A modification of the homotopy analysis method based on Chebyshev operational matrices

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1. Introduction

The homotopy analysis method (HAM) [1] is a powerful tool, and it has already been used for solving several nonlinear problems [2–18]. The homotopy analysis method is based on replacing a nonlinear equation by a system of ordinary differential equations which is solved via symbolic computation software such as Maple, Mathematica, or Matlab to obtain the solution of the system. This solution provides us with a convergent series which, as proved by Liao in [1,19], is the solution of the original nonlinear equation. In the HAM, the initial approximation, auxiliary linear operator, auxiliary function and the convergence controlling auxiliary parameter should be considered carefully to ensure the convergence of the solution series. Suggestions on how to select this combination of parameters are given in [20–23] for general nonlinear problems.

However, despite its many documented successes, the homotopy analysis method suffers from a number of deficiencies; see [24]. As pointed out in [24], one of the one of the main limitations of the HAM is the requirement that the solution sought ought to conform to the so-called rule of solution expression and the rule of coefficient ergodicity that guide us in choosing the appropriate initial approximations, the auxiliary linear operators and the auxiliary functions. In a recent study, Motsa et al. [24–27] proposed a spectral modification of the homotopy analysis method, the spectral-homotopy analysis method (SHAM), which seeks to remove some of the restrictive assumptions associated with the implementation of the standard homotopy analysis method. They applied the Chebyshev spectral collocation differentiation matrix to define the auxiliary linear operator. Also, they showed that the SHAM is more flexible than the HAM as it allows for a wider range of linear operators and one is not restricted to use the method of higher-order differential mapping.
In this paper, by constructing shifted Chebyshev functions and their operational matrices, we introduce a novel algorithm based on the Tau and homotopy analysis method (THAM) to solve the higher-order deformation equations. The operational matrix of derivatives obtained by the Tau method is more sparse than the collocation differentiation matrix used in the SHAM. The proposed method enables us to reduce these equations to a system of algebraic equations that can be used iteratively to obtain the solution of the problem.

The outline of the paper is as follows. In Section 2, the operational matrices of shifted Chebyshev functions are constructed. Section 3 presents the methodology of the Tau homotopy analysis method (THAM) along with error analysis. In Section 4, we apply the THAM with shifted Chebyshev functions to solve two boundary value problems. Finally, Section 5 presents our conclusion.

2. Chebyshev polynomials

2.1. Properties of shifted Chebyshev polynomials

The Chebyshev polynomials, denoted by \( T_i(z) \), are orthogonal with respect to the Chebyshev weight function \( w(z) = \frac{1}{\sqrt{1 - z^2}} \) over \((-1, 1)\), namely [28]

\[
\int_{-1}^{1} T_i(z) T_j(z) w(z) dz = c_i \delta_{i,j},
\]

where \( \delta_{i,j} \) is the Kronecker function, \( c_0 = \pi \) and \( c_i = \pi/2 \) for \( i > 0 \).

The Chebyshev polynomials are generated from the three-term recurrence relation:

\[
T_i(z) = 2z T_{i-1}(z) - T_{i-2}(z), \quad i \geq 2,
\]

where \( T_0(z) = 1 \) and \( T_1(z) = z \). Other useful relations to apply are

\[
2T_i(z) = \frac{T'_{i+1}(z)}{i+1} - \frac{T'_{i-1}(z)}{i-1}, \quad i \geq 2, \tag{1}
\]

\[
2T_i(z)T_j(z) = T_{i+j}(z) + T_{|j-i|}(z). \tag{2}
\]

In order to use these polynomials on the interval \( x \in [0, 1] \), we define the so-called shifted Chebyshev polynomials by introducing the change of variable \( z = 2x - 1 \). Let the shifted Chebyshev polynomials \( T_i(2x - 1) \) be denoted by \( \tilde{T}_i(x) \). The shifted Chebyshev polynomials are orthogonal with respect to the weight function \( \omega(x) = 1/\sqrt{x(1-x)} \) in the interval \([0, 1]\) with the orthogonality property:

\[
\int_{0}^{1} \tilde{T}_i(x) \tilde{T}_j(x) \omega(x) dx = c_i \delta_{i,j}. \tag{3}
\]

