On solutions of a quadratic integral equation of Hammerstein type

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Received 28 March 2005; accepted 21 April 2005

Abstract

We study the solvability of a nonlinear quadratic integral equation of Hammerstein type. Using the technique of measures of noncompactness we prove that this equation has solutions on an unbounded interval. Moreover, we also obtain an asymptotic characterization of these solutions. Several special cases of this integral equation are discussed and applications to real world problems are indicated.

Keywords: Quadratic integral equation; Banach space; Measure of noncompactness; Fixed point theorem; Asymptotic behaviour

1. Introduction

The main subject of this paper is the study of solutions of the following nonlinear quadratic integral equation:

\[ x(t) = g(t) + f(t, x(t)) \int_0^\infty K(t, s)h(s, x(s)) \, ds, \quad t \geq 0. \]  

(1)

In the case of a bounded interval this equation has the form

\[ x(t) = g(t) + f(t, x(t)) \int_a^b K(t, s)h(s, x(s)) \, ds, \quad t \in [a, b]. \]

It is worthwhile mentioning that the above equation appears very often, in a lot of applications to real world problems. For example, some problems considered in vehicular traffic theory, biology and queuing theory lead to the quadratic integral equations of this type (cf. [1]). Moreover, such integral equations are also applied in the theory of radiative transfer and the theory of neutron transport as well in the kinetic theory of gases (cf. [2–6], among others).

Thus a quadratic integral equation of the form (1) creates a generalization of several kinds of quadratic integral equations of the above-mentioned type. On the other hand, Eq. (1) contains, as a special case, the classical

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Hammerstein integral equation on an unbounded interval having the form
\[ x(t) = g(t) + \int_{0}^{\infty} K(t, s)h(s, x(s)) \, ds, \quad t \geq 0. \]

Indeed, putting \( f(t, x) \equiv 1 \) we obtain the above-mentioned integral equation.

Let us note that the Hammerstein integral equation plays a very significant role in several applications [7–9].

In this paper we will investigate the existence and asymptotic behaviour of solutions of Eq. (1). In our investigations we apply the method associated with the technique of measures of noncompactness and the fixed point theorem of Darbo type. This technique was mainly initiated in the book [10] and subsequently developed in many papers (cf. [1, 11–14], for instance).

In our study we will use a measure of noncompactness defined first in the book [10]. Up to now that measure of noncompactness has been applied to kinds of functional integral equations different to the quadratic integral equation (1).

2. Notation and basic facts

In this section we collect definitions and auxiliary facts which will be needed further on. Let \( (E, \| \cdot \|) \) be an infinite dimensional Banach space with zero element \( \emptyset \). Denote by \( B(x, r) \) the closed ball in \( E \) centered at \( x \) with radius \( r \). Let \( B_r \) stand for the ball \( B(\emptyset, r) \).

If \( X \) is a subset of \( E \), then \( \bar{X} \) and Conv \( X \) denote the closure and convex closure of \( X \), respectively. Moreover, the symbol \( \mathcal{M}_E \) denotes the family of all nonempty and bounded subsets of \( E \) while \( \mathcal{M}_E \) stands for its subfamily consisting of all relatively compact sets.

We will accept the following definition of the concept of a measure of noncompactness [10].

Definition 1. A mapping \( \mu : \mathcal{M}_E \rightarrow \mathbb{R}_+ = [0, +\infty) \) is said to be a measure of noncompactness in \( E \) if the following conditions are satisfied:

1° The family ker \( \mu = \{ X \in \mathcal{M}_E : \mu(X) = 0 \} \) is nonempty and \( \ker \mu \subset \mathcal{M}_E \).
2° \( X \subset Y \Rightarrow \mu(X) \leq \mu(Y) \).
3° \( \mu(\text{Conv } X) = \mu(X) \).
4° \( \mu(\bar{X}) = \mu(X) \).
5° \( \mu(\lambda X + (1 - \lambda) Y) \leq \lambda \mu(X) + (1 - \lambda) \mu(Y) \) for \( \lambda \in [0, 1] \).
6° If \( (X_n) \) is a sequence of sets from \( \mathcal{M}_E \) such that \( X_{n+1} \subset X_n \), \( \bar{X}_n = X_n \) (\( n = 1, 2, \ldots \)) and if \( \lim_{n \to \infty} \mu(X_n) = 0 \), then the intersection \( X_{\infty} = \bigcap_{n=1}^{\infty} X_n \) is nonempty.

