θ*-Relation on Hypermodules and Fundamental Modules Over Commutative Fundamental Rings

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To cite this Article
Anvariyeh, S. M., Mirvakili, S. and Davvaz, B. 'θ*-Relation on Hypermodules and Fundamental Modules Over Commutative Fundamental Rings', Communications in Algebra, 36: 2, 622 — 631

To link to this Article: DOI: 10.1080/00927870701724078
URL: http://dx.doi.org/10.1080/00927870701724078

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\( \theta^*-RELATION \) ON HYPERMODULES AND FUNDAMENTAL MODULES OVER COMMUTATIVE FUNDAMENTAL RINGS

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The main tools in the theory of hyperstructures are the fundamental relations. The fundamental relation on a hypermodule over a hyperring was already introduced by Vougiouklis. The fundamental relation on a hypermodule over a hyperring is defined as the smallest equivalence relation so that the quotient would be the module over a ring. Note that generally the commutativity with respect to both sum in the (fundamental) module and product in the (fundamental) ring are not assumed. In this article we introduce a new strongly regular equivalence relation on hypermodules so that the quotient is module (with abelin group) over a commutative ring. Also we state the conditions that is equivalent with the transitivity of this relation and finally we characterize the complete hypermodules over hyperrings.

Key Words: Fundamental relation; Hypermodule; Hyperring; Strongly regular relation.

2000 Mathematics Subject Classification: 16Y99; 20N20.

1. INTRODUCTION

A hypergroupoid \((H, \circ)\) is a nonempty set \(H\) with a hyperoperation \(\circ\) defined on \(H\), that is, a mapping of \(H \times H\) into the family of nonempty subsets of \(H\). If \((x, y) \in H \times H\), its image under \(\circ\) is denoted by \(x \circ y\). If \(A, B\) are nonempty subsets of \(H\), then \(A \circ B\) is given by \(A \circ B = \bigcup \{x \circ y \mid x \in A, y \in B\}\). \(x \circ A\) is used for \(\{x\} \circ A\) and \(A \circ x\) for \(A \circ \{x\}\). A hypergroupoid \((H, \circ)\) is called a hypergroup in the sense of Marty (1934) if for all \(x, y, z \in H\) the following two conditions hold: (i) \(x \circ (y \circ z) = (x \circ y) \circ z\), (ii) \(x \circ H = H \circ x = H\). The second condition is called the reproduction axiom, and it means that for any \(x, y \in H\) there exist \(u, v \in H\) such that \(y \in x \circ u\) and \(y \in v \circ x\). If \((H, \circ)\) satisfies only the first axiom, then it is called a semi-hypergroup. An exhaustive review updated to 1992 of hypergroup theory appears in Corsini (1993). A recent book (Corsini and Leoreanu, 2003) contains a wealth of applications.

If \(H\) is a semi-hypergroup and \(\rho \subseteq H \times H\) is an equivalence relation, then for all pairs \((A, B)\) of nonempty subsets of \(H\), we set \(\overline{A}\rho B\) if and only if \(a \rho b\) for all \(a \in A\) and \(b \in B\). The relation \(\rho\) is said to be strongly regular to the right if \(x \rho y\) implies \(x \circ a = y \circ a\) for all \((x, y, a) \in H^3\). Analogously, we can define strongly regular...
to the left. Moreover, $\rho$ is called strongly regular if it is strongly regular to the right and to the left. Let $H$ be a hypergroup and $\rho$ an equivalence relation on $H$. Let $\rho(a)$ be the equivalence class of $a$ with respect to $\rho$ and let $H/\rho = \{\rho(a) \mid a \in H\}$. A hyperoperation $\otimes$ is defined on $H/\rho$ by $\rho(a) \otimes \rho(b) = \{\rho(x) \mid x \in \rho(a) \circ \rho(b)\}$. If $\rho$ is strongly regular, then it readily follows that $\rho(a) \otimes \rho(b) = \{\rho(x) \mid x \in a \circ b\}$. It is well known for $\rho$ strongly regular that $(H/\rho, \otimes)$ is a group (see Theorem 31 in Corsini, 1993), that is, $\rho(a) \otimes \rho(b) = \rho(c)$ for all $c \in a \circ b$.