Then \( \tilde{T}_i(x) \) can be obtained as follows:

\[
\tilde{T}_i(x) = 2(2x - 1) \tilde{T}_{i-1}(x) - \tilde{T}_{i-2}(x), \quad i \geq 2
\]

\[
\tilde{T}_0(x) = 1, \quad \tilde{T}_1(x) = 2x - 1.
\]

A function \( f(x) \) defined over the interval \([0, 1]\) can be expanded as follows:

\[
f(x) = \sum_{i=0}^{\infty} a_i \tilde{T}_i(x),
\]

where the coefficients \( a_i \) are given by

\[
a_i = \frac{1}{c_i} \int_{0}^{1} \tilde{T}_i(x) f(x) \omega(x) dx, \quad i = 0, 1, \ldots.
\]

In practice, only the first \( m \) terms of the shifted Chebyshev polynomials are considered. Then, we have

\[
f(x) = \sum_{i=0}^{m-1} a_i \tilde{T}_i(x) = A^T \Phi(x),
\]

with

\[
A = [a_0, a_1, \ldots, a_{m-1}]^T.
\]

\[
\Phi(x) = [\tilde{T}_0(x), \tilde{T}_1(x), \ldots, \tilde{T}_{m-1}(x)]^T.
\]
2.2. The derivative operational matrix of a shifted Chebyshev polynomial

The derivative operator of the vector $\Phi(x)$ can be expressed by

$$\Phi'(x) = D\Phi(x),$$

where $D$ can be generated by using Eq. (1) as

$$\frac{T_{i+1}'(x)}{i+1} - \frac{T_{i-1}'(x)}{i-1} = 4T_i(x), \quad i \geq 2,$$

which can be considered in the following form:

$$\mathcal{B}\Phi'(x) = 4\Phi(x).$$

Two first rows of $\mathcal{B}$ are obtained by $T_0(x) = 1/2T_1'(x)$ and $T_1(x) = 1/8T_2'(x)$. $\mathcal{B}$ is a banded non-singular matrix, and its inverse can be simply found. Therefore, $\Phi'(x)$ is given as

$$\Phi'(x) = 4\mathcal{B}^{-1}\Phi(x).$$

By adding $T_0'(x)$ to $\Phi'(x)$ and deleting $T_m'(x)$ from $\Phi'(x)$, $\Phi'(x)$ is constructed. Finally, by adding the vector 0 in the first row of $4\mathcal{B}^{-1}$ and deleting the last row of it, the derivative operational matrix of the shifted Chebyshev polynomial for even or odd values of $m$ is obtained, respectively, as

$$D = \begin{bmatrix}
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
2 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 8 & 0 & 0 & \cdots & 0 & 0 \\
0 & 16 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
2(m-1) & 0 & 4(m-1) & 0 & \cdots & 4(m-1) & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
2 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 8 & 0 & 0 & \cdots & 0 & 0 \\
0 & 16 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 4(m-1) & 0 & 4(m-1) & \cdots & 4(m-1) & 0
\end{bmatrix}$$

or

$$D = \begin{bmatrix}
0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
2 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 8 & 0 & 0 & \cdots & 0 & 0 \\
0 & 16 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 4(m-1) & 0 & 4(m-1) & \cdots & 4(m-1) & 0
\end{bmatrix}$$

Remark 1. The derivative of order $p$ for the vector $\Phi(x)$ can be expressed by

$$\Phi^{(p)}(x) = D^p\Phi(x).$$

Subsequently, the derivative of order $p$ for the function $f(x)$ is obtained as

$$f^{(p)}(x) \simeq \hat{A}^T D^p \Phi(x).$$

2.3. The product operational matrix of a shifted Chebyshev polynomial

The following property of the product of two shifted Chebyshev vectors will also be applied.