The family ker \( \mu \) described in 1° is called the kernel of the measure of noncompactness \( \mu \).

A measure \( \mu \) is called sublinear if it satisfies the following two conditions:

7° \( \mu(\lambda X) = |\lambda| \mu(X) \) for \( \lambda \in \mathbb{R} \).
8° \( \mu(X + Y) \leq \mu(X) + \mu(Y) \).

Moreover, a measure \( \mu \) satisfying the condition

9° \( \mu(X \cup Y) = \max\{\mu(X), \mu(Y)\} \)

will be referred to as a measure with maximum property.

Other facts concerning measures of noncompactness and their properties may be found in [10]. For our purposes we will only need the following fixed point theorem [10].

Theorem 1. Let \( Q \) be nonempty bounded closed convex subset of the space \( E \) and let \( G : Q \rightarrow Q \) be continuous and such that \( \mu(GX) \leq k \mu(X) \) for any nonempty subset \( X \) of \( Q \), where \( k \) is a constant, \( k \in [0, 1] \). Then \( G \) has a fixed point in the set \( Q \).

Remark 1. Under the assumptions of the above theorem it can be shown that the set \( \text{Fix } G \) of fixed points of \( G \) belonging to \( Q \) is a member of the family ker \( \mu \).

This observation permits us to characterize solutions of the equations investigated.
In the sequel we will work in the Banach space $BC(\mathbb{R}_+)$ consisting of all real functions defined, bounded and continuous on $\mathbb{R}_+$. The space $BC(\mathbb{R}_+)$ is furnished with the standard norm

$$\|x\| = \sup\{|x(t)| : t \geq 0\}.$$

Now we recollect the definition of the measure of noncompactness which will be used further on. This measure was introduced in [10]. Fix a nonempty bounded subset $X$ of $BC(\mathbb{R}_+)$ and a positive number $T > 0$. For $x \in X$ and $\varepsilon \geq 0$ let us denote by $\omega^T(x, \varepsilon)$ the modulus of continuity of the function $x$ on the interval $[0, T]$, i.e.

$$\omega^T(x, \varepsilon) = \sup\{|x(t) - x(s)| : t, s \in [0, T], |t - s| \leq \varepsilon\}.$$

Further, let us put

$$\begin{align*}
\omega^T(X, \varepsilon) &= \sup\{\omega^T(x, \varepsilon) : x \in X\}, \\
\omega^T_0(X) &= \lim_{\varepsilon \to 0} \omega^T(X, \varepsilon), \\
\omega_0(X) &= \lim_{T \to \infty} \omega^T_0(X).
\end{align*}$$

Moreover, let us put

$$\beta(X) = \lim_{T \to \infty} \left\{\sup_{x \in X} \{\sup\{|x(t)| : t \geq T\}\} \right\}.$$

Finally, let us define the function $\mu$ on the family $\mathcal{M}_{BC(\mathbb{R}_+)}$ by the formula

$$\mu(X) = \omega_0(X) + \beta(X).$$

It can be shown [10] that the function $\mu$ is a sublinear measure of noncompactness with the maximum property in the space $BC(\mathbb{R}_+)$. The kernel $\ker \mu$ of this measure contains nonempty and bounded sets $X$ such that functions from $X$ are locally equicontinuous on $\mathbb{R}_+$ and they tend to zero at infinity uniformly with respect to the set $X$. This property of the kernel $\ker \mu$ allows us to characterize (in terms of asymptotic behaviour) solutions of the integral equation (1) and will be used in the rest of the paper.