A hyperring (Vougiouklis, 1987) is a multivalued system $(R, +, \circ)$ which satisfies the ring-like axioms in the following way:

(i) $(R, +)$ is a hypergroup in the sense of Marty;
(ii) $(R, \circ)$ is a semi-hypergroup;
(iii) The multiplication is distributive with respect to the hyperoperation $\circ$.


The fundamental relation $\beta^*$ was introduced on hypergroups by Koskas (1970), and studied by many authors, for example see Corsini and Leoreanu (1996, 2003), Davvaz (2003), Freni (1991), Pelea (2001), and Vougiouklis (1988, 1995). The fundamental relation $\beta^*$ is defined on a hypergroup as the smallest equivalence relation so that the quotient would be a group. Let $H$ be a hypergroup and $U$ be the set of all finite products of elements of $H$ and define the relation $\beta$ on $H$ as follows:

$$x \beta y \quad \text{if and only if } \{x, y\} \subseteq u \quad \text{for some } u \in U.$$  

Freni (1991) proved that for hypergroups we have $\beta^* = \beta$. Freni turned his research on the direction to find classes on semi-hypergroups, such that the above equality works. Among others he defined the $\gamma$-relation on semi-hypergroups and hypergroups, see Freni (2002, 2004). We recall the following definition from Freni (2002). If $H$ is a hypergroup, then we set $\gamma_1 = \{(x, x) \mid x \in H\}$ and, for every integer $n > 1$, $\gamma_n$ is the relation defined as follows: $x \gamma_n y$ if and only if there exist $(z_1, z_2, \ldots, z_n) \in H^n$ and $\sigma \in S_n$ such that

$$x = \prod_{i=1}^n z_i, \quad y = \prod_{i=1}^n z_{\sigma(i)}.$$

Obviously, for every $n \geq 1$, the relations $\gamma_n$ are symmetric, and the relation $\gamma = \bigcup_{n \geq 1} \gamma_n$ is reflexive and symmetric. Let $\gamma^*$ be the transitive closure of $\gamma$. In Freni (2002), it is proved that $\gamma^*$ is the smallest strongly regular equivalence such that $H/\gamma^*$ is an abelian group. Also, in Theorem 3.3 of Freni (2002), the author proved that in every hypergroup, the relation $\gamma$ is transitive, that is $\gamma^* = \gamma$, and in this case, according to Corollary 1.2 of Freni (2002), the quotient $H/\gamma^*$ is an abelian group. The $\gamma^*$-relation is, in some sense, a generalization of the relation $\beta^*$, also see Davvaz and Karimian (2007).

The fundamental relation on a hyperring was introduced by Vougiouklis (1991) at the fourth AHA congress (1990). The fundamental relation on a hyperring is defined as the smallest equivalence relation so that the quotient would be the (fundamental) ring. Note that the commutativity with respect to both sum and
product in the fundamental ring are not assumed. Davvaz and Vougiouklis (2007) 
introduced a new strongly regular equivalence relation on a hyperring such that the 
set of quotients is an ordinary commutative ring. We recall the following definition 
from Davvaz and Vougiouklis (2007).

**Definition 1.1.** Let $R$ be a hyperring. We define the relation $\sim$ as follows: 
$x \sim y$ if and only if there exist $n \in \mathbb{N}$, $(k_1, \ldots, k_n) \in \mathbb{N}^n$, 
$\sigma \in S_n$, and there exist 
$(x_{i_1}, \ldots, x_{i_k}) \in R^{k}$, $x_{i_j} \in S_{k_i}$, $(i = 1, \ldots, n)$] such that 

$$ 
x \in \sum_{i=1}^{n} \left( \prod_{j=1}^{k_i} x_{i_j} \right) \quad \text{and} \quad y \in \sum_{i=1}^{n} A_{\sigma(i)}, $$

where $A_i = \prod_{j=1}^{k_i} x_{i\sigma(j)}$.

The relation $\sim$ is reflexive and symmetric. Let $\sim^*$ be the transitive closure of $\sim$, 
then we have the following theorem.

**Theorem 1.2** (Davvaz and Vougiouklis, 2007). $\sim^*$ is a strongly regular relation 
both on $(R, +)$ and $(R, \cdot)$, and the quotient $R/\sim^*$ is a commutative ring.

