Reducing weight flexibility in fuzzy DEA

S. Saati a,*, A. Memariani b

a Department of Mathematics, Tehran-North Branch, Islamic Azad University, P.O. Box 14515-397, Tehran, Iran
b Department of Industrial Engineering, Tarbiat Modarres University, Tehran, Iran

Abstract

An important outcome of assessing relative efficiencies within a group of decision making units (DMUs) in fuzzy data envelopment analysis is a set of virtual multipliers or weights accorded to each (input or output) factor taken into account. These sets of weights are, typically, different for each of the participating DMUs, and in some cases it may be considered unacceptable that the same factor is accorded widely differing weights. Thus, it is important to find a common set of weights (CSW) across the set of DMUs. In this paper, by assessing upper bounds on factor weights and compaction the resulted intervals, a CSW is determined. Since resulted efficiencies by the proposed CSW are fuzzy numbers rather than crisp values, it is more informative for decision maker.

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1. Introduction

More typically processes and organizational units have multiple nonhomogeneous inputs and outputs and this complexity can be incorporated in an efficiency measure by defining the efficiency as:

\[ \text{Efficiency} = \frac{\text{weighted sum of outputs}}{\text{weighted sum of inputs}}. \]

*Corresponding author. E-mail address: ssaatim@yahoo.com (S. Saati).
This definition requires a set of weights to be defined and this can be difficult, particularly if a common set of weights (CSW) to be applied across the set of units is sought.

Charnes et al. [1] originally proposed data envelopment analysis (DEA) as a method for determining a set of weights for each decision making unit (DMU). These sets of weights are, typically, different for each of the participating DMUs and, some of the weights may be assigned an exceedingly small value. A possible answer to these difficulties lies in the notion of a CSW and specification of appropriate bounds within which weights are allowed to vary.

The fuzzy systems approach has many features, which are particularly suitable for the theory and practice of DEA models. Some fuzzy versions of DEA models are proposed in Sengupta [9], Cooper et al. [4], Kao and Liu [6], Guo and Tanaka [5] and Saati et al. [8]. The outcomes of these models are different sets of virtual multipliers or weights accorded to each (input or output) factor taken into account.

In this paper, a procedure is suggested to find a CSW in fuzzy DEA. In the proposed procedure first, upper levels of the weights are determined by solving some linear programming problems and then a CSW is determined by solving a linear programming problem.

The paper is organized as follows: Section 2 provides a short background about fuzzy CCR models. The suggested method to find a CSW in fuzzy DEA is presented in Section 3. To demonstrate the feasibility of the proposed procedure, an illustrated example is presented in Section 4. Section 5 closes with conclusion.

2. Fuzzy CCR model

The evaluation of a DMU has long been recognized to be a problem of considerable complexity. This evaluation becomes more difficult when it involves multiple inputs and multiple outputs, in that a set of weights has to be determined to aggregate the outputs and inputs separately to form a ratio as the efficiency. To do so, DEA approach is proposed, which allows every DMU to select their most favorable weights while requiring the resulted ratio of the aggregated outputs to the aggregated inputs of all DMUs to be less than or equal to 1.

Consider $n$ DMUs, each consumes varying amounts of $m$ different fuzzy inputs to produce $s$ different fuzzy outputs. In the model formulation, $\tilde{x}_{ip}$ ($i = 1, \ldots, m$) and $\tilde{y}_{rp}$ ($r = 1, \ldots, s$) denote, respectively, the input and output values for DMU$_p$, the DMU under consideration.

The programming statement for the (input oriented) fuzzy CCR model is:
\[
\begin{align*}
\text{max} \quad W &= \sum_{r=1}^{s} u_r \bar{y}_{rp} \\
\text{s.t.} \quad \sum_{i=1}^{m} v_i \bar{x}_{ip} &= \bar{1} \\
&\quad \sum_{r=1}^{s} u_r \bar{y}_{rj} - \sum_{i=1}^{m} v_i \bar{x}_{ij} \leq 0 \quad \forall j, \\
&\quad u_r, v_i \geq 0 \quad \forall r, i,
\end{align*}
\]

where \( v_i (i = 1, \ldots, m) \) and \( u_r (r = 1, \ldots, s) \) are the weights associated with input \( i \) and output \( r \), respectively.

