The Optimal Homotopy Asymptotic Method for the Solution of Fifth and Sixth Order Boundary Value Problems


Abstract: In this work we solve some fifth and sixth order boundary value problems by the optimal homotopy asymptotic method (OHAM). The presented is capable to handle both linear and nonlinear boundary value problems effectively. The numerical results given by OHAM are compared with the exact solution and with the existing results. The results show high accuracy and reliability of the method.

Key words: Fifth and Sixth order boundary value problems %Boundary value problems %Optimal Homotopy Asymptotic Method

INTRODUCTION

In science and engineering, only a limited class of differential equations can be solved exactly and for the rest of equations, either analytic or numerical methods are used to get the approximate solutions of acceptable accuracy. Recent analytic methods contain: Adomian decomposition method (ADM) [1-5], variational iteration [6], homotopy perturbation [7-8], homotopy Analysis [9], Differential Transform Method (DTM) [10] etc. Classical perturbation methods are based on small or large parameters and they cannot produce a general form of an approximate solution especially in nonlinear problems. The non perturbation methods like ADM and DTM can deal strongly nonlinear problems but the convergence region of their series solution is generally small. The HPM which is an elegant combination of homotopy and perturbation techniques overcomes the restrictions of small or large parameters in the problems. It deals a wide variety of nonlinear problems effectively. Recently Vasile Marinca et al. [11-14], introduced OHAM for approximate solution of nonlinear problems of thin film flow of a fourth grade fluid down a vertical cylinder. In their work they have used this method to understand the behavior of nonlinear mechanical vibration of electrical machine. They also used the same method for the solution of nonlinear equations arising in the steady state flow of a fourth-grade fluid past a porous plate and for the solution of nonlinear equation arising in heat transfer. This method is straightforward, reliable and explicitly defined. This method provides a convenient way to control the convergence of the series solution and allows adjustment of convergence region where it is needed. The residual is among all the beauties of this method by which the validity of the obtained solution can be confirmed and one needs no other method to insure the validity. Fifth-order boundary value problems arise in the mathematical modeling of viscoelastic flows [15]. Sixth-order boundary value problems are known to arise in astrophysics, the narrow convecting layers bounded by stable layers which are believed to surround A-type stars that may be modeled by sixth-order boundary value problems [16]. Glatzmaier [17] also studied that dynamo action in some stars may be modeled by such equations. Fifth and sixth order linear and nonlinear problems were solved by Wazwaz [4-5], while using decomposition method. Noor et al. [18-20] investigated these type of problems using Variational Iteration Method Using He’s Polynomials, Homotopy perturbation method and Variational iteration method. Recently Javed Ali et al. [21-22] used OHAM for the solution of multi-point and for the solution of special twelfth order boundary value problems. We use OHAM to find the approximate analytic solution of fifth and sixth order boundary value problems boundary value problems. The results of OHAM are compared with those of exact solution and the errors are compared with the existing results.

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The structure of this paper is organized as follows. Section 2 is devoted to the analysis of the proposed method. Some numerical examples are presented in Section 3. In Section 4, we concluded by discussing results of the numerical simulation by using Mathematica 5.2.

**MATERIAL AND METHOD**

Consider the Differential Equation:

\[ L(u(x)) + f(x) + N(u(x)) = 0, \quad B(u, du/dx) = 0, \quad (2.1) \]

Where \( L \) is a linear operator, \( x \) denotes independent variable, \( u(x) \) is an unknown function, \( f(x) \) is a known function, \( N \) is a nonlinear operator and \( B \) is a boundary operator.

According to OHAM we construct a homotopy, \( h((x, q); 0, 1) \) which satisfies

\[ (1 - q)[L((x, q)) + f(x)] = H(q)[L((x, q)) + f(x) + N((x, q))], \quad ((x, q), d(x, q)/dx) = 0, \quad (2.2) \]

Where \( x, S, q \in [0, 1] \) is an embedding parameter, \( H(q) \) is a nonzero auxiliary function for \( q \in [0, 1] \) and \( H(0) = 0 \) and \( H(1) \) is an unknown function. Obviously, when \( q = 0 \), and \( q = 1 \), it holds that \( R(x, 0) = u_0(x) \) and \( R(x, 1) = u(x) \) respectively. Thus, as \( q \) varies from 0 to 1, the function \( R(x, q) \) approaches from the initial value (approximation) \( u_0(x) \) to the solution function \( u(x) \) where \( u_0(x) \) is obtained from Eq.(2) for \( q = 0 \) and we have

\[ L(u_0(x)) + f(x) = 0, \quad B(u_0, du_0/dx) = 0. \quad (2.3) \]