3. Main result

In this section we will study the existence and asymptotic behaviour of solutions of the functional integral equation (1). Our investigations are situated in the Banach space $BC(\mathbb{R}_+)$ described in the previous section.

We will assume that the functions involved in Eq. (1) satisfy the following conditions:

(i) $g : \mathbb{R}_+ \to \mathbb{R}$ is a continuous function such that $g(t) \to 0$ as $t \to \infty$;

(ii) $f : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is continuous and the function $t \to f(t, 0)$ belongs to the space $BC(\mathbb{R}_+)$;

(iii) there exists a continuous function $m : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|f(t, x) - f(t, y)| \leq m(t)|x - y|$$

for all $x, y \in \mathbb{R}$ and $t \in \mathbb{R}_+$;

(iv) $h : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ is uniformly continuous on every rectangle of the form $\mathbb{R}_+ \times [-v, v]$;

(v) there exist a continuous function $a : \mathbb{R}_+ \to \mathbb{R}_+$ and a continuous and nondecreasing function $b : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$|h(t, x)| \leq a(t)b(|x|)$$

for $t \geq 0$ and $x \in \mathbb{R}$;

(vi) $K : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is a continuous function and there exist continuous functions $p, q : \mathbb{R}_+ \to \mathbb{R}_+$ such that the functions $q(t)$ and $a(t)q(t)$ are integrable over $\mathbb{R}_+$ and the following inequality:

$$|K(t, s)| \leq p(t)q(s)$$

is satisfied for $t, s \in \mathbb{R}_+$. Moreover, we assume that $\lim_{t \to \infty} p(t) = 0$ and the function $m(t)p(t)$ is bounded on the interval $\mathbb{R}_+$. 

Keeping in mind the above assumptions we can easily infer that the constants $F$, $M$, $P$ are defined by the formulas
\[
F = \sup \{|f(t,0)| : t \geq 0\}, \\
M = \sup \{m(t)p(t) : t \geq 0\}, \\
P = \sup \{p(t) : t \geq 0\}
\]
are finite.

For our further purposes we denote by $Q$ the constant
\[
Q = \int_0^\infty a(s)q(s)\,ds.
\]
Obviously $Q < \infty$.

In what follows we will also assume the following hypothesis:

(vii) The inequality
\[
\|g\| + b(r)(MQr + FPQ) \leq r
\]
has a positive solution $r_0$.

Now we can formulate our main result.

**Theorem 2.** Under the assumptions (i)–(vii) the Eq. (1) has at least one solution $x = x(t)$ in the space $BC(\mathbb{R}_+)$.

Moreover, all solutions of Eq. (1) belonging to the ball $B_{r_0}$ tend to zero at infinity uniformly with respect to the ball $B_{r_0}$.

**Proof.** Consider the operator $H$ defined on the space $BC(\mathbb{R}_+)$ by the formula
\[
(Hx)(t) = |g(t)| + f(t,x(t))\int_0^\infty K(t,s)h(s,x(s))\,ds, \quad t \geq 0.
\]

Observe that in view of the assumptions (i), (ii) and (iv)–(vi) the function $Hx$ is continuous on the interval $\mathbb{R}_+$ for any function $x \in BC(\mathbb{R}_+)$. Further, applying our assumptions we derive the following estimate:

\[
|(Hx)(t)| \leq |g(t)| + |f(t,x(t))|\int_0^\infty |K(t,s)| \cdot |h(s,x(s))|\,ds
\leq \|g\| + [\|f(t,x(t)) - f(t,0)\| + |f(t,0)||\int_0^\infty p(t)q(s)a(s)b(|x(s)|)\,ds
\leq \|g\| + [m(t)|x(t)| + |f(t,0)||\int_0^\infty p(t)q(s)a(s)\,ds
\leq \|g\| + m(t)p(t)\|x\|b(\|x\|)Q + FPQ(\|x\|)p(t).
\]

The above estimate allows us to infer that the function $Hx$ is bounded on the interval $\mathbb{R}_+$. Moreover, we obtain that the following inequality holds:
\[
\|Hx\| \leq \|g\| + MQ\|x\|b(\|x\|) + FPQ(\|x\|).
\]

This inequality in conjunction with the assumption (vii) ensures that there exists a positive number $r_0$ for which the operator $H$ transforms the ball $B_{r_0}$ into itself.