Model (1) is a fuzzy linear programming problem. There are several methods to solve it (e.g. see [5,6,8,9]). In continuation, we will use the proposed method in [8] to solve the fuzzy linear programming problems.

Apart from the restriction that no weight may be zero, weights on inputs and outputs are only restricted by the requirement that they must not make the efficiency of any DMU more than 1. The advantage of allowing such freeness on the weights is that, a best efficiency rating is associated to each DMU. However, in this flexibility, some of the weights may be assigned an exceedingly small value. Also, as a different model is run for each DMU, the set of weights will typically be different for each DMU, and it is unacceptable that the same factor has widely different weights. To control the flexibility of weights, we consider the bounded fuzzy DEA models.

The fuzzy CCR model, assuming bounds on factor weights, is as follows:

\[
\begin{align*}
\text{max} \quad W &= \sum_{r=1}^{s} u_r \bar{y}_{rp} \\
\text{s.t.} \quad \sum_{i=1}^{m} v_i \bar{x}_{ip} &= \bar{1} \\
&\quad \sum_{r=1}^{s} u_r \bar{y}_{rj} - \sum_{i=1}^{m} v_i \bar{x}_{ij} \leq 0 \quad \forall j, \\
&\quad U_r^l \leq u_r \leq U_r^u \quad \forall r, \\
&\quad V_i^l \leq v_i \leq V_i^u \quad \forall i,
\end{align*}
\]

where \( U_r^l, U_r^u, V_i^l \) and \( V_i^u \) are lower and upper bounds on output and input weights, respectively.

This formulation restricts the flexibility of weights within certain bounds. These bounds are same for all DMUs and care should be taken in their selection, because unsuitable bounds makes (2) infeasible.

The bounded fuzzy CCR model cannot fully restrict the flexibility of weights, and restricts it in given bounds. When no flexibility is allowed in fuzzy
DEA for assigning the individual set of weights to each of the participating DMUs, a CSW is determined.

3. Proposed model

As can be seen in (1), in fuzzy DEA the relative efficiency of a DMU is determined by assigning such weights to the DMU’s inputs and outputs that its relative efficiency is maximized. Solving a programming problem for each DMU separately does this. Solving separate problems for each DMU allows different weights to be used in computing the relative efficiency of different DMUs, thus obviating the need to negotiate a CSW across the set of DMUs.

In what follows, a procedure is suggested to assess a CSW. This is done in two steps. In the first step, by solving a linear programming problem for each factor, an upper bound is determined for each of them. A CSW is determined in the second step by compacting the weight intervals via solving a linear programming problem.

**Step 1: Bounds determination**

To determine the upper bounds on input and output weights, consider the following problems:

**Upper bounds of output weights (A)**

\[
\max u_p \\
\text{s.t.} \quad \sum_{i=1}^{m} v_i \bar{x}_{ij} \leq \bar{1} \quad \forall j, \\
\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \bar{x}_{ij} \leq 0 \quad \forall j, \\
u_r, v_i \geq 0 \quad \forall r, i.
\]

**Upper bounds of input weights (B)**

\[
\max v_i \\
\text{s.t.} \quad \sum_{i=1}^{m} v_i \bar{x}_{ij} \leq \bar{1} \quad \forall j, \\
\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \bar{x}_{ij} \leq 0 \quad \forall j, \\
u_r, v_i \geq 0 \quad \forall r, i.
\]

By solving \(m + s\) linear programming problems, the upper bounds of output and input weights are determined.

Among the various types of fuzzy numbers, triangular fuzzy numbers are of the more important. In the sequel, we consider the inputs and outputs of DMUs as triangular fuzzy numbers. Let \(\bar{x}_{ij} = (x_{ij}^m, x_{ij}^a, x_{ij}^b)\) and \(\tilde{y}_{rj} = (y_{rj}^m, y_{rj}^a, y_{rj}^b)\).
Assumption 1. Each DMU has at least one output with positive $y^m_{rj} - y^a_{rj}$ (i.e. two sided fuzzy number).