### 2. $\theta$-RELATION ON HYPERMODULES

Let $(M, +)$ be a hypergroup and $(R, +, \cdot)$ be a hyperring. According to 
Vougiouklis (1994) $M$ is said to be a hypermodule over a hyperring $R$ if there exists 

$$ 
\cdot : R \times M \rightarrow \wp(M); \quad (a, m) \mapsto a \cdot m
$$

such that for all $a, b \in R$ and $m_1, m_2, m \in M$, we have:

1) $a \cdot (m_1 + m_2) = a \cdot m_1 + a \cdot m_2$;
2) $(a + b) \cdot m = (a \cdot m) + (b \cdot m)$;
3) $(a \cdot b) \cdot m = a \cdot (b \cdot m)$.

The fundamental relation $\epsilon^*$ on $M$ can be defined as the smallest equivalence 
relation such that the quotient $M/\epsilon^*$ be a module over the corresponding 
fundamental ring such that $M/\epsilon^*$ as a group is not abelian. Moreover, the 
fundamental ring is not commutative with respect to both sum and product. 
Now, we would like the fundamental module as a group to be abelian and the 
fundamental ring to be commutative with respect to both sum and product.

**Definition 2.1.** Let $R$ be a hyperring and $M$ be a hypermodule over $R$. We define 
the relation $\theta$ as follows: $x \theta y$ if and only if there exist $n \in \mathbb{N}$, 
$(m_1, \ldots, m_n) \in M^n$, $(k_1, k_2, \ldots, k_n) \in \mathbb{N}^n$, $\sigma \in S_n$, 
$(x_{i_1}, x_{i_2}, \ldots, x_{i_k}) \in R^{k}$, $\sigma_i \in S_{k_i}$, $\sigma_{ij} \in S_{k_{ij}}$ such that 

$$ 
x \in \sum_{i=1}^{n} m'_i, \quad m'_i = m_i \quad \text{or} \quad m'_i = \sum_{j=1}^{n_i} \left( \prod_{k=k_{ij}}^{k_j} x_{ijk} \right) m_i $$

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and \( y \in \sum_{i=1}^{n} m'_{\sigma(i)} \) where

\[
m'_{\sigma(i)} = \begin{cases} 
m_{\sigma(i)} & \text{if } m'_i = m_i \\
B_{\sigma(i)m_{\sigma(i)}} & \text{if } m'_i = \sum_{j=1}^{n_i} \left( \prod_{k=1}^{k_i} x_{ijk} \right) m_i
\end{cases}
\]

such that \( B_i = \sum_{j=1}^{n_i} A_{i\sigma(j)} \) and \( A_{ij} = \prod_{k=1}^{k_i} x_{ij\sigma_k(k)} \).

The relation \( \theta \) is reflexive and symmetric. Let \( \theta^* \) be transitive closure of \( \theta \).

**Lemma 2.2.** \( \theta^* \) is a strongly regular relation both on \((M, +)\) and \(M\) as an \(R\)-hypermodule.

**Proof.** Clearly, \( \theta^* \) is an equivalence relation. In order to prove that it is strongly regular, it is enough to show that for every \((x, y) \in M^2\)

\[
x \theta^* y \iff \begin{cases} 
x + a \overline{\theta} y + a, & a \neq x \theta^* y, \ \forall a \in M \\
x \cdot r \theta^* r \cdot y, & \forall r \in R.
\end{cases}
\]

If \( x \theta^* y \), then \( x \in \sum_{i=1}^{n} m'_i \) such that \( m'_i = m_i \) or \( m'_i = \sum_{j=1}^{n_i} \left( \prod_{k=1}^{k_i} x_{ijk} \right) m_i \) and \( y \in \sum_{i=1}^{n} m'_{\sigma(i)} \). Therefore if \( m'_i = m_i \), then \( m'_{\sigma(i)} = m_{\sigma(i)} \) and if \( m'_i = \sum_{j=1}^{n_i} \left( \prod_{k=1}^{k_i} x_{ijk} \right) m_i \), then \( m'_{\sigma(i)} = B_{\sigma(i)m_{\sigma(i)}}, B_i = \sum_{j=1}^{n_i} A_{i\sigma(j)}, A_{ij} = \prod_{k=1}^{k_i} x_{ij\sigma_k(k)} \), therefore \( x + a \in \sum_{i=1}^{n} m'_i + a \) and \( y + a \in \sum_{i=1}^{n} m'_{\sigma(i)} + a \).

