On the Number and the Distribution of the Nash Equilibria in SuperModular Games and their Impact on the Tipping Set *

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Abstract

In this paper we analyze a class of \( n \)-person supermodular games that arise in the context of interdependent security analysis. More specifically, we quantify the number and the distribution of Nash equilibria in pure strategies and their impact on the tipping set.

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1 Introduction

Interdependent security analysis has been successfully modeled using the framework of the $n$-person (non-cooperative) game theory [Owen (1995)]. In a series of seminal papers Kunreuther and Heal [2003] and Heal and Kunreuther [(2005), (2006), and (2007)] describe these models and their applications to airline security, vaccination problem to control the spread of an infectious disease, etc. There are a number of features that distinguish this class of models from the general game models. First, each player is endowed with only two pure strategies or actions – invest (denoted by 1) and not invest (denoted by 0) in security measures. Further, players are not allowed to use randomized strategies. Consequently, we are only interested in the analysis of Nash equilibria in pure strategies. Secondly, the utility function (which is the negative of the cost function) for each player has two explicit components – first due to the self action (invest or not-invest) and the second due to the actions (invest or not-invest) of the rest of the players. This second component is also known as the externality component. Thirdly, a subgroup of players, by clever choice of their own actions, can affect the externalities of players not in their subgroup in such a way as to force them to choose a pre-specified action. This phenomenon whereby a group can enforce a certain type of behavior by other is commonly known as tipping [Schilling (1978), Gladwill (2000)]. While tipping refers to a sudden change in the equilibrium of a game, such a change may also be brought about rather slowly in a step by step fashion - a phenomenon called cascading [Dixit (2002)].

Mathematically, the phenomenon of tipping and cascading are captured by requiring that the utility functions of each player satisfy strictly increasing property. It turns out that this latter property is closely related to the property of supermodularity of utility functions.

An appropriate mathematical framework for defining supermodular functions and functions with increasing differences rests on the theory of lattices. This theory has been very well developed by Topkis [(1978) and
(1979)] and his book [Topkis (1998)] provides a very readable summary of results in this area. For related work, refer to Milgrom and Roberts [1990], Vives [(1990) and (2005)], and Zhou [1994]. For purposes of easy reference, Appendix A contains all the basic definitions and concepts needed for our development.

Heal and Kunreuther [(2006) and (2007)] have analyzed the tipping process in the context of interdependent security games. In particular, they characterize the properties of the tipping set when there are exactly two Nash equilibria – all investing or no one investing. It turns out that the number and the distribution of the Nash equilibria affect the nature of the tipping set. In this paper, our goal is to characterize the number and distribution of Nash equilibria in supermodular games and analyze their impact on the tipping set.

In Section 2 we formally define the game model. Our main results are contained in Section 3. Section 4 covers some discussion of tipping. Appendix A provides an overview of concepts and definitions from the theory of lattices and Appendix B contains a derivation of the cost functions that arise in the context of airline security games.

## 2 The Game Model

There are \( n \) players each endowed with two pure actions or strategies denoted by 1 and 0. In interdependent security games, 1 denotes the action to **invest** and 0 denotes the action **not to invest**. A play is defined by the \( n \)-tuple \( \mathbf{a} = (a_1, a_2, \cdots, a_n) \) where \( a_i \in \{0, 1\} \) denotes the pure strategy chosen by player \( i \), for \( 1 \leq i \leq n \).

Thus, there is a total of \( 2^n \) distinct plays denoted by the set

\[
S = \{ \mathbf{a} | \mathbf{a} = (a_1, a_2, \cdots, a_n), a_i \in \{0, 1\}, 1 \leq i \leq n \} \tag{2.1}
\]

Clearly, the elements of \( S \) denote the \( 2^n \) distinct corners of the \( n \)-dimensional unit cube.

Let \( \mathbf{a}, \mathbf{b} \in S \). Define a binary relation \( \leq \) on \( S \) as follows. We say \( \mathbf{a} \leq \mathbf{b} \) exactly when \( a_i \leq b_i \) for \( 1 \leq i \leq n \). It
can be verified that \((S, \leq)\) is a poset and indeed a complete lattice [Topkins (1998)]. Example of the lattices for \(n = 3\) and \(n = 4\) are given in Figure 2.1 and Figure 2.2, respectively. In this lattice, we will refer the plays with exactly \(k\) 1’s, \(0 \leq k \leq n\) (that is the plays in which exactly \(k\) players invest and the \(n - k\) do not invest) as the level-\(k\) plays.