Therefore, (3A) can be written as follows:

$$\text{max} \quad u_p$$

s.t. \hspace{1cm} \sum_{i=1}^{m} v_i(x^m_{ij}, x^x_{ij}, x^b_{ij}) \leq \tilde{1} \quad \forall j,$$

$$\sum_{r=1}^{s} u_r(y^m_{rj}, y^x_{rj}, y^b_{rj}) - \sum_{i=1}^{m} v_i(x^m_{ij}, x^x_{ij}, x^b_{ij}) \leq 0 \quad \forall j,$$

$$u_r, v_i \geq 0 \quad \forall r, i.$$

Applying the suggested two phases method in [8], in the end of the first phase, the following model would be achieved:

$$\text{max} \quad u_p$$

s.t. \hspace{1cm} \sum_{i=1}^{m} v_i\hat{x}_{ij} \leq 1 \quad \forall j,$$

$$\sum_{r=1}^{s} u_r\hat{y}_{rj} - \sum_{i=1}^{m} v_i\hat{x}_{ij} \leq 0 \quad \forall j,$$

$$x^m_{ij} - (1 - \gamma)x^x_{ij} \leq \hat{x}_{ij} \leq x^m_{ij} + (1 - \gamma)x^b_{ij} \quad \forall i, j,$$

$$y^m_{rj} - (1 - \gamma)y^x_{rj} \leq \hat{y}_{rj} \leq y^m_{rj} + (1 - \gamma)y^b_{rj} \quad \forall r, j,$$

$$u_r, v_i \geq 0 \quad \forall r, i,$$

which is a nonlinear programming problem.

Applying the second phase (5) will be converted into the following model:

$$\text{max} \quad u_p$$

s.t. \hspace{1cm} \sum_{i=1}^{m} \bar{x}_{ij} \leq 1 \quad \forall j,$$

$$\sum_{r=1}^{s} \bar{y}_{rj} - \sum_{i=1}^{m} \bar{x}_{ij} \leq 0 \quad \forall j,$$

$$v_i(x^m_{ij} - (1 - \gamma)x^x_{ij}) \leq \bar{x}_{ij} \leq v_i(x^m_{ij} + (1 - \gamma)x^b_{ij}) \quad \forall i, j,$$

$$u_r(y^m_{rj} - (1 - \gamma)y^x_{rj}) \leq \bar{y}_{rj} \leq u_r(y^m_{rj} + (1 - \gamma)y^b_{rj}) \quad \forall r, j,$$

$$\bar{x}_{ij}, \bar{y}_{rj} \geq 0 \quad \forall r, i, j,$$

$$u_r, v_i \geq 0 \quad \forall r, i,$$
which is a parametric linear programming problem, while $\gamma \in [0, 1]$ is a parameter. A similar linear programming problem can be attained for (3B).

**Theorem 1.** The problem (6) is feasible and its optimal value is bounded and positive.

**Proof.** Obviously $(X, Y, U, V) = (0, 0, 0, 0)$ is a feasible solution of (6). To prove the boundedness of (6), consider its dual problem:

$$
\begin{align*}
\min \quad & \sum_{j=1}^{n} x_j \\
\text{s.t.} \quad & x_j - \beta_j + \lambda_{ij} - \theta_{ij} \geq 0 \quad \forall i, j, \\
& \beta_j + \phi_{rj} - \rho_{rj} \geq 0 \quad \forall r, j, \\
& - \sum_{j=1}^{n} (x_{ij}^m + (1 - \gamma)x_{ij}^n) \lambda_{ij} + \sum_{j=1}^{n} (x_{ij}^m - (1 - \gamma)x_{ij}^n) \theta_{ij} \geq 0 \quad \forall i, \\
& - \sum_{j=1}^{n} (y_{rj}^m + (1 - \gamma)y_{rj}^n) \phi_{rj} + \sum_{j=1}^{n} (y_{rj}^m - (1 - \gamma)y_{rj}^n) \rho_{rj} \geq 0 \quad \forall r \neq p, \\
& - \sum_{j=1}^{n} (y_{pj}^m + (1 - \gamma)y_{pj}^n) \phi_{pj} + \sum_{j=1}^{n} (y_{pj}^m - (1 - \gamma)y_{pj}^n) \rho_{pj} \geq 1
\end{align*}
$$