Now, let \( \tau \) be the permutation of \( S_{n+1} \) such that \( \tau(i) = \sigma(i), i = 1, 2, \ldots, n \) and \( \tau(n+1) = n + 1 \) and \( m_{n+1} = m'_{n+1} = a \) and therefore \( m'_{\tau(n+1)} = a \). Thus \( x + a \in \sum_{i=1}^{n+1} m'_i \), and \( y + a \in \sum_{i=1}^{n+1} m'_{\sigma(i)} \). Therefore for all \( u \in x + a \) and \( v \in y + a \), we have \( u \theta v \). Thus \( x + a \overline{\theta} y + a \). In the same way we can show that \( a + x \overline{\theta} y + a \). It is easy to see that \( x + a \overline{\theta} y + a \) and \( a + x \overline{\theta} a + y \).

Now we prove if \( x \theta^* y \), then \( r \cdot x \overline{\theta} r \cdot y \). Let \( x \in \sum_{i=1}^{n} m'_i \) and \( y \in \sum_{i=1}^{n} m'_{\sigma(i)} \) hence \( r \cdot x \in r \cdot \sum_{i=1}^{n} m'_i \subseteq \sum_{i=1}^{n} r \cdot m'_i \) and \( r \cdot y \in r \cdot \sum_{i=1}^{n} m'_{\sigma(i)} \subseteq \sum_{i=1}^{n} r \cdot m'_{\sigma(i)} \). Now if \( u \in r \cdot x \) and \( v \in r \cdot y \), then \( u \in \sum_{i=1}^{n} r \cdot m'_i \) and \( v \in \sum_{i=1}^{n} r \cdot m'_{\sigma(i)} \). Because \( u \theta v \), we have \( r \cdot x \overline{\theta} r \cdot y \).

**Theorem 2.3.** The (abelian group) \( M/\theta^* \) is an \( R/x^* \)-module where \( R/x^* \) is a commutative ring.

**Proof.** According to Theorem 1.2, \( R/x^* \) is a commutative ring. Let \( a, b \in M \) and \( x \in R \). We define \( \oplus \) and \( \ominus \) on \( M/\theta^* \) in the usual manner:

\[
\theta^*(a) \oplus \theta^*(b) = \theta^*(c) \quad \text{for all } c \in \theta^*(a) + \theta^*(b),
\]

\[
\alpha^*(x) \ominus \theta^*(b) = \theta^*(d) \quad \text{for all } d \in \alpha^*(x) \cdot \theta^*(b).
\]

We prove both \( \oplus \) and \( \ominus \) are well defined. We have \( a' \theta^* a \) if and only if there exist \( a_1, \ldots, a_{p+1}, a_1 = a', a_{p+1} = a \) such that \( a_i \theta a_{i+1} \) (\( r = 1, \ldots, p \)), and \( b' \theta^* b \) if and
only if there exist \(b_1, \ldots, b_{q+1}, b_i = b', b_{q+1} = b\) such that \(b_i \equiv b_{q+1} (s = 1, \ldots, q)\). Then \(a_i \equiv b_{r+1}\) if and only if there exist \(n_r \in \mathbb{N}, \sigma \in \mathbb{S}_{n_r}\), and \(\sigma_{st} \in \mathbb{S}_{st} (i = 1, \ldots, n_r)\) such that

\[
a_r \equiv \sum_{i=1}^{n_r} m'_{r(i)}, \quad \text{and} \quad a_{r+1} \equiv \sum_{i=1}^{n_r} m'_{r(i)}.
\]

\(b_i \equiv b_{q+1}\) if and only if there exist \(n_s \in \mathbb{N}, \tau \in \mathbb{S}_{n_s}\), and \(\tau_{st} \in \mathbb{S}_{st} (i = 1, \ldots, n_s)\) such that

\[
b_s \equiv \sum_{i=1}^{n_s} m''_{s(i)}, \quad \text{and} \quad b_{q+1} \equiv \sum_{i=1}^{n_s} m''_{s(i)}.
\]