Let \(U : S \to R^n\), where \(U(a) = (u_1(a), u_2(a), \cdots, u_n(a))\) denote the \(n\)-tuple of values of the utility functions where \(u_i : S \to R\) denotes the utility of the player \(i\), \(1 \leq i \leq n\), for the play \(a\). The game is completely specified by \((S, U)\). An example of the derivation of a specific utility (negative of the cost) function for the airline problem is given in Heal and Kunreuther (2005). It turns out that each \(u_i(a)\) has two components - one due to self-action by player \(i\), and the other due to the actions by other players \(j \neq i\).

Let \(a = (a_1, a_2, \cdots, a_i, \cdots, a_n) \in S\). Then, let \(a/i = (a_1, a_2, \cdots, \overline{a_i}, \cdots, a_n)\). Given the game \((S, U)\), the
pair \((a, U(a))\) is called a Nash equilibrium (NE) \(^{1}\) for the game, if for all \(1 \leq i \leq n\)

\[
u_i(a/i) < u_i(a)
\]  

We are particularly interested in a special class of games called supermodular games. Refer to Topkins (1998) for the definitions. A game \((S, U)\) is called supermodular if the utility function \(U\) defined on \(S\) satisfies the following condition. Let \(a, b \in S\), and \(a \lor b\) and \(a \land b\) denote the bitwise join and the meet of \(a\) and \(b\). The function \(u_i : S \rightarrow \mathbb{R}\) is said to be strictly supermodular when, for each \(1 \leq i \leq n\),

\[
u_i(a) + u_i(b) < u_i(a \lor b) + u_i(a \land b)
\]  

Let \(a_{-i}\) denote the strategy \(a = (a_1, a_2, \cdots, a_{i-1}, *, a_{i+1}, \cdots, a_n)\), and \(a = (1_i, a_{-i}) = (a_1, a_2, \cdots, a_{i-1}, 1, a_{i+1}, \cdots, a_n)\) and \(a = (0_i, a_{-i}) = (a_1, a_2, \cdots, a_{i-1}, 0, a_{i+1}, \cdots, a_n)\). Consider a pair \(a_{-i}\) and \(b_{-i}\) such that \(a_{-i} \leq b_{-i}\).

\(^{1}\)In general, the Nash equilibrium is defined by \(u_i(a) \leq u_i(a/i)\) for all \(1 \leq i \leq n\). Since we are interested in characterizing the number of distinct equilibria, we require strict inequality in (2.2)
Then, if \( u_i \) is supermodular, by (2.3), we get
\[
 u_i((1_i, a_i) + u_i(0_i, b_i) < u_i(1_i, b_i) + u_i(0_i, a_i) \tag{2.4}
\]
Rearranging, we get
\[
 u_i(1_i, a_i) - u_i(0_i, a_i) < u_i(1_i, b_i) - u_i(0_i, b_i) \tag{2.5}
\]
Any utility function satisfying (2.5) is called a function of (strictly) increasing differences [Topkins(1998)].

Since (2.4) and (2.5) are equivalent, the reader can see that there is a strong relation between supermodular functions and functions with increasing differences.

In this paper our goal is to analyze the number and distributions of the NE in supermodular games. This calls for the conditions under which the two systems of inequalities (2.2) and (2.5) co-exist.

### 3 Distribution of Nash equilibria

In this section we analyze the distribution of the Nash equilibria in a supermodular game. It is well known that the set of all Nash equilibria form a complete lattice (Tokis[1978], Vives[1990], and Zhou[1994]). However, the number of such equilibria is not known. It turns out that the presence of multiple (more than two) equilibria affects the tipping set.

It is assumed that \( 0^n \) and \( 1^n \) are always two Nash equilibria. As \( n \) increases the number of possible equilibria also increases. Our goal is to uncover the pattern of occurrences of these equilibria. In the following, two plays will be called neighbors of each other if they differ only in one component.

**Lemma 3.1.** In an \( n \)-person supermodular game, if \( 0^n \) and \( 1^n \) are NE, then no node of the form \( (1_i, 0^{n-1}_i) \) and \( (0_i, 1^{n-1}_i) \) can be a NE.
Proof: Since node of type \((1, 0^{n-1})\) are neighbors of nodes of type \(0^n\), and since neighbor of a NE node cannot be a NE, none of these nodes can be a NE. Similarly, for nodes of type \((0, 1^{n-1})\) which are neighbors of node \(1^n\).