By Assumption 1, it is evident that there exist a $q, 1 \leq q \leq n$, such that $z = y_{pq}^m - (1 - \gamma)y_{pq}^n \neq 0$. The following point is a feasible solution of dual problem:

$$
W = \left\{ \begin{array}{ll}
x_j = 0 & \forall j \neq q, \\
x_q = \frac{1}{z}, \\
\beta_j = 0 & \forall j \neq q, \\
\beta_q = \frac{1}{z}, \\
\lambda_{ij} = \theta_{ij} = 0 & \forall i, j, \\
\phi_{rj} = 0 & \forall r, j, \\
\rho_{rj} = 0 & \forall r \neq p, j \neq q, \\
\rho_{pq} = \frac{1}{z}.
\end{array} \right.
$$

Therefore, optimal value of (6) is bounded.

To prove the positivity of optimal value of (6), consider the problem (5). Evidently, the following point is a feasible point of (5):
\[
Z = \left\{
\begin{align*}
\hat{x}_{ij} &= x^m_{ij} + (1 - \gamma)\hat{x}_{ij} & \forall i, j, \\
\hat{y}_{rj} &= y^m_{rj} + (1 - \gamma)\hat{y}_{rj} & \forall r, j, \\
v_i &= \frac{1}{mwi} & \forall i, \\
u_r &= \frac{1}{szr} & \forall r,
\end{align*}
\right.
\]

where
\[
w_i = \max_j \left\{x^m_{ij} + (1 - \gamma)\hat{x}_{ij}\right\} \neq 0 \quad \text{and} \quad z_r = \max_j \left\{y^m_{rj} + (1 - \gamma)\hat{y}_{rj}\right\} \neq 0,
\]

and, \(m\) and \(s\) are the input and output numbers, respectively.

Since \(u_r = \frac{1}{szp} > 0\), the optimal value of (6) is positive. \(\square\)

**Step 2: Determining a CSW**

Starting from bounded model (2), a CSW can be achieved by expressing the deviation from either bound as a fraction of the range between the upper and lower bounds. Assuming the same deviation from bounds across all DMUs, we get:

\[
\max \phi
\]

s.t. \[
\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq 0 \quad \forall j,
\]

\[
U_r^l + \phi(U_r^u - U_r^l) \leq u_r \leq U_r^u - \phi(U_r^u - U_r^l) \quad \forall r,
\]

\[
V_i^l + \phi(V_i^u - V_i^l) \leq v_i \leq V_i^u - \phi(V_i^u - V_i^l) \quad \forall i.
\]

Compacting the weight intervals in (7) is done by a proportion of each corresponding interval length, since the upper bound of factor weights are not same.

Applying the results of models (3A) and (3B), and setting the lower bounds of factor weights equal to zero, the following model is deduced:

\[
\max \phi
\]

s.t. \[
\sum_{r=1}^{s} u_r \tilde{y}_{rj} - \sum_{i=1}^{m} v_i \tilde{x}_{ij} \leq 0 \quad \forall j,
\]

\[
\phi U_r \leq u_r \leq (1 - \phi)U_r \quad \forall r,
\]

\[
\phi V_i \leq v_i \leq (1 - \phi)V_i \quad \forall i,
\]

where \(U_r \ (r = 1, \ldots, s)\) and \(V_i \ (i = 1, \ldots, m)\) are optimal values of models (3A) and (3B), respectively.