In what follows we show that $H$ is continuous on the ball $B_{r_0}$. To do this let us fix $\varepsilon > 0$ and take $x, y \in B_{r_0}$ such that $\|x - y\| \leq \varepsilon$. Then, for arbitrarily fixed $t \in \mathbb{R}_+$ we get
\[
|(Hx)(t) - (Hy)(t)| \leq |f(t,x(t)) - f(t,y(t))|\int_0^\infty |K(t,s)| \cdot |h(s,x(s))|\,ds
\leq m(t)|x(t) - y(t)|\int_0^\infty p(t)q(s)a(s)b(|x(s)|)\,ds
\leq m(t)|x(t) - y(t)|\int_0^\infty p(t)q(s)a(s)b(|x(s)|)\,ds.
\]
we conclude that the operator take
\[ \omega_0(\epsilon) = \sup \{|h(s, x) - h(s, y)| : x \geq 0, \ y \in [-r_0, r_0], \ |x - y| \leq \epsilon \}. \]
Observe that in view of the assumption (iv) we infer that \( \omega_0(\epsilon) \to 0 \) as \( \epsilon \to 0 \). Thus from the estimate (3) we conclude that the operator \( H \) is continuous on the ball \( B_{r_0} \).

Now, let us take a nonempty set \( X, X \subset B_{r_0} \). Next, fix arbitrarily \( T > 0 \) and \( \epsilon > 0 \). Choose a function \( x \in X \) and take \( t, s \in [0, T] \) such that \( |t - s| \leq \epsilon \). Then, by virtue of the accepted assumptions we have
\[
\left| (Hx)(t) - (Hx)(s) \right| \leq |g(t) - g(s)| + \left| f(t, x(t)) - f(s, x(s)) \right| \int_0^\infty |K(t, \tau)| |h(\tau, x(\tau))| \, d\tau \\
+ \left| f(s, x(s)) \right| \int_0^\infty |K(t, \tau) - K(s, \tau)| \cdot |h(\tau, x(\tau))| \, d\tau \leq |g(t) - g(s)| \\
+ |f(s, x(s)) - f(s, 0)| \middle| + |f(f, x(s)) - f(s, x(s))| \middle| \int_0^\infty p(t)q(\tau)a(\tau)b(|x(\tau)|) \, d\tau \\
+ |f(f, x(s)) - f(f, 0)| \middle| + |f(f, 0)| \middle| \int_0^\infty |K(t, \tau) - K(s, \tau)| |a(\tau)\cdot b(|x(\tau)|)\middle| \middle| \int_0^\infty p(t)q(\tau)a(\tau)\cdot b(|x(\tau)|) \, d\tau \\
\leq \omega^T(g, \epsilon) + \left\{ m(t)|x(t) - (x(s)| + \omega^T_0(f, \epsilon) \middle| b(r_0)p(t) \middle| a(\tau)q(\tau) \, d\tau \\
+ |m(s)| |x(s)| + |f(s, 0)| \middle| b(r_0) \middle| \int_0^\infty |K(t, \tau) - K(s, \tau)| a(\tau) \, d\tau \leq \omega^T(g, \epsilon) + MQb(r_0) \omega^T(x, \epsilon) + P Qb(r_0) \omega^T_0(f, \epsilon) \\
+ (M_T r_0 + F) \middle| b(r_0) \middle| \int_0^\infty |K(t, \tau) - K(s, \tau)| a(\tau) \, d\tau, \right.
\]
where we defined
\[
M_T = \max \{ m(t) : t \leq T \}, \\
\omega^T_0(f, \epsilon) = \sup \{|f(t, x) - f(s, x)| : t, s \in [0, T], \ |t - s| \leq \epsilon, \ |x| \leq r_0 \}. \]