$$\Phi(x)\Phi(x)^T V \simeq \hat{V}\Phi(x),$$

(4)
where $\tilde{V}$ is an $m \times m$ product operational matrix for the vector $V = [v_0, v_1, \ldots, v_{m-1}]$. Using above equation, and by the orthogonal property Eq. (3), the elements $\{\tilde{V}_{ij}\}_{i,j=0}^{m-1}$ can be calculated from

$$
\tilde{V}_{ij} = \frac{1}{c_i} \sum_{k=0}^{m-1} v_k g_{ijk},
$$

where $g_{ijk}$ is given by

$$
g_{ijk} = \int_0^1 T_i(x) T_j(x) T_k(x) \omega(x) \, dx.
$$

To achieve $g_{ijk}$ more simply, we describe the product of two shifted Chebyshev polynomials $T_i(x) T_j(x)$ as Eq. (2), and then we have

$$
g_{ijk} = \frac{1}{2} c_k (\delta_{i+j,k} + \delta_{|i-j|,k}).
$$

Thus, the product operational matrix for the vector $V$ is given by

$$
\tilde{V} = \begin{bmatrix}
v_0 & v_1 / 2 & v_2 / 2 & v_3 / 2 & \cdots & v_{m-1} / 2 \\
v_1 / 2 & v_0 + v_2 / 2 & v_1 + v_3 / 2 & v_2 + v_4 / 2 & \cdots & v_{m-2} / 2 \\
v_2 / 2 & v_1 + v_3 / 2 & v_0 + v_4 / 2 & v_1 + v_5 / 2 & \cdots & v_{m-3} / 2 \\
v_3 / 2 & v_2 + v_4 / 2 & v_1 + v_5 / 2 & v_0 + v_6 / 2 & \cdots & v_{m-4} / 2 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
v_{m-1} / 2 & v_{m-2} / 2 & v_{m-3} / 2 & v_{m-4} / 2 & \cdots & v_0
\end{bmatrix}.
$$

**Remark 2.** The function $f(x)^k$ can be expressed by

$$
f(x)^k \simeq A^T \tilde{A}^{k-1} \Phi(x).
$$

### 3. The methodology

Prior to an outline of the Tau homotopy analysis method, let us present a brief description of the standard homotopy analysis method. Readers may refer to [1] for a more general description of the HAM.

#### 3.1. The homotopy analysis method

We consider the following differential equation

$$
F(f(x)) = 0
$$

where $F$ is a nonlinear operator, $x$ denotes the independent variables, and $f(x)$ is an unknown function, respectively. With $f_0(x)$ denoting an initial guess of $f(x)$, $h \neq 0$ an auxiliary parameter, $H(x) \neq 0$ an auxiliary function, $\mathcal{L}$ an auxiliary linear operator, $q \in [0, 1]$ an embedding parameter and $\psi(x; q)$ an unknown function, Liao [1] constructed the so-called zero-order deformation equation,

$$
(1 - q) \mathcal{L} [\psi(x; q) - f_0(x)] = q h H(x) F[\psi(x; q)].
$$

(5)

It should be emphasized that we have great freedom to choose the initial guess, the auxiliary linear operator $\mathcal{L}$, the non-zero auxiliary parameter $h$, and the auxiliary function $H(x)$. Obviously, when $q = 0$ and $q = 1$, both

$$
\psi(x; 0) = f_0(x) \quad \text{and} \quad \psi(x; 1) = f(x)
$$

hold. Thus, as $q$ increases from 0 to 1, the solution $\psi(x; q)$ varies from the initial guess $f_0(x)$ to the solution $f(x)$. Expanding $\psi(x; q)$ in Taylor series with respect to $q$, one has

$$
\psi(x; q) = f_0(x) + \sum_{n=1}^{+\infty} f_n(x) q^n,
$$

(6)
where
\[ f_n(x) = \frac{1}{n!} \frac{\partial^n \psi(x; q)}{\partial q^n} |_{q=0}. \]

If the initial guess, the auxiliary linear operator, the non-zero auxiliary parameter \( h \), and the auxiliary function \( H(x) \) are properly chosen, then the series (6) converges at \( q = 1 \), so, by using \( \psi(x; 1) = f(x) \) one has the so-called homotopy series solution,
\[ f(x) = f_0(x) + \sum_{n=1}^{+\infty} f_n(x), \]
which must be one of solutions of the original nonlinear equation, as proved by Liao [1]. Define the vector
\[ \tilde{f} = [f_0(x), f_1(x), \ldots, f_l(x)]. \]