Similarly, with triangular fuzzy data, (8) can be converted to a crisp programming problem using the proposed method in Saati et al. [8] as follows:
\[ \text{max} \quad \phi \]
\[ \text{s.t.} \quad \sum_{r=1}^{s} \tilde{y}_{rj} - \sum_{i=1}^{m} x_{ij} \leq 0 \quad \forall j, \]
\[ v_i(x_{ij}^m - (1 - \gamma)x_{ij}^s) \leq x_{ij} \leq v_i(x_{ij}^m + (1 - \gamma)x_{ij}^s) \quad \forall i, j, \]
\[ u_r(y_{rj}^m - (1 - \gamma)y_{rj}^s) \leq y_{rj} \leq u_r(y_{rj}^m + (1 - \gamma)y_{rj}^s) \quad \forall r, j, \]
\[ \phi U_r \leq u_r \leq (1 - \phi) U_r \quad \forall r, \]
\[ \phi V_i \leq v_i \leq (1 - \phi) V_i \quad \forall i, \]

where, \( U_r \) (\( r = 1, \ldots, s \)) and \( V_i \) (\( i = 1, \ldots, m \)) are optimal values of (3) for a given \( \gamma \in [0, 1] \).

A CSW is obtained by solving (9) and the efficiency of each DMU can be evaluated as follows:
\[ e_j = \frac{\sum_{r=1}^{s} u_r^s \tilde{y}_{rj}}{\sum_{i=1}^{m} v_i^s \tilde{x}_{ij}} \quad \forall j, \]

where, \( u_r^s \) (\( r = 1, \ldots, s \)) and \( v_i^s \) (\( i = 1, \ldots, m \)) are optimal values of (9).

Applying the division of triangular fuzzy numbers (by approximation) calculates the fuzzy efficiency measures as:
\[ e_j = (e_j^m, e_j^s, e_j^0) \quad \forall j, \]

where
\[ e_j^m = \frac{\sum_{r=1}^{s} u_r^s y_{rj}^m}{\sum_{i=1}^{m} v_i^s x_{ij}^m} \]
\[ e_j^s = \frac{\sum_{r=1}^{s} u_r^s y_{rj}^s \sum_{i=1}^{m} v_i^s x_{ij}^s + \sum_{r=1}^{s} u_r^s y_{rj}^0 \sum_{i=1}^{m} v_i^s x_{ij}^0}{(\sum_{i=1}^{m} v_i^s x_{ij}^m)^2} \]

### Table 1
Data for numerical example

<table>
<thead>
<tr>
<th>DMUs</th>
<th>Inputs</th>
<th>Outputs</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( I_1 )</td>
<td>( I_2 )</td>
</tr>
<tr>
<td>( D01 )</td>
<td>(18, 1.2)</td>
<td>(39, 1.5, 3)</td>
</tr>
<tr>
<td>( D02 )</td>
<td>(11, 2.1)</td>
<td>(94, 1.1)</td>
</tr>
<tr>
<td>( D03 )</td>
<td>(9, 1.5, 1.5)</td>
<td>(45, 2.3)</td>
</tr>
<tr>
<td>( D04 )</td>
<td>(13, 2.3)</td>
<td>(100, 4.1)</td>
</tr>
<tr>
<td>( D05 )</td>
<td>(7, 0.5, 1)</td>
<td>(30, 1.2)</td>
</tr>
<tr>
<td>( D06 )</td>
<td>(10, 1, 1.5)</td>
<td>(45, 3.2, 5)</td>
</tr>
<tr>
<td>( D07 )</td>
<td>(12, 2.2)</td>
<td>(110, 3.3)</td>
</tr>
<tr>
<td>( D08 )</td>
<td>(7, 1.5, 0.5)</td>
<td>(30, 4.1, 5)</td>
</tr>
<tr>
<td>( D09 )</td>
<td>(19, 2.1)</td>
<td>(174, 5.6)</td>
</tr>
<tr>
<td>( D10 )</td>
<td>(10, 3.2)</td>
<td>(185, 3, 1)</td>
</tr>
</tbody>
</table>
Table 2
Efficiencies by fuzzy DEA model