Therefore we obtain

\[
a_r + b_1 \equiv \sum_{n=1}^{n_r} m'_{r(i)} + \sum_{i=1}^{n_1} m''_{l(i)},
\]

\[
a_{r+1} + b_1 \equiv \sum_{n=1}^{n_r} m'_{r(i)} + \sum_{i=1}^{n_1} m''_{l(i)}
\]

and

\[
a_{p+1} + b_s \equiv \sum_{i=1}^{n_p} m'_{p(i)} + \sum_{i=1}^{n_s} m''_{s(i)},
\]

\[
a_{p+1} + b_{q+1} \equiv \sum_{i=1}^{n_p} m'_{p(i)} + \sum_{i=1}^{n_s} m''_{s(i)}
\]

Now, pick up elements \(c_1, \ldots, c_{p+q}\) such that \(c_r \in a_r + b_1 (r = 1, \ldots, p), c_{p+q} \in a_{p+1} + b_{q+1}\), and using the above relations we get \(c_r \equiv b_{r+1}\). Therefore every element \(c_1 \in a_1 + b_1 = a' + b'\) is \(\theta^*\)-equivalent to every element \(c_{p+q} \in a_{p+1} + b_{q+1} = a + b\). Therefore

\[
\theta^*(a) \oplus \theta^*(b) = \theta^*(c) \quad \text{for all} \quad c \in \theta^*(a) + \theta^*(b).
\]

In a similar way, it is proved that \(\alpha^*(x) \cap \theta^*(b) = \theta^*(c)\) for all \(c \in \alpha^*(x) \cdot \theta^*(b)\). Suppose that \(\sigma\) is the permutation of \(\mathbb{S}_2\) such that \(\sigma(1) = 2\). For every \(x \in a_1 + a_2, y \in a_{\sigma(1)} + a_{\sigma(2)}\), we have \(x \theta^* y\), thus \(x \theta^* y\) and so

\[
\theta^*(a_1) \oplus \theta^*(a_2) = \theta^*(x) = \theta^*(a_2) \oplus \theta^*(a_1).
\]

Therefore \(M/\theta^*\) is an abelian group. Since \(M\) is an \(R\)-hypermodule, the properties of \(M\) as an \(R\)-hypermodule, grantee that the abelian group \(M/\theta^*\) is an \(R/\alpha^*\)-module.

\[\square\]

**Theorem 2.4.** The relation \(\theta^*\) is the smallest equivalence relation such that the (abelian) quotient \(M/\theta^*\) is an \(R/\alpha^*\)-module.
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\textbf{Proof.} Let \( \rho \) be an equivalence relation such that \( M/\rho \) is an \( R/x^* \)-module and let \( \Phi : M \to M/\rho \) be the canonical projection. If \( x \theta y \), then there exist \( n \in \mathbb{N} \) and \( \sigma \in \mathbb{S}_n \) such that

\[
x \in \sum_{i=1}^n m_i' \quad \text{and} \quad y \in \sum_{i=1}^n m_{\sigma(i)}'.
\]

So \( \Phi(x) \in \Phi(\sum_{i=1}^n m_i') = \bigoplus \sum_{i=1}^n m_i' \in M/\rho \) and \( \Phi(y) \in \Phi(\sum_{i=1}^n m_{\sigma(i)}') = \bigoplus \sum_{i=1}^n m_{\sigma(i)}' \in M/\rho \). By the commutativity of \( M/\rho \) it follows that \( \Phi(x) = \Phi(y) \). Thus \( x \theta y \) implies that \( x \rho y \) thus \( \theta \subseteq \rho \). Finally, since \( \rho \) is transitively closed, we obtain \( \theta^* \subseteq \rho \). □

\textbf{Definition 2.5.} If \( M \) is a hypermodule over a hyperring \( R \), then we set

\[
\theta_0 = \{(m, m) \mid m \in M\}
\]

and for every integer \( n \geq 1 \), \( \theta_n \) is the relation defined as follows:

\[
x \theta_n y \quad \text{if and only if} \quad x \in \sum_{i=1}^n m_i' \quad \text{and} \quad y \in \sum_{i=1}^n m_{\sigma(i)}' \quad \text{with} \quad \sigma \in \mathbb{S}_n.
\]

Obviously, for every \( n \geq 1 \), the relation \( \theta_n \) is symmetric, and the relation \( \theta = \bigcup_{n \geq 0} \theta_n \) is reflexive and symmetric.

\textbf{Lemma 2.6.} If \( M \) is a hypermodule over a hyperring \( R \) and \( n \geq 1 \), then \( \theta_n \subseteq \theta_{n+1} \).