Differentiating Eq. (5) \( n \) times with respect to the embedding parameter \( q \) and then setting \( q = 0 \) and finally dividing them by \( n! \), we have the so-called \( m \)-th order deformation equation,
\[ \mathcal{L}[f_n(x) - \chi_n f_{n-1}(x)] = hH(x)R_n(\tilde{f}_{n-1}), \quad (7) \]
where
\[ R_n(\tilde{f}_{n-1}) = \frac{1}{(n-1)!} \frac{\partial^{n-1} \psi(x; q)}{\partial q^{n-1}} |_{q=0} \quad (8) \]
and
\[ \chi_n = \begin{cases} 0, & n \leq 1, \\ 1, & n > 1. \end{cases} \]

According to the definition (8), the right-hand side of Eq. (7) is only dependent upon \( f_{n-1}(x) \). Thus, we gain \( f_1(x), f_2(x), \ldots \) by means of solving the linear high-order deformation equation (7) iteratively.

3.2. The Tau homotopy analysis method

Consider the nonlinear equation of the form
\[ F(x, f, f', f'', \ldots, f^{(p-1)}, f^{(p)}) = 0, \quad x \in [a, b], \quad (9) \]
subject to the boundary condition
\[ f^{(k)}(e_i) = \alpha_k, \quad k = 1, \ldots, p, \quad i \in \{0, 1, \ldots, p-1\}, \]
where \( e_i \in [0, 1] \). To illustrate the basic concepts of the Tau homotopy analysis method, by considering the boundary conditions \( A_0^T \Phi(e_i) = \alpha_k \), we construct \( A_0 \). Also, we define the unknown functions \( f_n(x) \) for \( i \geq 1 \) as
\[ f_n(x) = A^T_n \Phi(x). \]

We aim to obtain the unknown \( A_n \) vectors by using the THAM to determine \( f_n(x) \). Now, from Eq. (7), we have
\[ \mathcal{L}[A_{n}^T \Phi(x) - \chi_n A_{n-1}^T \Phi(x)] = hH(x)R_n(f_{n-1}(x)), \quad (10) \]
so one can define
\[ H(x) = H^T \Phi(x), \quad H^T = [h_0, h_1, \ldots, h_{m-1}], \]
and also we have
\[ R_n(f_{n-1}(x)) = \tilde{R}_n \Phi(x), \]
where \( \tilde{R}_n \) is an \( m \times m \) matrix which can be constructed by using the operational matrices.

Therefore, Eq. (10) by use of Eq. (4) is simply obtained as
\[ \mathcal{L}[A_{n}^T \Phi(x) - \chi_n A_{n-1}^T \Phi(x)] = hH(x) \Phi(x) \Phi^T(x)H = h\tilde{R}_n \Phi(x), \]
which yields
\[ A_{n}^T \Phi(x) = (\chi_n A_{n-1}^T L + h\tilde{R}_n) \Phi(x), \quad (11) \]
where \( L \) is the matrix form of linear differential operator \( \mathcal{L} \), subject to the boundary conditions
\[ A_n^T \Phi(e_i) = 0. \quad (12) \]
While applying the Tau method [29] on Eq. (11), one has
\[
A_0^T L = X_0 A_{\text{ref}}^T L + h R_0 \hat{H} = Q^T. \tag{13}
\]
The boundary conditions (12) are imposed on the last columns of the L matrix on the left-hand side of the above equation, and we set their values in the right-hand side vector of the mentioned equation. Obviously, this gives
\[
A_0^T L = \hat{Q}^T \hat{L}^{-1}, \tag{14}
\]
where \( \hat{L} \) and \( \hat{Q} \) are the modified matrix L and vector Q, respectively, after implementing boundary conditions.