**Lemma 3.2.** In an \(n\)-person supermodular game, \(n \geq 2\), if \((0, 1^{n-1})\) is a NE, then no node of the type \((0, 1^{n-1}), i \neq j\), can be NE.

Proof: Assume \((1, 0_j, 1^{n-2})\) is a NE. From Figure 3.1, because of supermodularity,

\[
u_i(1^n) - u_i(0, 1^{n-1}) > u_i(1, 0_j, 1^{n-2}) - u_i(0, 0_j, 1^{n-2})
\]

This is, however, not possible, since left hand side of this inequality is negative, because \((0, 1^{n-1})\) is given to be NE, and the right hand side of this inequality is positive because \((1, 0_j, 1^{n-2})\) is assumed to be NE.

**Corollary 3.3.** In a two-person supermodular game \(01\) and \(10\) cannot simultaneously be NE.

**Corollary 3.4.** In a two-person supermodular game, the only two NE that can exist simultaneously are when both invest \((11)\) or no one invests \((00)\).

**Lemma 3.5.** In an \(n\)-person supermodular game, \(n \geq 3\), if \((0, 1^{n-1})\) is a NE, no node of the type \((0, 0_j, 1^{n-2})\) can be a NE.
Figure 3.2.

Proof: If \( j = i \) or \( k = i \), then \((0,0_k,1^{n-2})\) is a neighbor of \((0,1^{n-1})\) and hence cannot be a NE. So, assume, \( j \neq k \neq i \) and \((1,0_j,0_k,1^{n-3})\) is a NE. From Figure 3.2, by supermodularity,

\[
\begin{align*}
&u_i(1^n) - u_i(0,1^{n-1}) > u_i(1,0_j,1^{n-2}) - u_i(0,0_j,1^{n-2}) \\
&> u_i(1,0_j,0_k,1^{n-3}) - u_i(0,0_j,0_k,1^{n-3})
\end{align*}
\]

Since \(0,1^{n-1}\) is a NE, the first term on the left in this inequality is negative, and, since \(1,0_j,0_k,1^{n-3}\) is a NE, the last term in the inequality is positive. Thus, this inequality cannot hold.

**Corollary 3.6.** In a 3-person supermodular game, there cannot be more than 2 NE simultaneously. The simultaneous NE can occur when every one is investing \((111)\) and when no one is investing \((000)\).

**Lemma 3.7.** In an \(n\)-person supermodular game, let the play \((0_i,x)\) be a NE. In the lattice of \(2^n\) plays, let \((1_i,x)\) be the parent of the node \((0_i,x)\). Let "\(x\)" contain \(k \leq n-1\) 1's. Then none of the nodes that can be obtained from the parent \((1_i,x)\) by replacing 1, 2, \ldots, or \(k\) 1's by 0's can be a NE.

**Proof:** Consider a node obtained from \((1_i,x)\) by replacing \(j \leq k\) 1's in \((1_i,x)\). Let \(x_r\) denote the substring
of string $x$ obtained from $x$ by deleting the element 1 at position $r$. Because of supermodularity, we should have

$$u_i(1, x) - u_i(0, x) > u_i(1, 0_j, x_j) - u_i(0, 0_j, x_j)$$

$$> u_i(1, 0_j, 0_j, x_j) - u_i(0, 0_j, 0_j, x_j)$$

$$> u_i(1, 0_j, 0_j \cdots 0_j, x_{j1}x_{j2} \cdots x_{jk}) - u_i(0, 0_j, 0_j \cdots 0_j, x_{j1}x_{j2} \cdots x_{jk})$$

If $(0, x)$ is a NE, then the first term is negative. However, if any one of $(1, 0_j, x_j)$, $(1, 0_j, 0_j, x_{j1}x_{j2})$, $\cdots$ $u_i(1, 0_j, 0_j \cdots 0_j, x_{j1}x_{j2} \cdots x_{jk})$ is also a NE, then the corresponding term must be positive. This will violate the supermodularity condition.

Lemma 3.8. In a four-person supermodular game, a maximum of four NE can coexist simultaneously.