Further, let us note that we can obtain the following chain of estimates:
\[
\int_0^\infty |K(t, \tau) - K(s, \tau)| a(\tau) \, d\tau \leq \int_0^T |K(t, \tau) - K(s, \tau)| a(\tau) \, d\tau + \int_T^\infty |K(t, \tau) - K(s, \tau)| a(\tau) \, d\tau \\
\leq \int_0^T |K(t, \tau) - K(s, \tau)| a(\tau) \, d\tau \right. \]
we obtain

\[ A_T \int_0^T \left| K(t, \tau) - K(s, \tau) \right| \, d\tau + \int_T^\infty (p(t) + p(s))q(\tau)a(\tau) \, d\tau \leq A_T \cdot T\omega^T(K, \varepsilon) + 2P_T \int_T^\infty a(\tau)q(\tau) \, d\tau, \tag{5} \]

where we defined

\[ A_T = \max \{ a(t) : t \in [0, T]\}, \]
\[ P_T = \max \{ p(t) : t \in [0, T]\}, \]
\[ \omega_T(K, \varepsilon) = \{ |K(t, \tau) - K(s, \tau)| : t, s, \tau \in [0, T], |t - s| \leq \varepsilon \}. \]

Now, linking the estimates (4) and (5) we get

\[ \omega^T(Hx, \varepsilon) \leq \omega^T(g, \varepsilon) + MQb(r_0)\omega^T(x, \varepsilon) + PQb(r_0)\omega^T(x, \varepsilon) + PQb(r_0)\omega^T(f, \varepsilon) + (MTr_0 + F)b(r_0) \left[ T\omega_T(K, \varepsilon) + 2P_T \int_T^\infty a(\tau)q(\tau) \, d\tau \right]. \tag{6} \]

The assumption (i) yields that \( \omega^T(g, \varepsilon) \to 0 \) as \( \varepsilon \to 0 \). Moreover, in view of the assumption (iii) we deduce that the function \( f = f(t, x) \) is uniformly continuous on the set \([0, T] \times [-r_0, r_0]\). This implies

\[ \lim_{\varepsilon \to 0} \omega^T(f, \varepsilon) = 0. \]

Similarly, from the assumption (vi) we derive

\[ \lim_{\varepsilon \to 0} \omega_T(K, \varepsilon) = 0. \]

Now, taking into account the above facts, from the inequality (6) we obtain

\[ \omega^T_0(HX) \leq MQb(r_0)\omega^T_0(X) + 2(MTr_0 + F)b(r_0)P_T \int_T^\infty a(\tau)q(\tau) \, d\tau. \]

This inequality in conjunction with the well-known property of improper integrals yields the following estimate:

\[ \omega_0(HX) \leq MQb(r_0)\omega_0(X). \tag{7} \]

Furthermore, take an arbitrary function \( x \in X \) and a number \( T > 0 \). Then, from the estimate (2) we deduce easily the following inequality:

\[ \sup \{ |(Hx)(t)| : t \geq T \} \leq \sup \{ |g(t)| : t \geq T \} + MQb(r_0)\sup \{ |x(t)| : t \geq T \} + FQb(r_0)\sup \{ p(t) : t \geq T \}. \]

Hence, taking into account that \( p(t) \to 0 \) and \( g(t) \to 0 \) as \( t \to \infty \), we derive the following estimate:

\[ \beta(HX) \leq MQb(r_0)\beta(X). \tag{8} \]

Finally, linking the inequalities (7) and (8) and keeping in mind the definition of the measure of noncompactness \( \mu \) from Section 2, we get

\[ \mu(HX) \leq MQb(r_0)\mu(X). \tag{9} \]

Now, let us observe that the assumption (vii) yields the following inequality:

\[ MQb(r_0) \leq 1 - \frac{FPQb(r_0)}{r_0} + \| g \| < 1. \tag{10} \]

Thus, joining the estimates (9) and (10) and taking into account Theorem 1 we infer that the operator \( H \) has at least one fixed point in the ball \( B_{r_0} \).