Now, by starting from \( A_0^T \) as the initial approximation, the above equation is ripe to be iteratively solved to obtain \( \{A_1, A_2, \ldots, A_l\} \), and, consequently, it gives
\[
f_i(x) = \sum_{\pi=0}^{l} A_{\pi}^T \Phi(x). \tag{15}
\]

3.3. Error analysis

We can check the accuracy of the method by defining the residual function by using Eqs. (9) and (15) as
\[
R_{m,l}(x, h) = F(x, f_i(x), f_i(x'), f_i(x''), \ldots, f_i(x)^{(p-1)}, f_i(x)^{(p)}) \tag{17}
\]
Now, by defining the norm of \( \| \cdot \|_{u(x)} \) and assuming that \( R_{m,l}(x, h) \in P_N \), \( \| R_{m,l}(x, h) \|_{u(x)} \) can be obtained by the well-known Chebyhev Gauss quadrature formulation as [28]
\[
\| R_{m,l}(x, h) \|_{u(x)} = \int_{0}^{1} R_{m,l}^2(x, h) dx = \frac{\pi}{N + 1} \sum_{i=0}^{N} R_{m,l}^2(x_i, h), \tag{16}
\]
where
\[
x_i = \cos \left( \frac{(2i + 1)\pi}{2N + 2} \right), \quad i = 0, 1, \ldots, N. \tag{18}
\]
It is clear that the minimum of \( \| R_{m,l}(x, h) \|_{u(x)} \) might be determined conveniently by the auxiliary parameter \( h \). Suppose that
\[
\| R_{m,l}(x, h) \|_{u(x)} \leq 10^{-r}, \quad r \in \mathbb{N}, \tag{18}
\]
where \( r \) is an appropriate guessed value. Thus, we can obtain the valid interval of convergence that determines the optimal values of \( h \).

4. Application of the Tau homotopy analysis method (THAM)

In order to assess the accuracy of the THAM for solving nonlinear equations we will consider the two following examples.

4.1. Example 1: Fin temperature distribution

Consider the following example [30]:
\[
\frac{\partial^2 \theta}{\partial \xi^2} + \beta\frac{\partial \theta}{\partial \xi^2} + \beta \left( \frac{\partial \theta}{\partial \xi} \right)^2 - \psi^2 \theta = 0, \tag{17}
\]
subject to boundary conditions
\[
\frac{d \theta}{d \xi} = 0 \quad \text{at} \quad \xi = 0, \quad \text{and} \quad \theta = 1 \quad \text{at} \quad \xi = 1, \tag{18}
\]
where \( \psi \) is the thermo-geometric fin parameter and \( \beta \) is the thermal conductivity parameter.

A considerable amount of research has been conducted in variable thermal conductivity. Arslanturk [30] and Rajabi [31] respectively used the ADM and the homotopy perturbation method (HPM) to evaluate the efficiency of straight fins with temperature-dependent thermal conductivity. Kim and Huang [32] applied the Taylor series expansion method to the fin problem with a temperature-dependent thermal conductivity. Ganji et al. [33] analyzed convective straight fins with temperature-dependent thermal conductivity using the variational iteration method (VIM) and the HPM. Also, VIM and FEM analyzes of the problem have been conducted in [34]. Also, Inc [35] and Khani et al. [36] investigated the problem by applying the homotopy analysis method.
Table 1
The dimensionless temperature distribution.

<table>
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<th>ξ</th>
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<th>(\beta = 0, \psi = 1)</th>
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</tr>
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</table>

4.1.1. Applying the THAM

In order to solve Eq. (17) with boundary conditions (18), we construct \(A_0\) from \(\theta_0 = 1\) as

\[ A_0 = [1, 0, \ldots, 0]^T, \]

which satisfies the boundary conditions, and the auxiliary vector, \(H\), is defined such that

\[ H = [1, 0, \ldots, 0]^T. \]

Also, we choose an auxiliary matrix operator as

\[ L = D^2. \]

In order to obtain \(\tilde{R}_n\), by using (4), we have

\[ R_n(\theta_{n-1}(\xi)) = A_{n-1}^T D^2 \Phi(\xi) - \psi^2 A_{n-1}^T \Phi(\xi) + \beta \sum_{i=0}^{n-1} (A_{n-i}^T D^2 \tilde{A}_i + A_{n-i}^T D \tilde{A}_i) \Phi(\xi), \]

where \(\Lambda_i = D^2 \tilde{A}_i\). Thus \(\tilde{R}_n\) is given by

\[ \tilde{R}_n = A_{n-1}^T (D^2 - \psi^2 I) + \beta \sum_{i=0}^{n-1} A_{n-i-1}^T (D^2 \tilde{A}_i + D \tilde{A}_i), \]

where \(I\) is an identity matrix of order \(m\). According to Eqs. (12) and (18), one has

\[ D \Phi(0) = 0, \quad \Phi(1) = 0. \]