Next we choose the auxiliary function \( h(q) \) in the form

\[ h(q) = \sum_{i=1}^{m} q^i C_i \quad (2.4) \]

Where \( C_i \)s are constant to be determined. \( H(q) \) can be expressed in many forms as reported by V. Marinc et al. [10-13]. To get an approximate solution, we expand \( R(x,q; C_i) \) in the Taylor’s series about \( q \),

\[ \psi (x,q; C) = u_0(x) + \sum_{k=1}^{m} u_k (x, C_k) q^k \quad (2.5) \]

Substituting Eq.(2.4) and Eq.(2.5) into Eq.(2.2) and equating the coefficient of like powers of \( q \), we obtain the following linear equations which are directly integrable.

Zeroth order problem is given by Eq.(3) and the first order problem is given by Eq.(2.6):

\[ L(u_t(x)) + f(x) = C_i N (u_0(x)), \quad B(u_t, du_t/dx) = 0. \quad (2.6) \]

The general governing equations for \( u_i(x) \) are given by:

\[ L(u_k(x)) - L(u_{k-1}(x)) = C_k N(u(x)) + \sum_{i=1}^{k-1} C_i [L(u_{k-i}(x)) + N_{k-i}(u(x), u_i(x), \ldots, u_{k-i}(x))] \quad k = 2, 3, \ldots, \quad B(u_k, du_k/dx) = 0. \quad (2.7) \]

Where \( N_m(u_0(x)), u_1(x), \ldots, u_m(x) \) is the coefficient of \( q_m \) in the expansion of \( N(R(x,q)) \) about the embedding parameter \( q \):

\[ N(\psi(x, q)) = N(0) + \sum_{m=1}^{k} N_m(u_0, u_1, \ldots, u_m) q^m \quad (2.8) \]

It has been observed that the convergence of the series (2.5) depends upon the auxiliary constants \( C_i \)'s. If it is convergent at \( q = 1 \), one has

\[ \psi (x, C_i) = u_0(x) + \sum_{k=1}^{m} u_k (x, C_i) \quad (2.9) \]

The result of the mth-order approximations are given by

\[ \tilde{u}(x, C_i) = u_0(x) + \sum_{k=1}^{m} u_k (x, C_i). \quad (2.10) \]

Substituting Eq.(2.10) into Eq.(2.1), the following residual is obtained:

\[ R(x, C_i) = L(\tilde{u}(x, C_i)) + f(x) + N(\tilde{u}(x, C_i)). \quad (2.11) \]

If \( R = 0 \), then \( \tilde{E} \) will be the exact solution. In general it does not so. However we can minimize \( R \). Now we have to look for those values of \( C_i \)'s which minimizes \( R \). For this purpose we first construct the functional

\[ J(C_i) = \int_a^b R^2(x, C_i) dx, \quad (2.12) \]

and then minimizing it, we have

\[ \frac{\partial J}{\partial C_i} = 0, \quad i = 1, 2, \ldots, m. \quad (2.13) \]

Where \( a \) and \( b \) are in the domain of the problem. Knowing these constants, the approximate solution \( \tilde{E} \) (of order \( m \)) is obtained.
Numerical Examples

Example 1: Fifth order linear BVP

\[ y^{(v)}(x) = y - 15e^x - 10xe^x, \quad 0 < x < 1, \]

with boundary conditions,

\[ y(0) = 0, \quad y'(0) = 1, \quad y''(0) = 0, \quad y(1) = 0, \quad y'(1) = -e. \]

The exact solution of this problem is \( y(x) = x(1-x)e^x \).

Following the OHAM procedure in section 2, we obtain the following linear problems:

Zeroth Order Problem:

\[ \frac{dy_0}{dx} = 0, \]

\[ y_0(0) = 0, \quad y'_0(0) = 0, \quad y''_0(0) = 0, \quad y_0(1) = 0, \quad y'_0(1) = -e. \]  

First Order Problem:

\[ \frac{d^{(5)}}{dx^{(5)}}(y_1(x, C_1)) = 15C_1e^x + 10C_1xe^x - C_1y_0(x) + (1 + C_1)y_0^{(5)}(x), \]

\[ y_1(0) = 0, \quad y'_1(0) = 0, \quad y''_1(0) = 0, \quad y_1(1) = 0, \quad y'_1(1) = 0. \]  