Thus the proof of our theorem is complete. \( \square \)
Remark 2. In view of Remark 1 we deduce that the set $\text{Fix } H$ of fixed points of the operator $H$ belonging to the ball $B_{r_0}$ contains all solutions of Eq. (1) which belong to $B_{r_0}$. Moreover, those solutions tend to zero at infinity uniformly with respect to $B_{r_0}$.

4. Final remarks

This section is devoted to discussing a few facts concerning the assumptions accepted in our main existence result, i.e. in Theorem 2.

Let us initially consider the case $m(t) = k = \text{const.}$ (cf. assumption (iii)). Then the function $f(t, x)$ satisfies the classical Lipschitz condition

$$|f(t, x) - f(t, y)| \leq k|x - y|.$$  

Observe that assumptions of this kind are typical if we consider Eq. (1) for describing problems in traffic theory (cf. [1]).

Further, let us assume that $a : \mathbb{R}^+ \to \mathbb{R}^+$ is a bounded function and the function $b : \mathbb{R}^+ \to \mathbb{R}^+$ has the form $b(r) = r$. Then the estimate from assumption (v) has the form

$$|h(t, x)| \leq a(t)|x|.$$  

In this case the inequality from assumption (vii) has the form

$$\|g\| + r(M QR + FPQ) \leq r.$$  

In order to guarantee that this inequality has a positive solution if is sufficient to assume that $FPQ < 1$ and $FPQ \leq 1 - 2\sqrt{MQ\|g\|}$. Now, let us take the function $b = b(r)$ having the form $b(r) = r^\alpha$, where $\alpha$ is a positive constant. In this case the inequality from (vii) has the form

$$\|g\| + r^\alpha(M QR + FPQ) \leq r.$$  

Let us consider two cases.

1º $0 < \alpha < 1$.

In this case we can transform the inequality (11) to the form

$$\|g\| \leq r^\alpha \left(r^{1-\alpha} - M QR - FPQ\right).$$

The above inequality admits a positive solution for

$$FPQ < \left(1 - \frac{\alpha}{MQ}\right)^{(1-\alpha)/\alpha}$$

and for $g$ such that

$$\|g\| < \left(1 - \frac{\alpha}{MQ}\right)^{(1-\alpha)/\alpha} - FPQ.$$  

2º $\alpha > 1$.

In this case it is easy to verify that the inequality (11) has a positive solution for $\|g\|$ sufficiently small. We omit the details.

In order to illustrate our investigations let us consider the following quadratic integral equation:

$$x(t) = te^{-4t} + \left(tx(t) + \frac{t}{16 + t^2}\right) \int_0^\infty \frac{t^2e^{-s}}{1 + t^2} \sqrt{|x(s)|} \, ds.$$  

(12)
Obviously Eq. (12) is a special case of Eq. (1). Indeed, if we put
\[ g(t) = te^{-4t}, \]
\[ f(t, x) = tx + \frac{t}{16 + t^2}, \]
\[ K(t, s) = \frac{te^{-s}}{1 + t^2}, \]
\[ h(t, x) = t|x|, \]
then it is easily seen that the assumptions of Theorem 2 are satisfied.

In fact, in this case we have that \( f(t, 0) = \frac{t}{16 + t^2}. \) Consequently \( F = 1/8. \) Further we have that \( p(t) = \frac{t}{1 + t^2}, \)
\[ q(t) = e^{-t}, \]
\[ b(r) = \sqrt{r}, \]
\[ a(t) = t \text{ and } m(t) = t. \]
This yields that \( M = 1, P = 1/2, Q = 1 \) and \( \|g\| = 1/4e. \) Moreover, we can check that the inequality from assumption (vii) has the form
\[ \frac{1}{4e} + \sqrt{r} \left( r + \frac{1}{16} \right) \leq r. \]

Obviously this inequality has a positive solution \( r_0. \) For example, \( r_0 = 1/4. \) We can also verify other hypotheses of Theorem 2. The details are omitted.

References