In order to implement these boundary conditions, while we impose these boundary conditions on the two last columns of \(L\) in Eq. (19) and set their values in \(Q\) in Eq. (13), \(\hat{L}\) and \(\hat{Q}\) are constructed.

Now, by starting from \(A_0^T\) as the initial approximation and using Eq. (14), \(\{A_1, A_2, \ldots, A_l\}\) can be iteratively obtained, and consequently we obtain

\[ \theta_l(\xi) = \sum_{n=0}^{l} A_n^T \Phi(\xi). \]

The calculations presented in this example use \(l = 10\) and \(n = 40\) or \(n = 20\) in some cases. The results indicate that the supposed values for \(l\) and \(n\) are sufficient to provide an accurate solution. As can be seen from Table 1, the comparison between our results and those of [37] indicates tremendously accurate results in our proposed method (THAM). In Fig. 1, the dimensionless temperature distribution is displayed for \(\beta = 0.5\) and various \(\psi\) values.

According to Section 3.1, we should ensure that the solution series (20) converges. Note that this series contains the auxiliary parameter \(h\), which influences its convergence region and rate. In general, by means of the so-called \(h\)-curve, it is straightforward to choose an appropriate range for \(h\) which ensures the convergence of the solution series. The appropriate region of \(h\) to get a convergent solution of \(\theta(0)\) in linear (\(\beta = 0\)) is shown in Fig. 2. Also, the valid region of \(h\) for \(\theta(0)\) of nonlinear cases when \(\psi = 0.5\) and for some values of \(\beta\) is obvious in Fig. 3. The residual error in Eq. (16) is displayed in Fig. 8, which indicates the interval for the admissible values of \(h\).
Fig. 1. Dimensionless temperature variation for $\beta = 0.5$ in Example 1.

Fig. 2. $h$-curve of $\theta(0)$ when $\beta = 0$ for the 40th-order approximation in Example 1.

Fig. 3. $h$-curve of $\theta(0)$ for the 20th-order approximation in Example 1.
4.2. Example 2: Darcy–Brinkman–Forchheimer equation

Let us consider the Darcy–Brinkman–Forchheimer equation for steady fully developed fluid flow in a horizontal channel filled with a porous medium [24].

\[
\frac{d^2u(x)}{dx^2} - s^2 u(x) - F s u^2(x) + \frac{1}{M} = 0, \tag{21}
\]

subject to the boundary conditions

\[
u'(0) = 0, \quad u(1) = 0 \tag{22}
\]

where \( M \) is the viscosity ratio, \( F \) is the Forchheimer number and \( s \) is the porous medium shape parameter.

The Darcy–Brinkman–Forchheimer has been studied numerically using a second-order accurate finite difference scheme [38]. The asymptotic solutions to Eq. (21) were found by perturbation methods for limiting cases in [39]. Motsa et al. [24] have applied a kind of modification of the homotopy analysis method [21] in order to obtain a solution with acceptable accuracy. In [40], an exact solution in an implicit form is given by Abbasbandy et al. In the following, this problem is used to demonstrate the utility of the Tau homotopy analysis method (THAM).

4.2.1. Applying the THAM

In order to solve Eq. (21) with boundary conditions (22), we begin by choosing \( u_0(x) = \frac{1}{\rho} - \frac{1}{2\rho}x^2 \), to construct the general form of \( A_0 \) satisfying the boundary conditions,

\[
A_0 = \left[ \begin{pmatrix} 1 - \frac{1}{\rho} \\ \frac{1}{2\rho} \end{pmatrix}, \begin{pmatrix} -1 \\ \frac{1}{2\rho} \end{pmatrix}, 0, \ldots, 0 \right]^T,
\]

where \( \rho \) is a known parameter to adjust in order to have a suitable candidate for the initial approximation.

