begin{cases} 
(c_1, 0.2, -0.5), \\
(c_2, 0.3, -0.6), \\
(c_3, 0.4, -0.3), \\
(c_4, 0.7, -0.2)
\end{cases}, \\
F(e_2) = & \begin{cases} 
(c_1, 0.1, -0.6), \\
(c_2, 0.3, -0.5), \\
(c_3, 0.6, -0.1), \\
(c_4, 0.4, -0.4)
\end{cases}, \\
F(e_3) = & \begin{cases} 
(c_1, 0.1, -0.6), \\
(c_2, 0.3, -0.5), \\
(c_3, 0.6, -0.1), \\
(c_4, 0.4, -0.4)
\end{cases}, \\
F(e_4) = & \begin{cases} 
(c_1, 0.1, -0.6), \\
(c_2, 0.3, -0.5), \\
(c_3, 0.6, -0.1), \\
(c_4, 0.4, -0.4)
\end{cases},
\end{align*}
\]
in which number of rows and number of columns are equal and both are labeled by the object name of the universe such as \( e_1, e_2, e_3, \ldots, e_n \) and the entries \( d_{ij} \) where \( d_{ij} \) is the number of parameters for which the value of \( d_j \) exceeds or equal to the value of \( d_i \).

**Algorithm.**

1. Input the set \( A \subseteq E \) of choice of parameters of Mr. X.
2. Consider the bipolar fuzzy soft set in tabular form.
3. Compute the comparison table of positive information function and negative information function.
4. Compute the positive information score and negative information score.
5. Compute the final score by subtracting positive information score from negative information score.

Find the maximum score, if it occurs in i-th row, then Mr. X will buy \( d_i \) due to certain reason, his second choice will be \( c_2 \) or \( c_3 \), so the score of \( c_2 \) or \( c_3 \) are same.

**Table 1**

<table>
<thead>
<tr>
<th>Membership score function</th>
<th>( e_1 )</th>
<th>( e_2 )</th>
<th>( e_3 )</th>
<th>( e_4 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( c_1 )</td>
<td>0.4</td>
<td>0.5</td>
<td>0.7</td>
<td></td>
</tr>
<tr>
<td>( c_2 )</td>
<td>0.6</td>
<td>0.3</td>
<td>0.5</td>
<td></td>
</tr>
<tr>
<td>( c_3 )</td>
<td>0.8</td>
<td>0.4</td>
<td>0.6</td>
<td></td>
</tr>
<tr>
<td>( c_4 )</td>
<td>0.5</td>
<td>0.7</td>
<td>0.4</td>
<td></td>
</tr>
</tbody>
</table>

**Table 2**

<table>
<thead>
<tr>
<th>Comparison table of the above table</th>
</tr>
</thead>
<tbody>
<tr>
<td>( e_1 )</td>
</tr>
<tr>
<td>-----------</td>
</tr>
<tr>
<td>( c_1 )</td>
</tr>
<tr>
<td>( c_2 )</td>
</tr>
<tr>
<td>( c_3 )</td>
</tr>
<tr>
<td>( c_4 )</td>
</tr>
</tbody>
</table>

**Table 3**

<table>
<thead>
<tr>
<th>Positive membership score function</th>
</tr>
</thead>
<tbody>
<tr>
<td>Row</td>
</tr>
<tr>
<td>-----</td>
</tr>
<tr>
<td>sum(a)</td>
</tr>
<tr>
<td>( e_1 )</td>
</tr>
<tr>
<td>( e_2 )</td>
</tr>
<tr>
<td>( e_3 )</td>
</tr>
<tr>
<td>( e_4 )</td>
</tr>
</tbody>
</table>
We study basic operations on bipolar fuzzy soft set.

We combine the concept of bipolar fuzzy set and soft set to introduced the concept of bipolar fuzzy soft sets. We examine some operations on bipolar fuzzy soft set. We study basic operations on bipolar fuzzy soft set. We also give an application of bipolar fuzzy soft set into decision making problem. We give a general algorithm to solved decision making problems by bipolar fuzzy soft set. Therefore, this paper gives an idea for the beginning of a new study for approximations of data with uncertainties. We will focus on the following problems: bipolar fuzzy soft relations, bipolar fuzzy soft matrix, bipolar fuzzy soft function and bipolar fuzzy soft graphs, and applications in artificial intelligence and general systems.

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References


Table 4 Tabular representation of negative information function.

<table>
<thead>
<tr>
<th>e1</th>
<th>e2</th>
<th>e3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 5 Comparison table of the above table.

<table>
<thead>
<tr>
<th>e1</th>
<th>e2</th>
<th>e3</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.5</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.3</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0.2</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 6 Negative membership score table.

<table>
<thead>
<tr>
<th>row</th>
<th>column</th>
<th>Non-membership</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.2</td>
<td>5.1</td>
<td>0.2</td>
</tr>
<tr>
<td>1.2</td>
<td>5.1</td>
<td>0.2</td>
</tr>
<tr>
<td>1.2</td>
<td>5.1</td>
<td>0.2</td>
</tr>
<tr>
<td>1.2</td>
<td>5.1</td>
<td>0.2</td>
</tr>
</tbody>
</table>

Table 7 Final score table.

<table>
<thead>
<tr>
<th>Positive</th>
<th>Negative</th>
<th>Final score</th>
</tr>
</thead>
<tbody>
<tr>
<td>e1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>e2</td>
<td>3</td>
<td>3</td>
</tr>
<tr>
<td>e3</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

9. Conclusions

We combine the concept of bipolar fuzzy set and soft set to introduced the concept of bipolar fuzzy soft sets. We examine some operations on bipolar fuzzy soft. We define extended union, intersection of two bipolar fuzzy soft set. We also give an application of bipolar fuzzy soft set into decision making problem. We give a general algorithm to solved decision making problems by bipolar fuzzy soft set. Therefore, this paper gives an idea for the beginning of a new study for approximations of data with uncertainties. We will focus on the following problems: bipolar fuzzy soft relations, bipolar fuzzy soft matrix, bipolar fuzzy soft function and bipolar fuzzy soft graphs, and applications in artificial intelligence and general systems.

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