\textbf{Proof.} If \( x \theta_n y \), then there exist \( (m_1', \ldots, m_n') \) and \( \sigma \in \mathbb{S}_n \) such that \( x \in \sum_{i=1}^n m_i' \), \( y \in \sum_{i=1}^n m_{\sigma(i)}' \). Since \( M \) is a hypergroup, so there exists \( (t_1, t_1) \) such that \( m_i' \subseteq t_1 + t_i \). Let \( m_i'' = m_i' \) for \( 1 \leq i \leq n-1 \) and \( m_n'' = t_1, m_{n+1}'' = t_2 \), so \( x \in m_1'' + m_2'' + \cdots + m_n'' + m_{n+1}'' \). Now let \( \sigma' = (n \ n+1) \sigma \), so \( \sigma' \in \mathbb{S}_{n+1} \) and \( y \in \sum_{i=1}^{n+1} m_{\sigma'(i)}'' \). Therefore \( x \theta_{n+1} y \). □

\textbf{Lemma 2.7.} For every \( x, y, a \in M \) and \( r \in R \), if \( x \theta_n y \), then

\[
(x + a)\overline{\theta}_{n+1}(y + a), \quad (a + x)\overline{\theta}_{n+1}(a + y)
\]

and

\[
r \cdot a \overline{\theta}_{n} b, \quad a \cdot r \overline{\theta}_{n} b \cdot r.
\]

\textbf{Proof.} Let \( x \theta_n y \), then \( x \in \sum_{i=1}^n m_i' \) and \( y \in \sum_{i=1}^n m_{\sigma(i)}' \). Therefore \( x + a \subseteq \sum_{i=1}^n m_i' + a \) and \( y + a \subseteq \sum_{i=1}^n m_{\sigma(i)}' + a \). Now, let \( \tau \) be the permutation of \( \mathbb{S}_{n+1} \) such that \( \tau(i) = \sigma(i) \), \( i = 1, 2, \ldots, n \) and \( \tau(n+1) = n+1 \) and \( m_{n+1} = m_{n+1} + a \). Thus \( x + a \subseteq \sum_{i=1}^{n+1} m_i' \), and \( y + a \subseteq \sum_{i=1}^{n+1} m_{\sigma(i)}' \). Therefore for all \( u \in x + a \) and \( v \in y + a \), we have \( u \overline{\theta}_{n+1} v \). Thus \( x + a \overline{\theta}_{n+1} y + a \). In a same way we can show that \( a + x \theta_{n+1} a + y \).
Now, we prove that if $x\theta_n y$, then $r \cdot x \theta \overline{r} \cdot y$. Let $x \in \sum_{i=1}^n m'_i$ and $y \in \sum_{i=1}^n m'_{\sigma(i)}$. Then $r \cdot x \subseteq r \cdot \sum_{i=1}^n m'_i \subseteq \sum_{i=1}^n r \cdot m'_i$ and $r \cdot y \subseteq r \cdot \sum_{i=1}^n m'_{\sigma(i)} \subseteq \sum_{i=1}^n r \cdot m'_{\sigma(i)}$.

Now if $u \in r \cdot x$ and $v \in r \cdot y$, then $u \in \sum_{i=1}^n r \cdot m'_i$ and $v \in \sum_{i=1}^n r \cdot m'_{\sigma(i)}$. Therefore $u \theta_n v$ and we have $r \cdot x \theta \overline{r} \cdot y$.

3. TRANSITIVITY CONDITION OF $\theta$

**Definition 3.1.** Let $H$ be a nonempty subset of $M$. We say that $H$ is a $\theta$-part of $M$ if for every $n \in \mathbb{N}$, for every $\sigma \in S_n$ and for every $(m'_1, \ldots, m'_n)$

\[
\sum_{i=1}^n m'_i \cap H \neq \emptyset \Rightarrow \sum_{i=1}^n m'_{\sigma(i)} \subseteq H.
\]

**Lemma 3.2.** Let $H$ be a nonempty subset of a $R$-hypermodule $M$. The following conditions are equivalent:

1) $H$ is a $\theta$-part of $M$;
2) $x \in H$, $x\theta y \Rightarrow y \in H$;
3) $x \in H$, $x\theta^* y \Rightarrow y \in H$.