Proof: Assume $(1111)$ is a NE. Then since $(0111)$, $(1011)$, $(1101)$, and $(1110)$ are all neighbors of $(1111)$, none of them can be a NE. Now, assume $(1100)$ is also a NE, then by the previous lemma none of $(0110), (1010), (0011), (1100)$ can be a NE. Also, $(1000)$, being a neighbor of $(1100)$, cannot be a NE. The only other possibilities for NE that are left are $(0011), (0001), (0000)$. If $(0001)$ is a NE, then neither $(0011)$ nor $(0000)$, being neighbor of $(0001)$, can be a NE. This choice will give a maximum choice of three NE. However, both $(0011)$ and $(0000)$ can coexist as NE. This gives a maximum of four simultaneous NE. Note that the choice of $(1100)$ is arbitrary. Any other choice for NE in which two players invest and two players do not invest, will only give a maximum of four simultaneous NE.

If $(1111)$ is not a NE. Let us assume $(0111)$ is a NE. Then, none of $(1011)$, $(1101)$, $(1011)$, $(1010)$, $(1100)$, and $(1000)$ can be a NE because of the previous lemma. Also, none of $(0111), (0101), (0110), and$
(1111) being neighbors of (0111), can be a NE. This leaves (0100), (0010), (0001), and (0000) as the possibilities for a NE, out of which only one can be a NE. Thus, if (1111) is not a NE, at most two NE can exist simultaneously.

Thus, no more than four NE can exist simultaneously. In the following airline example (1111), (1100), (0011), and (0000) are NE.

**Example 3.9.** The following table gives the amount of loss an airline may suffer in case of a mishap, and the amount of investment the airline needs to make to avoid any mishap on its own.

<table>
<thead>
<tr>
<th></th>
<th>c1 = 388.0</th>
<th>L1 = 20000.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>c2</td>
<td>288.0</td>
<td>15000.0</td>
</tr>
<tr>
<td>c3</td>
<td>770.0</td>
<td>40000.0</td>
</tr>
<tr>
<td>c4</td>
<td>576.0</td>
<td>30000.0</td>
</tr>
</tbody>
</table>

The following matrix gives the probability of a mishap in case the appropriate investment is not made. The entry $Q_{ij}$ is the probability that airline $i$ will cause mishap for airline $j$ if it does not make the appropriate investment.

<table>
<thead>
<tr>
<th></th>
<th>0.020</th>
<th>0.050</th>
<th>0.020</th>
<th>0.015</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.040</td>
<td>0.020</td>
<td>0.015</td>
<td>0.020</td>
<td></td>
</tr>
<tr>
<td>0.010</td>
<td>0.015</td>
<td>0.020</td>
<td>0.050</td>
<td></td>
</tr>
<tr>
<td>0.010</td>
<td>0.020</td>
<td>0.040</td>
<td>0.020</td>
<td></td>
</tr>
</tbody>
</table>

The following matrices are the possible utilities (negative of the possible losses) of the four airlines under various scenarios. In each of the tables, row 1 represents the case when airlines 1 and 2 do not make the investment; row 2 represents the case when airline 1 does not invest but airline 2 invests; row 3 represents
the case when airline 1 makes the investment but airline 2 does not invest; and row 4 represents the case when both airlines 1 and 2 invest. Similarly, column 1 represent the case when airline 3 and airline 4 do not invest; column 2 represents the case when airline 3 does not invest, but airline 4 invests; column 3 represents the case where airline three invests, but airline 4 does not; and column 4 represents the case when both airlines 3 and 4 invest. The aim for each airline is to minimize its loss.

<table>
<thead>
<tr>
<th></th>
<th>Airline 1</th>
<th>Airline 2</th>
<th>Airline 3</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>-1558.4384</td>
<td>-1372.1600</td>
<td>-3673.8304</td>
</tr>
<tr>
<td></td>
<td>-790.0400</td>
<td>-596.0000</td>
<td>-3120.6400</td>
</tr>
<tr>
<td></td>
<td>-1570.0800</td>
<td>-1380.0000</td>
<td>-2932.4800</td>
</tr>
<tr>
<td></td>
<td>-786.0000</td>
<td>-588.0000</td>
<td>-2368.0000</td>
</tr>
<tr>
<td></td>
<td>111.0000</td>
<td>1372.1600</td>
<td>800.0000</td>
</tr>
<tr>
<td></td>
<td>400.0000</td>
<td>1380.0000</td>
<td>2370.0000</td>
</tr>
<tr>
<td></td>
<td>388.0000</td>
<td>1188.0000</td>
<td>770.0000</td>
</tr>
<tr>
<td></td>
<td>1184.0000</td>
<td>1035.0000</td>
<td>2158.0000</td>
</tr>
<tr>
<td></td>
<td>400.0000</td>
<td>1038.0000</td>
<td>1570.0000</td>
</tr>
<tr>
<td></td>
<td>388.0000</td>
<td>288.0000</td>
<td>770.0000</td>
</tr>
<tr>
<td></td>
<td>1184.0000</td>
<td>1035.0000</td>
<td>2158.0000</td>
</tr>
</tbody>
</table>
It can be easily seen that in this case (0000), (0011), (1100), and (1111) are all NE.