Second order problem:

\[ \frac{d^{5}}{dx^{5}}(y_2(x, C_1, C_2)) = 15C_2e^x + 10C_2xe^x - C_2y_0(x) - C_1y_1(x, C_1) + (1 + C_1)y_0^{(5)}(x), \]

\[ y_2(0) = 0, \quad y'_2(0) = 0, \quad y''_2(0) = 0, \quad y_2(1) = 0, \quad y'_2(1) = 0. \]  

Solutions of the problems (3.1),(3.2) and (3.3), are given by (3.4), (3.5) and (3.6).

\[ y_0(x) = x - 3x^3 + ex^3 + 2x^4 - ex^4. \]  

\[ y_0(x, C_1) = C_1(35-35e^x + 25x + 10xe^x + 7.5x^2 + 1.051582859x^3 - 0.59323767x^4 - 0.00138889x^6 + 0.00004192x^8 + 0.000047505x^9) \]  

\[ y(x, C_1, C_2) = C_1(35-35e^x + 25x + 10xe^x + 7.5x^2 + 1.051583x^3 - 0.593237685x^4 - 0.00138889x^6 + 0.00004192x^8 + 0.000047505x^9) \]

\[ + 0.00004192x^8 + 0.000047505x^9) + C_2(35 - 35e^x + 25x + 10xe^x + 7.5x^2 + 1.051583x^3 - 0.5932377x^4 - 0.00138889x^6 \]

\[ - 0.00138889x^6 + 0.00004192x^8 + 0.000047505x^9) + C_1^2(50e^x - 50 - 50x - 25x^2 - 8.1151161x^3 \]

\[ - 2.468193x^4 + 0.0361111x^6 - 0.0029761x^7 - 0.00011456x^8 + 0.00008674x^9 \]

\[ + 0.0000008674x^9 + 2.5052\times10^{-8}x^{11} - 2.714\times10^{-10}x^{13} - 1.9774\times10^{-10}x^{14}). \]

To obtain the OHAM 2nd order solution,

\[ \tilde{y}(x) = y_0(x) + y_1(x, C_1) + y_2(x, C_1, C_2) \]  

We determine the auxiliary constants. Taking \( a = 0 \) and \( b = 1 \) and following the procedure in section 2, the following values are obtained.

\[ C_1 = -0.912019009, \quad C_2 = -0.0077753602 \]
By considering these values (3.7) becomes:

$$\hat{y}(x) = x - 0.5x^3 - 0.333334x^4 - 0.124999x^5 - 0.0333332x^6 - 0.00694441x^7$$
$$-0.00119137x^8 - 0.000127901x^9 - 0.0000221613x^{10} - 2.49696 \times 10^{-6} x^{11}$$
$$-2.5263 \times 10^{-7} x^{12} - 2.29636 \times 10^{-8} x^{13} - 2.05622 \times 10^{-9} x^{14} + O(x^{15}).$$

(3.8)

Numerical results of the solution (3.8) are displayed in Table 1.

**Example 2: Fifth order non-linear BVP**

$$y^{(v)}(x) = y^2(x)e^{-x}, \quad 0 < x < 1,$$

with boundary conditions,

$$y(0) = 1, \quad y'(0) = 1, \quad y''(0) = 1, \quad y(1) = e, \quad y'(1) = e.$$

The exact solution for this problem is, $y(x) = e^x$.

We consider the second order solution,

$$\hat{y}(x) = y_0(x) + y_1(x, C_1) + y_2(x, C_1, C_2) + O(x^{13}).$$

Using procedure in section 2, we obtain the following values of $C_i$'s for $a = 0$ and $b = 1$.

$$C_1 = -0.97647733, \quad C_2 = -0.031756606$$

The second order approximate solution is

$$\hat{y}(x) = 1 + x + x^2 / 2 + 0.166664439x^3 + 0.041667135x^4 + 0.0083327684x^5$$
$$+ 0.001389521x^6 + 0.00019810282x^7 + 2.4837 \times 10^{-5} x^8 + 2.6754 \times 10^{-6} x^9$$
$$+ 3.4474 \times 10^{-7} x^{10} - 4.2926 \times 10^{-9} x^{11} + 1.4626 \times 10^{-8} x^{12} + O(x^{13}).$$

(3.9)

Numerical results of the solution (3.9) are displayed in Table 2.

**Example 3: Sixth order linear BVP**

$$y^{(v)}(x) = y(x) - 6e^x, \quad 0 < x < 1,$$

Table 1: In this table we compare the exact solution with the OHAM second order solution (3.8) and the errors obtained by using the homotopy perturbation method (HPM), variational iteration method (VIM), decomposition method (ADM), iterative method (ITM) and the variational iteration method using He’s polynomials (VIMHP)[18].