**Proof.** (1 $\Rightarrow$ 2) If $(x, y) \in M^2$ is a pair such that $x \in H$ and $x\theta y$, then there exist $(m'_1, \ldots, m'_n) \in M^n$ and $\sigma \in S_n$ such that $x \in (\sum_{i=1}^n m'_i) \cap H$ and $y \in \sum_{i=1}^n m'_{\sigma(i)}$. Since $H$ is a $\theta$-part of $M$, we have $\sum_{i=1}^n m'_{\sigma(i)} \subseteq H$ and $y \in H$.

(2 $\Rightarrow$ 3) Let $(x, y) \in M^2$ such that $x \in H$ and $x \in \theta^*(y)$. Obviously, there exist $m \in \mathbb{N}$ and $(w_0 = x, w_1, \ldots, w_m = y) \in M^{m+1}$ such that $x = w_0 \theta w_1 \ldots \theta w_m = y$. Since $x \in H$, applying (2) $m$ times, we obtain $y \in H$.

(3 $\Rightarrow$ 1) Let $\sum_{i=1}^n m'_i \cap H \neq \emptyset$ and $x \in \sum_{i=1}^n m'_i \cap H$. For every $\sigma \in S_n$ and for every $y \in \sum_{i=1}^n m'_{\sigma(i)}$, we obtain $y \in H$, whence $\sum_{i=1}^n m'_{\sigma(i)} \subseteq H$. □

Before proving the next theorem we introduce the following notation:

\[
[x, z]_{k_1, \ldots, k_n} = \{(x_{i_1}, \ldots, x_{i_{k_i}}), (z_{i_1}, \ldots, z_{i_{k_i}}) \mid x_{i_j} \in R, z_{i_j} \in M, (i = 1, \ldots, n)\};
\]

\[
T_n(u) = \{[x, z]_{k_1, \ldots, k_n} \mid u \in \sum_{i=1}^n m'_i, m'_i = z_i \text{ or } m'_i = \sum_{j=1}^{k_i} (\prod_{k=1}^{j} x_{i_j}) z_i\};
\]

\[
P_n(u) = \bigcup_n \{\sum_{i=1}^n m'_{\sigma(i)} \mid [x, z]_{k_1, \ldots, k_n} \in T_n(u), \sigma \in S_n\};
\]

\[
P_n(u) = \bigcup_n P_n(u).
\]

**Lemma 3.3.** For every $u \in M$, $P_n(u) = \{v \in M \mid u\theta v\}$.

**Proof.** For every pair $(u, v)$ of elements of $M$ we have

\[
u \theta v \Leftrightarrow \text{there exists } n \in \mathbb{N} \text{ such that } u \in \sum_{i=1}^n m'_i, v \in \sum_{i=1}^n m'_{\sigma(i)}
\]
\( \theta^*\)-RELATION ON HYPERMODULES

\[ \iff \text{there exists } n \in \mathbb{N} \text{ such that } v \in P_n(u) \]
\[ \iff v \in P_\sigma(u). \]

**Theorem 3.4.** Let \( M \) be a hypermodule over a hyperring \( R \). The following conditions are equivalent:

1) \( \theta \) is transitive;
2) For every \( u \in M \), \( \theta^*(u) = P_\sigma(u) \);
3) For every \( u \in M \), \( P_\sigma(u) \) is \( \theta^* \)-part of \( M \).

**Proof.** (1 \( \Rightarrow \) 2) By Lemma 3.3 for every pair \((u, v)\) of elements of \( M \) we have

\[ v \in \theta^*(u) \iff u \theta^* v \iff u \theta v \iff v \in P_\sigma(u). \]

(2 \( \Rightarrow \) 3) By Lemma 3.2 if \( H \) is a nonempty subset of \( M \), then \( H \) is a \( \theta^* \)-part of \( M \) if and only if it is union of equivalence classes modulo \( \theta^* \). Particularly, every equivalence class modulo \( \theta^* \) is a \( \theta^* \)-part.