The auxiliary vector, \( H \), is defined as

\[
H = [1, 0, \ldots, 0]^T.
\]

and we choose an auxiliary matrix operator as

\[
L = D^2 - I, \tag{23}
\]

where \( I \) is an identity matrix of order \( m \).

In order to obtain \( R_n \), by using (4), we have

\[
R_n(u_{n-1}(x)) = A_{n-1}^T D^2 \Phi(x) - s^2 A_{n-1}^T \Phi(x) - F s \sum_{i=0}^{n-1} A_{n-i-1}^T \hat{A}_i \Phi(x) + \gamma (I - \chi_n) \Phi(x),
\]

where \( \gamma = \frac{1}{M}, 0, \ldots, 0 \)^T. Thus \( \hat{R}_n \) is given by

\[
\hat{R}_n = A_{n-1}^T (D^2 - s^2 I) - F s \sum_{i=0}^{n-1} A_{n-i-1}^T \hat{A}_i + \gamma (I - \chi_n).
\]

According to Eqs. (12) and (22), one has

\[
D \Phi(0) = 0, \quad \Phi(1) = 0.
\]

In order to implement these boundary conditions, while we impose these boundary conditions on the two last columns of \( L \) in Eq. (23) and set their values in \( Q \) in Eq. (13), \( \hat{L} \) and \( \hat{Q} \) are constructed.

Now, by starting from \( A_0^T \) as the initial approximation and using Eq. (14), \( \{A_1, A_2, \ldots, A_l\} \) can be iteratively obtained, and consequently we obtain

\[
u_l(x) = \sum_{n=0}^{l} A_n^T \Phi(x). \tag{24}
\]

In this example, the calculations are carried out for the values \( l = 6, n = 30 \) and \( \rho = \frac{1}{2} \), which supply good accuracy for the solution. In Fig. 4, the solution of the problem is displayed for \( F = M = 1 \) and various values of the porous medium shape parameter, \( s \). Also, Table 2 provides a comparison between our results of filtration velocity at \( y = 0 \), i.e. \( u(0) \) and \( u'(1) \) obtained by the THAM, and exact ones [40] for various values of the porous medium shape parameter \( s \) when \( M = F = 1 \). It is worth noting that the THAM results seem to have this accuracy at the 30th order of approximation.

The same comparison for various values of viscosity ratio \( M \) when \( s = F = 1 \) is made in Table 3. Additionally, Table 4 gives the THAM results of filtration velocity at \( y = 0 \), i.e. \( u(0) \) and \( u'(1) \), against exact results for fixed values of \( s \) and \( M \) when the Forchheimer number is varied.
Fig. 4. THAM solution of velocity profile $u(x)$ with the 30th order of approximation for various $s$ when $F = M = 1$ in Example 2.

Table 2
A comparison between present method results and similar exact values obtained by [40] for $u(0)$ and $u'(1)$ for various values of $s$ when $M = F = 1$.

<table>
<thead>
<tr>
<th>Porous medium shape parameter $s$</th>
<th>$u(0)$</th>
<th>$u'(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>THAM</td>
<td>Exact</td>
</tr>
<tr>
<td>0</td>
<td>0.5</td>
<td>0.5</td>
</tr>
<tr>
<td>1</td>
<td>0.32384748</td>
<td>0.32384737</td>
</tr>
<tr>
<td>1.5</td>
<td>0.23838513</td>
<td>0.23838511</td>
</tr>
<tr>
<td>2</td>
<td>0.17443255</td>
<td>0.17443249</td>
</tr>
<tr>
<td>2.5</td>
<td>0.12917335</td>
<td>0.12917323</td>
</tr>
</tbody>
</table>

Table 3
A comparison between present method results and similar exact values obtained by [40] for $u(0)$ and $u'(1)$ for various values of $M$ when $s = F = 1$.