**Lemma 3.10.** In an $n$-person supermodular game, there can be at most $n/2$ different plays in which two players do not invest and all other players invest that can simultaneously be NE.

**Proof:** Let $n \geq 4$. Let $(0, i_1, 0, 1^{n-2})$ be a NE. Then, by Lemma 3.7, no play of the type $(1, 0, 0, 1^{n-3})$ or $(0, 1, 0, 1^{n-3})$, where $1 \leq k \leq n$, $k \neq i_1, i_2$, can be a NE. There are $2(n-2)$ such plays. A possible NE that can simultaneously exist with the above NE is a play where $(0, i_1, 0, 1^{n-2})$. Again, by Lemma 3.7 no play of the type $(1, 0, 0, 1^{n-3})$ or $(0, 1, 0, 1^{n-3})$, where $1 \leq k \leq n$, $k \neq i_3, i_4$, can be a NE. Again, there are $2(n-2)$ such plays. However, 4 plays – $(1, 0, 0, 1^{n-4})$, $(1, 0, 1, 0, 1^{n-4})$, $(0, 1, 1, 0, 1^{n-4})$, and $(0, 1, 1, 0, 1^{n-4})$ – are common. Thus, we get only $2(n-2) - 4 = 2(n-4)$ additional plays that cannot be NE with the two chosen so far. Proceeding like this, we see that possibly $n/2$ plays satisfying the conditions of this Lemma can simultaneously be NE. Taking all the plays that can and that cannot be NE simultaneously, we have

\[
n/2 + [2(n-2) + 2(n-4) + 2(n-6) + \cdots + 2(n-(n-2))] = n/2 + 2[(n-2) + (n-4) + \cdots + 2] = n/2 + 4[1 + 2 + \cdots + (n/2 - 1)] = n/2 + 4((n/2 - 1)n/2)/2 = n(n-1)/2
\]

plays, accounting for all the plays that fall in the category defined by the Lemma.
Lemma 3.11. Assume that in an \( n \)-person (\( n \) even) supermodular game, \( n/2 \) plays in which all but two players invest are simultaneously NE. Then no play in which all but three players invest can be a NE.

Proof: Without loss of generality, we can assume that simultaneous NE plays are 001111 \cdots 1111, 110011 \cdots 1111, 11110011 \cdots 1111, \cdots, 1111 \cdots 0011, 1111 \cdots 1100. Consider, the play 001111 \cdots 1111, in which all but the first two players invest. According to Lemma 3.7, the plays obtained from this play by replacing one of the zeros with 1, and replacing two of the one’s with zeros, cannot be a NE. In any such play, all but three players are investing. There are \( 2^{n-2}C_2 \) such plays. By similar arguments, corresponding to the play 110011 \cdots 11, there are also \( 2^{n-2}C_2 \) plays with all but three players not investing that cannot be NE. However, any play of the type 0101x, or 0110x, or 1001x, or 1010x are plays that are common with the plays obtained from the play 0011 \cdots 1111. Here \( x \) is a string of length \( n-4 \) with one zero and \( n-5 \) one’s. There are \( 4(n-4) \) such plays. Thus, we obtain \( 2^{n-2}C_2 - 4(n-4) \) new plays with all but three players investing that cannot be NE. Next, corresponding to the play 11110011 \cdots, there are \( 2^{n-2}C_2 - 4(n-4) - 4(n-6) \) new plays with all but three players investing that cannot be NE. This is so, because there are \( 4(n-4) \) plays of the type 0101x, 0110x, 1001x, and 1010x common with the first group of plays. Plays of the type 110101x, 110110x, 111001x, and 111010x, in which \( x \) is a string of length \( n-6 \) with one zero and the rest one’s, are common with the second group. There are \( 4(n-6) \) plays of this type. Using similar arguments, \( 2^{n-2}C_2 + [2^{n-2}C_2 - 4(n-4)] + [2^{n-2}C_2 - 4(n-4) - 4(n-6)] + \cdots + [2^{n-2}C_2 - 4(n-4) - 4(n-6) - \cdots - 4(n-(n-2))] + [2^{n-2}C_2 - 4(n-4) - 4(n-6) - \cdots - 4(n-(n-2))] \) plays with all
but three investing cannot be NE. All these are equal to