(3 \( \Rightarrow \) 1) Let \( u \theta v \) and \( u \theta w \), then there exist \((n, m) \in \mathbb{N} \times \mathbb{N}\), \([x, z]_{k_1, \ldots, k_s} \in T_n(u)\), \([y, z']_{k_1', \ldots, k_s'} \in T_m(v)\), \( \sigma \in \mathbb{S}_n \), and \( \tau \in \mathbb{S}_m \) such that \( v \in \sum_{i=1}^n m_{\sigma(i)}' \) and \( w \in \sum_{i=1}^m z_{\tau(i)}' \). Since \( P_\sigma(u) \) is a \( \theta^* \)-part of \( M \) we have

\[ v \in \sum_{i=1}^m z_{\tau(i)}' \cap P_\sigma(u) \Rightarrow \sum_{i=1}^n m_{\sigma(i)}' \subseteq P_\sigma(u) \Rightarrow v \in P_\sigma(u) \]
\[ \Rightarrow \sum_{i=1}^m z_{\tau(i)}' \subseteq P_\sigma(u) \Rightarrow w \in P_\sigma(u) \Rightarrow \exists k \in \mathbb{N} \]
\[ w \in P_k(u) \Rightarrow u \theta w. \]

**Definition 3.5.** A hypermodule \( M \) over a hyperring \( R \) is said to be \( n \)-complete if for every \((m_1', \ldots, m_n') \in M^n\),

\[ \theta \left( \sum_{i=1}^n m_i' \right) = \sum_{i=1}^n m_i'. \]

**Definition 3.6.** A hypermodule \( M \) is \( \theta_n \)-complete if for every \((m_1', \ldots, m_n') \in M^n\) and \( \sigma \in \mathbb{S}_n\),

\[ \theta \left( \sum_{i=1}^n m_i' \right) = \sum_{i=1}^n m_{\sigma(i)}'. \]

**Theorem 3.7.** A hypermodule \( M \) over a hyperring \( R \) is \( \theta_n \)-complete if and only if for every \((m_1', \ldots, m_n') \in M^n\), \( \sigma \in \mathbb{S}_n \), and \( z \in \sum_{i=1}^n m_i' \),

\[ \theta(z) = \sum_{i=1}^n m_{\sigma(i)}'. \]
\textbf{Proof.} Let $M$ is $\theta_n$-complete, and let $z \in \sum_{i=1}^{n} m_i$, we have

$$\theta(z) \subseteq \bigcup_{z \in \sum_{i=1}^{n} m_i'} \theta(z) = \theta\left(\sum_{i=1}^{n} m_i'\right) = \sum_{i=1}^{n} m_{\sigma(i)}'.$$

Hence $\theta(z) \subseteq \sum_{i=1}^{n} m_{\sigma(i)}'$. Now if $z \in \sum_{i=1}^{n} m_i'$, then $\theta(z) \subseteq \sum_{i=1}^{n} m_{\sigma(i)}'$. Therefore if $y \in \sum_{i=1}^{n} m_{\sigma(i)}'$, then

$$z \theta_n y \implies z \theta y \implies y \in \theta(z).$$

Conversely, for every $(m_1', \ldots, m_n')$, and every $z \in \sum_{i=1}^{n} m_i'$, we have $\theta(z) = \sum_{i=1}^{n} m_{\sigma(i)}'$, therefore

$$\theta\left(\sum_{i=1}^{n} m_i'\right) = \bigcup_{z \in \sum_{i=1}^{n} m_i'} \theta(z) = \sum_{i=1}^{n} m_{\sigma(i)}'$$

and hence $M$ is $\theta_n$-complete. \hfill \Box

\textbf{Proposition 3.8.} If $M$ is a $\theta_n$-complete hypermodule, then $\theta = \theta_n$.

\textbf{Proof.} It is suffices to prove that $\theta \subseteq \theta_n$. Suppose $x \theta y$ thus there exists $m \in \mathbb{N}$ such that $x \theta_m y$. If $m \leq n$, then by Lemma 2.6, $\theta_m \subseteq \theta_n$. If $m > n$, then there exist $(m_1', \ldots, m_n')$ and $\sigma \in S_n$ such that $x \in \sum_{i=1}^{m} m_i'$, and $y \in \sum_{i=1}^{m} m_{\sigma(i)}'$. Since $(M, +)$ is a hypergroup, so there exists $s \in M$ such that $x \in \sum_{i=1}^{m-1} m_i' + s$ and we have $y \in \theta(x) = \sum_{i=1}^{n} m_{\sigma(i)}'$. Therefore $y \in \sum_{i=1}^{n} m_{\sigma(i)}'$, so $x \theta_n y$. \hfill \Box

\textbf{REFERENCES}