<table>
<thead>
<tr>
<th>Viscosity ratio $M$</th>
<th>$u(0)$</th>
<th>$u'(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>THAM</td>
<td>Exact</td>
</tr>
<tr>
<td>1</td>
<td>0.32384748</td>
<td>0.32384746</td>
</tr>
<tr>
<td>2</td>
<td>0.16840089</td>
<td>0.16840085</td>
</tr>
<tr>
<td>3</td>
<td>0.11385781</td>
<td>0.11385781</td>
</tr>
<tr>
<td>4</td>
<td>0.08601438</td>
<td>0.08601437</td>
</tr>
<tr>
<td>5</td>
<td>0.06911628</td>
<td>0.06911628</td>
</tr>
<tr>
<td>6</td>
<td>0.05776861</td>
<td>0.05776862</td>
</tr>
</tbody>
</table>

Table 4
A comparison between present method results and similar exact values obtained by [40] for $u(0)$ and $u'(1)$ for various values of $F$ when $s = M = 1$.

<table>
<thead>
<tr>
<th>Forchheimer number $F$</th>
<th>$u(0)$</th>
<th>$u'(1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact</td>
<td>THAM</td>
<td>Exact</td>
</tr>
<tr>
<td>0</td>
<td>0.35194573</td>
<td>0.35194571</td>
</tr>
<tr>
<td>1</td>
<td>0.32384748</td>
<td>0.32384751</td>
</tr>
<tr>
<td>2</td>
<td>0.30260920</td>
<td>0.30260920</td>
</tr>
<tr>
<td>3</td>
<td>0.28566735</td>
<td>0.28566733</td>
</tr>
<tr>
<td>4</td>
<td>0.27166879</td>
<td>0.27166875</td>
</tr>
<tr>
<td>5</td>
<td>0.25980413</td>
<td>0.25980411</td>
</tr>
</tbody>
</table>

The appropriate region of $h$ that controls the convergence lies on the horizontal segment of the $h$-curve. In Fig. 5 the $h$-curves of $u(0)$ obtained at the 30th order of the THAM approximation for various values of $s$ and fixed values of $M$ and $F$ is displayed. It can be seen, by increasing $s$, the length of the horizontal segment of the $h$-curve shrinks, which shows rapid convergence of the THAM solutions for large values of the porous medium shape parameter.
**Fig. 5.** The 30th-order THAM $\bar{h}$-curve of $u(0)$ for various values of $s$ when $F = M = 1$ in Example 2.

**Fig. 6.** The 30th-order THAM $\bar{h}$-curve of $u(0)$ for various values of $F$ when $s = M = 1$ in Example 2.

**Fig. 7.** The 30th-order THAM $\bar{h}$-curve of $u(0)$ for various values of $M$ when $s = F = 1$ in Example 2.
5. Conclusion

In this study we have proposed the Tau homotopy analysis method for the solution of nonlinear boundary value problems. In order to test the applicability, accuracy and efficiency of this new Tau modification of the HAM, we considered two examples, for which the comparisons verify our present method. Also, the following advantages are worth noting.

(1) In the Tau homotopy analysis method (THAM), the rule of solution expression and the rule of ergodicity are not workable, unlike the standard HAM approach.

(2) In using the THAM, one may use any form of initial guess (even complicated ones) as long as it satisfies the boundary conditions, whereas in the HAM one is restricted to choosing an initial approximation that will make the integration of the higher-order deformation equations possible.

(3) Using operational matrices and the Tau method changes the standard homotopy analysis method to an iterative matrix method which is capable of greatly reducing the size of calculation while maintaining the accuracy of the solution. Actually our described method is not computationally expensive.

Fig. 8. The residual error for various values of $\beta$ and $\psi = 0.5$ in Example 1.

Fig. 9. The residual error for various values of $M$ when $s = F = 1$ in Example 2.

Fig. 6 shows the $h$-curves of $u(0)$ at the 30th order of approximation of the THAM for various values of $F$ when $M = s = 1$. For various values of $M$ while $F = s = 1$, in order to determine the optimal range of the values of $h$ for which the series solution (24) is convergent, we plot the so-called $h$-curves of $u(0)$, as shown in Fig. 7. The residual error in Eq. (16) is depicted in Fig. 9, which indicates the appropriate region of $h$ in which our series solution is convergent.

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Acknowledgments

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References