<table>
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<tr>
<th>x</th>
<th>Exact Sol.</th>
<th>OHAM Sol.(3.11)</th>
<th>$E^*(3.8)$</th>
<th>$E^*(18)$</th>
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</table>

$E^*$=Exact-Approx.
Table 2: Comparison of the second order OHAM solution (3.9) with the exact solution and the error estimates. Last column of Table 2, are the errors in the solutions of HPM, VIM, ADM, ITM and VIMHP for the same problem [18]. The accuracy of the proposed method can be improved further by evaluating more components of \( \tilde{y}(x) \).

<table>
<thead>
<tr>
<th>x</th>
<th>Exact solution</th>
<th>OHAM Sol.</th>
<th>( E^*(3.9) )</th>
<th>( E^*(18) )</th>
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\( E^* = \text{Exact}-\text{Approx.} \)

with boundary conditions,

\[ y(0) = 1, \; y''(0) = -1, \; y^{(iv)}(0) = -3, \; y(1) = 0, \; y''(1) = -2e, \; y^{(iv)}(1) = -4e. \]

The exact solution for this problem is \( y(x) = (1 - x)e^x \).

Now following the OHAM procedure, the second order solution

\[
\tilde{y}(x) = y_0(x) + y_1(x, C_1) + y_2(x, C_1, C_2) + O(x^{13})
\]

is obtained by determining the values of \( C_1, C_2 \). Following the procedure for \( C_1 \)'s in section 2, we get

\[ C_1 = 0, \; C_2 = -0.999035462. \]

The OHAM second order solution is

\[
\tilde{y}(x) = 1 - 2.062484 \times 10^{-7} x - x^2/2 - 0.333333 x^3 - x^4/8 - 0.0333325 x^5 - 0.00694578 x^6 \]
\[ -0.00118984 x^7 - 0.000173587 x^8 - 0.0000221313 x^9 - 2.47936 \times 10^{-6} x^{10} \]
\[ -2.39498 \times 10^{-7} x^{11} - 2.504 \times 10^{-8} x^{12} + O(x^{13}). \]  

(3.10)

Numerical results of the solution (3.10) are displayed in Table 3.

**Example: 4: Sixth order non-linear BVP**

\[
y^{(vii)}(x) = e^{-x} y^2(x), \quad 0 < x < 1,
\]

with boundary conditions,

\[ y(0) = 1, \; y''(0) = 1, \; y^{(iv)}(0) = 1, \; y(1) = e, \; y''(1) = e, \; y^{(iv)}(1) = e. \]

The exact solution is \( y(x) = e^x \).

Let us try a different auxiliary function \( h(p) = p(C_1 + C_2 e^p) \) and consider the second order solution,

\[
\tilde{y}(x) = y_0(x) + y_1(x, C_1, C_2) + y_2(x, C_1, C_2) + O(x^{13}).
\]
Table 3: In this table we compare the exact solution with the OHAM solution (3.10) and the errors obtained by decomposition method (ADM) [5], homotopy perturbation method (HPM) [19] and the variational iteration method [20].

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E*=Exact-Approx.

Table 4: Comparison of the OHAM solution (3.11) with the exact solution and the errors obtained by decomposition method (ADM) [5], homotopy perturbation method (HPM) [19] and the variational iteration method [20].

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E*=Exact-Approx.

For \( a = 0 \) and \( b = 1 \); and following the procedure for the values of \( C_i \)'s in section 2, we get

\[
C_1 = -0.99142072, \quad C_2 = -0.015535939.
\]

Knowing these values the approximate solution is

\[
y(x) = 1 + x + 0.5x^2 + 0.16666666667x^3 + 0.04166666667x^4 + 0.0083333658x^5 + 0.001388217x^6 + 0.0001984188x^7 + 2.4802 \times 10^{-5}x^8 + 2.7589 \times 10^{-6}x^9 + 2.7427 \times 10^{-7}x^{10} + 2.4703 \times 10^{-8}x^{11} + 2.1384 \times 10^{-9}x^{12} + O(x^{13}).
\]

(3.11)

Numerical results of the solution (3.11) are displayed in Table 4.

CONCLUSIONS

In this paper, we have used OHAM, for finding the solution of fifth and sixth-order, linear and nonlinear boundary value problems. In this method all the recursive relations are explicitly defined and the values of the convergence controlling constants are optimally determined. No restrictive assumptions are needed and one feels very comfortable as the convergence of the method is not dependent on the initial guess. The low order solutions show excellent agreement with the exact solution and a remarkable low error is notable. The method gives more realistic series solutions that converge very rapidly in physical problems. It may be concluded that the method is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems.
REFERENCES