\[ n^{n-2}C_2 - 4\left(\frac{n}{2} - 1\right)(n - 4) - 4\left(\frac{n}{2} - 2\right)(n - 6) - \cdots - 4.2 - 4.2 \]

\[ = n^{n-2}C_2 - 4\left[\left(\frac{n}{2} - 1\right)(n - 4) + \left(\frac{n}{2} - 2\right)(n - 6) - \cdots - 3.4 - 2.2\right] \]

\[ = n^{n-2}C_2 - 8\left[\left(\frac{n}{2} - 1\right)\left(\frac{n}{2} - 2\right) + \left(\frac{n}{2} - 2\right)\left(\frac{n}{2} - 3\right) - \cdots - 3.2 - 2.1\right] \]

\[ = \frac{n(n - 2)(n - 3)}{2} - 8\left[\frac{n^3}{24} - \frac{n^2}{4} + \frac{n}{3}\right] \]

\[ = \frac{n^3 - 5n^2 + 6n}{2} - \frac{n^3}{3} + 2n^2 - 8\frac{n}{3} \]

\[ = \frac{n^3 - 3n^2 + 2n}{6} = n^{-3}C_3 \]

which is the number of all the plays in which all but three players invest. This proves the Lemma.

The following is the main result of this Section.

**Theorem 3.12.** In an \(n\)-person supermodular game, no more than \(2^{n/2}\) plays can be NE simultaneously.

**Proof:** Here, we prove the result for \(n\) even. The case for odd \(n\) can be handled similarly.

We have shown that when \(n = 2\) and \(n = 4\), number of plays that can be NE simultaneously are 2 and 4, respectively. Thus, the theorem holds when, \(n = 2\) and \(n = 4\). Assume, theorem holds for some even \(n > 4\).

Consider the case of a \(n + 2\)-person supermodular game. As proved in Lemma ??, there are at most \(n/2 + 1\) plays in which all but two players invest that can be NE simultaneously. In each of such plays, the players who do not invest are different. Consider the lattice of all the \(2^{n+2}\) plays. Without loss of generality, let one of the NE play be \(0011\cdots11\). Consider the set of all the plays in which the same two players do not invest. This set of plays is a sublattice of the original lattice with \(2^n\) plays. This has at most \(2^{n/2}\) NE. Next consider the NE play \(1100\cdots11\) in which all but the two players invest. The sublattice of plays in which the same two players do not invest also has at most \(2^{n/2}\) simultaneous NE plays. The two sublattices will
have $2^{n/2-1}$ are common to both the sublattices. The common plays will be the one’s in which the same four players do not invest. Thus, these two sublattices have at most $2^{n/2} + 2^{n/2-1}$ NE plays. Next consider the NE play 11110011\ldots11. The sublattice of all the plays in which the same two players do not invest. This sublattice also has at most $2^{n/2}$ plays that can be NE. Out of these $2^{n/2-1}$ are common with the NE plays of the first sublattice and $2^{n/2-2}$ additional plays that are common with the NE plays of the second sublattice. Thus, from this third sublattice we get at most $2^{n/2-2}$ additional plays. Proceeding this way, we get at most

$$2^{n/2} + 2^{n/2-1} + 2^{n/2-2} + \ldots + 2 + 1 = 2^{n/2+1} - 1$$

plays that can be NE simultaneously. These plays do not include the play in which all players invest. Including this play in the set of plays that can be NE simultaneously proves the theorem.