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$D_1$</th>
<th>$D_2$</th>
<th>$D_3$</th>
<th>$D_4$</th>
<th>$D_5$</th>
<th>$D_6$</th>
<th>$D_7$</th>
<th>$D_8$</th>
<th>$D_9$</th>
<th>$D_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.733</td>
<td>0.974</td>
<td>0.798</td>
<td>0.854</td>
<td>1</td>
<td>1</td>
<td>0.805</td>
<td>1</td>
<td>0.474</td>
<td>1</td>
</tr>
<tr>
<td>0.3</td>
<td>0.704</td>
<td>0.885</td>
<td>0.747</td>
<td>0.775</td>
<td>1</td>
<td>0.936</td>
<td>0.752</td>
<td>0.977</td>
<td>0.451</td>
<td>1</td>
</tr>
<tr>
<td>0.5</td>
<td>0.676</td>
<td>0.805</td>
<td>0.699</td>
<td>0.704</td>
<td>1</td>
<td>0.868</td>
<td>0.704</td>
<td>0.902</td>
<td>0.428</td>
<td>1</td>
</tr>
<tr>
<td>0.7</td>
<td>0.650</td>
<td>0.742</td>
<td>0.655</td>
<td>0.639</td>
<td>1</td>
<td>0.805</td>
<td>0.659</td>
<td>0.833</td>
<td>0.407</td>
<td>0.977</td>
</tr>
<tr>
<td>0.9</td>
<td>0.624</td>
<td>0.692</td>
<td>0.614</td>
<td>0.580</td>
<td>1</td>
<td>0.746</td>
<td>0.617</td>
<td>0.771</td>
<td>0.387</td>
<td>0.873</td>
</tr>
<tr>
<td>1</td>
<td>0.611</td>
<td>0.669</td>
<td>0.595</td>
<td>0.552</td>
<td>1</td>
<td>0.718</td>
<td>0.597</td>
<td>0.742</td>
<td>0.377</td>
<td>0.826</td>
</tr>
</tbody>
</table>

Table 3
The weights by fuzzy DEA model

<table>
<thead>
<tr>
<th>DMUs</th>
<th>$\gamma = 0.3$</th>
<th>$\gamma = 0.5$</th>
<th>$\gamma = 0.7$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$v_1$</td>
<td>$v_2$</td>
<td>$u_1$</td>
</tr>
<tr>
<td>$D_1$</td>
<td>0</td>
<td>0.026</td>
<td>0.022</td>
</tr>
<tr>
<td>$D_2$</td>
<td>0.104</td>
<td>0</td>
<td>0.022</td>
</tr>
<tr>
<td>$D_3$</td>
<td>0.126</td>
<td>0</td>
<td>0.002</td>
</tr>
<tr>
<td>$D_4$</td>
<td>0.086</td>
<td>0</td>
<td>0.018</td>
</tr>
<tr>
<td>$D_5$</td>
<td>0.130</td>
<td>0</td>
<td>0.027</td>
</tr>
<tr>
<td>$D_6$</td>
<td>0.108</td>
<td>0</td>
<td>0.022</td>
</tr>
<tr>
<td>$D_7$</td>
<td>0.094</td>
<td>0</td>
<td>0.002</td>
</tr>
<tr>
<td>$D_8$</td>
<td>0.168</td>
<td>0</td>
<td>0.003</td>
</tr>
<tr>
<td>$D_9$</td>
<td>0.057</td>
<td>0</td>
<td>0.001</td>
</tr>
<tr>
<td>$D_{10}$</td>
<td>0.127</td>
<td>0</td>
<td>0.007</td>
</tr>
</tbody>
</table>
\[ e_j^\beta = \frac{\sum_{r=1}^{s} u_r^s y^m_{ij} \sum_{i=1}^{m} v_i^s x^2_{ij} + \sum_{r=1}^{s} u_r^s y^\beta_{ij} \sum_{i=1}^{m} v_i^s x^m_{ij}}{\left(\sum_{i=1}^{m} v_i^s x^m_{ij}\right)^2}. \]