4 Tipping set

The tipping set is a subgroup of players, who, by a clever choice of their own action, can affect the externalities of players outside of their subgroup in such a way as to force them to choose a pre-specified action. In Heal and Kunreuther [2006], it is proved that in a supermodular $n$-person game in which $0^n$ and $1^n$ are two NE plays, a tipping set from equilibrium $0^n$ to $1^n$ can be found by looking at the chain of inequalities

$$u_i(0^{n-1}, i) - u_i(0^{n-1}, 0_i) \ < \ u_i(0^{n-2}, 1_i, 1_i) - u_i(0^{n-2}, 1_i, 0_i)$$

$$\ < \ u_i(0^{n-3}, 1_i, 1_i, 1_i) - u_i(0^{n-3}, 1_i, 1_i, 0_i)$$

$$\ < \ \ldots$$

$$\ < \ u_i(1_i, 1_i, \ldots, 1_i) - u_i(1_i, 1_i, \ldots, 1_i, 0_i)$$

In this chain, the first term is negative and the last term is positive since both $0^n$ and $1^n$ are NE. If
\[ u_i(0^{n-k-1}, 1_1, 1_2, \cdots, 1_k, 1_i) \] is the first positive term, then \( \{1, 2, \cdots, k \} \) is the tipping set. Note that if we are interested in finding the smallest tipping set then we have to look at all \( n! \) such chains.

Now consider a supermodular game that contains other NE in addition to these two. Note that each term in (4.1) is a difference of two numbers. These numbers are the utilities of player \( i \) for different plays. If there are more than two NE, one or more of the terms in (4.1) may contain utilities of player \( i \) at NE plays. If the first number in a term pertains to the utility of player \( i \) at a NE, that term will be positive and if it appears as a second number, that term will be negative. Now the first positive term in the inequality (4.1) signals a tipping set. If this term happens to be positive because its first number is the utility for a NE play, the tipping set so obtained will not tip the system to \( 1^n \), but rather it will tip it to this NE play. If such a tipping set happens to be the smallest one, it will not be the true desired smallest tipping set.

Finding smallest tipping set from \( 0^n \) to \( 1^n \) in the presence of other NE in a supermodular game needs further analysis. This problem will be dealt with in another paper.
References


APPENDIX A

Lattices and Supermodular Functions

For purposes of easy reference, in this appendix we collect many of the useful concepts and definitions from the theory of lattices and supermodular functions that are need in the main body of this paper. For more details, refer to Topkis[1978], [1979], and [1998]. Also, refer to Vives[1990], [2005], and Zhou [1994]. The following development is adapted from Topkis[1998].

A Partially Ordered Set

Let $\chi$ be a set and let $\preceq$ be a binary relation on $\chi$. By definition, for any two elements $x$ and $y$ in $\chi$, either $x \preceq y$ or $y \preceq x$ (that is $x$ and $y$ are related) or $x \not\preceq y$ ($x$ and $y$ are not related). When $x \preceq y$ and $x \neq y$, we write $x \prec y$. We are particularly interested in binary relations satisfying the following properties. A binary relation ($\preceq$) is said to be

- **reflexive** if $x \preceq x$ for all $x \in \chi$;
- **antisymmetric** if $x \preceq y$ and $y \preceq x$ implies $x = y$; and
- **transitive** if $x \preceq y$ and $y \preceq z$, then $x \preceq z$.

Given the pair $(\chi, \preceq)$, if the binary relation $\preceq$ is reflexive, antisymmetric and transitive on $\chi$ then $(\chi, \preceq)$ is called a **partially ordered set** or simply as **poset**. Given a pair of elements $x$ and $y$ in a poset $(\chi, \preceq)$, if either $x \preceq y$ or $y \preceq x$, then $x$ and $y$ are said to be **ordered**; otherwise, $x$ and $y$ are **unordered**. If every pair of elements in a poset is ordered, then it is called a **chain**. Posets come in various shapes and forms and here are some examples.
Example A.1. Let $\mathbb{R}$ denote the set of all real numbers and $\mathbb{Z}$ denote the set of all integers, where $\mathbb{Z} \subseteq \mathbb{R}$. Let $\preceq$ denote the usual "less than or equal to" relation on numbers. Then, $(\mathbb{R}, \preceq)$ and $(\mathbb{Z}, \preceq)$ are posets. Similarly, let $\mathbb{Z}_2 = \{0, 1\}$ and $\preceq$ is such that $0 \preceq 0 \preceq 1$ and $1 \preceq 1$. Thus, $(\mathbb{Z}_2, \preceq)$ is also a poset. Further, since for any two numbers $a$ and $b$ either $a \preceq b$ or $b \preceq a$, it immediately follows that $(\mathbb{R}, \preceq)$, $(\mathbb{Z}, \preceq)$ and $(\mathbb{Z}_2, \preceq)$ are all chains.