In cases where \( e^m_p + e^\beta_p > 1 \), at least for one \( p, 1 \leq p \leq n \), or for all DMUs \( e^m_p + e^\beta_p < 1 \), to scale the efficiencies in \((0,1]\), all output and/or input weights can be changed by minimal proportion. For this, among alternatives, one option is the following substitutions:

\[
M_r = \frac{u_r^s}{e}, \quad N_i = v_i^s \quad \forall r, i,
\]

where,

\[
e = \max_{1 \leq j \leq n} \left\{ e^m_j + e^\beta_j \right\}.
\]

The resulted weights \( M_r \) \((r = 1, \ldots, s)\) and \( N_i \) \((i = 1, \ldots, m)\) are the proposed CSW.

After determining the CSW, the fuzzy efficiencies of DMUs can be calculated. To rank these fuzzy scores one of the ranking methods \([2,3,7,10,11]\) for fuzzy numbers can be applied.

### 4. A numerical example

In this example, the variation between the weights of each factor (input, output) is minimized as much as possible using the method suggested in Section 3.

Consider 10 DMUs as Table 1, each DMU consuming two fuzzy inputs to produce two fuzzy outputs.

Running (1) by different \( \gamma \)-cuts for each DMU resulted in efficiencies and a weight matrix as shown in Tables 2 and 3, respectively.

In Table 3, as seen, there are large differences in weights accorded to the same factor, with 19–20 out of the 40 weights being virtually zero.

#### Table 4
The CSW by proposed model

<table>
<thead>
<tr>
<th>( \gamma )</th>
<th>( Iu_1 )</th>
<th>( Iu_2 )</th>
<th>( Ou_1 )</th>
<th>( Ou_2 )</th>
<th>( v_1 )</th>
<th>( v_2 )</th>
<th>( u_1 )</th>
<th>( u_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>0.0581</td>
<td>0.0055</td>
<td>0.0125</td>
<td>0.0012</td>
<td>0.0344</td>
<td>0.0033</td>
<td>0.0040</td>
<td>0.0004</td>
</tr>
<tr>
<td>0.3</td>
<td>0.0568</td>
<td>0.0055</td>
<td>0.0117</td>
<td>0.0011</td>
<td>0.0335</td>
<td>0.0032</td>
<td>0.0048</td>
<td>0.0005</td>
</tr>
<tr>
<td>0.5</td>
<td>0.0556</td>
<td>0.0055</td>
<td>0.0110</td>
<td>0.0011</td>
<td>0.0326</td>
<td>0.0032</td>
<td>0.0043</td>
<td>0.0004</td>
</tr>
<tr>
<td>0.7</td>
<td>0.0543</td>
<td>0.0054</td>
<td>0.0104</td>
<td>0.0010</td>
<td>0.0318</td>
<td>0.0032</td>
<td>0.0037</td>
<td>0.0004</td>
</tr>
<tr>
<td>0.9</td>
<td>0.0532</td>
<td>0.0054</td>
<td>0.0097</td>
<td>0.0010</td>
<td>0.0310</td>
<td>0.0031</td>
<td>0.0036</td>
<td>0.0004</td>
</tr>
<tr>
<td>1.0</td>
<td>0.0526</td>
<td>0.0054</td>
<td>0.0095</td>
<td>0.0009</td>
<td>0.0306</td>
<td>0.0031</td>
<td>0.0035</td>
<td>0.0003</td>
</tr>
</tbody>
</table>
Table 5
The corresponding efficiencies by CSW

<table>
<thead>
<tr>
<th>DMU</th>
<th>$\gamma$</th>
<th>0.1</th>
<th>0.3</th>
<th>0.5</th>
<th>0.7</th>
<th>0.9</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$D1$</td>
<td>$0.29,0.05,0.02$</td>
<td>$0.29,0.05,0.02$</td>
<td>$0.29,0.05,0.02$</td>
<td>$0.29,0.05,0.02$</td>
<td>$0.30,0.05,0.02$</td>
<td>$0.30,0.05,0.02$</td>
<td></td>
</tr>
<tr>
<td>$D2$</td>
<td>$0.45,0.04,0.07$</td>
<td>$0.45,0.04,0.07$</td>
<td>$0.45,0.04,0.07$</td>
<td>$0.45,0.04,0.07$</td>
<td>$0.45,0.04,0.07$</td>
<td>$0.45,0.04,0.07$</td>
<td></td>
</tr>
<tr>
<td>$D3$</td>
<td>$0.47,0.09,0.09$</td>
<td>$0.47,0.09,0.09$</td>
<td>$0.47,0.09,0.09$</td>
<td>$0.47,0.09,0.09$</td>
<td>$0.47,0.09,0.09$</td>
<td>$0.47,0.09,0.09$</td>
<td></td>
</tr>
<tr>
<td>$D4$</td>
<td>$0.37,0.07,0.07$</td>
<td>$0.37,0.07,0.07$</td>
<td>$0.37,0.07,0.07$</td>
<td>$0.37,0.07,0.07$</td>
<td>$0.36,0.07,0.06$</td>
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<tr>
<td>$D5$</td>
<td>$0.89,0.14,0.11$</td>
<td>$0.89,0.14,0.11$</td>
<td>$0.90,0.14,0.10$</td>
<td>$0.90,0.14,0.10$</td>
<td>$0.90,0.14,0.10$</td>
<td>$0.90,0.14,0.10$</td>
<td></td>
</tr>
<tr>
<td>$D6$</td>
<td>$0.63,0.10,0.08$</td>
<td>$0.63,0.10,0.08$</td>
<td>$0.63,0.09,0.08$</td>
<td>$0.63,0.09,0.08$</td>
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<td>$0.63,0.09,0.08$</td>
<td></td>
</tr>
<tr>
<td>$D7$</td>
<td>$0.38,0.05,0.05$</td>
<td>$0.38,0.05,0.05$</td>
<td>$0.38,0.05,0.05$</td>
<td>$0.38,0.05,0.05$</td>
<td>$0.38,0.05,0.05$</td>
<td>$0.38,0.05,0.05$</td>
<td></td>
</tr>
<tr>
<td>$D8$</td>
<td>$0.62,0.05,0.17$</td>
<td>$0.62,0.05,0.17$</td>
<td>$0.62,0.05,0.17$</td>
<td>$0.62,0.05,0.17$</td>
<td>$0.62,0.05,0.17$</td>
<td>$0.62,0.05,0.17$</td>
<td></td>
</tr>
<tr>
<td>$D9$</td>
<td>$0.24,0.02,0.02$</td>
<td>$0.24,0.02,0.02$</td>
<td>$0.24,0.02,0.02$</td>
<td>$0.24,0.02,0.02$</td>
<td>$0.24,0.02,0.02$</td>
<td>$0.24,0.02,0.02$</td>
<td></td>
</tr>
<tr>
<td>$D10$</td>
<td>$0.34,0.03,0.05$</td>
<td>$0.33,0.03,0.05$</td>
<td>$0.33,0.03,0.05$</td>
<td>$0.33,0.03,0.05$</td>
<td>$0.33,0.03,0.05$</td>
<td>$0.33,0.03,0.05$</td>
<td></td>
</tr>
</tbody>
</table>
The results of proposed method for finding CSW by different $\gamma$-cuts are presented in Table 4.

As seen, the wide variation of weights between DMUs in Table 3 is limited by suggested method. For instance, $v_1$, the weight of the first input in Table 3 is varied in $[0, 0.168]$ by $\gamma = 0.3$ and the achieved weight by proposed method when $\gamma = 0.3$ is 0.0335, which is unique for all DMUs. The efficiencies corresponding to these weights are shown in Table 5.

5. Conclusion

One of the prominent features of standard and fuzzy DEA is the representation of each of the participating DMUs in the best possible light, relative to the others. To this end, factor weights are allowed to vary freely within the general constraints in each run of the model. Weight flexibility in DEA assessments is such that it can lead to some DMUs having all but their most favorable input(s) and output(s) ignored in their assessment. To overcome this problem, the paper suggests a method to find a CSW.

References