Example A.2. Let $(\chi_i, \preceq_i)$ be posets for $1 \leq i \leq n$. Let $\chi = \prod_{i=1}^{n} \chi_i$ be the cartesian produce of $\chi_i$, $1 \leq i \leq n$. Define a binary relation $\preceq$ on $\chi$ induced by the $n$ individual binary relations $\chi_i$ on $\chi_i$, $1 \leq i \leq n$ as follows. If $a = (a_1, a_2, \cdots, a_n)$ and $b = (b_1, b_2, \cdots, b_n)$ are two elements in $\chi$ with $a_i, b_i \in \chi_i$, for $1 \leq i \leq n$, then define

$$a \preceq b \text{ exactly when } a_i \preceq b_i, 1 \leq i \leq n. \quad (A.1)$$

It can be verified that $(\chi, \preceq)$ is a poset.

Example A.3. Let $\chi_i \equiv \mathbb{R}$ and identify $\preceq_i$ with the usual "less than or equal to" relation $\leq$ on $\mathbb{R}$. It follows that $(\mathbb{R}^n, \preceq)$ is a poset consisting of all $n$ real vectors. Thus, for $n = 2$, we immediately have $a = (1, 2) \preceq (2, 2) = b$, but $b$ is not related to $c = (1, 3)$.

Example A.4. Let $\mathbb{Z}_2^n$ denote the set of all binary strings of length $n$. Let $a = a_na_{n-1} \cdots a_2a_1$ and $b = b_nb_{n-1} \cdots b_2b_1$ be two elements of $\mathbb{Z}_2^n$. Then,

$$a \preceq b \text{ exactly when } a_i \preceq b_i \text{ for } 1 \leq i \leq n. \quad (A.2)$$

Thus, $(\mathbb{S}_2^n, \preceq)$ is a poset.

Any (finite) poset can be graphically represented. Refer to Figure A1 for a representation of $(\mathbb{Z}_2^3, \preceq)$.

Let $(\chi, \preceq)$ be a poset and $A \subseteq \chi$. If $x \in \chi$ is such that $a \preceq x (x \preceq a)$ for all $a \in A$, then $x$ is called an upper (lower) bound for $A$. If an element $a^* \in A(a^* \in A)$ is an upper (lower) bound for $A$, then
Figure A1. A graphical representation of posets.
$a^*(a_*)$ is called the greatest (least) element of $A$. If, for $a_1 \in A$, there is no element $a_2 \in A$ such that $a_1 \preceq a_2 (a_2 \preceq a_1)$, then $a_1$ is called a maximal (minimal) element of $A$. The following properties are easily verified.

1. The greatest (least) element is a maximal (minimal) element but not vice versa.

2. While a poset can have at most one greatest (least) element, it can have a number maximal (minimal) elements.

3. Distinct maximal (minimal) elements in a poset are unordered.

4. If a set of upper (lower) bounds of $A \subseteq \chi$ has a least (greatest) element in $\chi$, then this least upper (greatest lower) bound of $A$ is denoted by $\text{lub}_{\chi}(A)$ ($\text{glb}_{\chi}(A)$).

Example A.5. Referring to Figure A1, consider the set $A = \{110, 101, 011, 100, 010, 001\} \subseteq Z_2^3$. The $\text{lub}(A) = 111$ and $\text{glb}(A) = 000$ both of which do not belong to $A$. The elements $\{110, 101, 011\}$ are maximal elements and $\{100, 010, 001\}$ are minimal elements.

B Lattice

Let $(\chi, \preceq)$ be a poset. For any two elements $x$ and $y$ in $\chi$, define two binary operations denoted by $\lor$ (called the join) and by $\land$ (called the meet) as

\[ x \lor y = \text{glb}_{\chi}(x, y) \]
\[ x \land y = \text{lub}_{\chi}(x, y) \]

A poset $(\chi, \subseteq)$ that contains the meet and join of every pair of elements in it is called a lattice.

Example B.1. 1. The poset $(\mathbb{R}, \preceq)$ is a lattice with $x \lor y = \text{max}(x, y)$ and $x \land y = \text{min}(x, y)$. 21
2. The poset $(\mathbb{R}^n, \preceq)$ is a lattice, where

\[ x \lor y = (x_1 \lor y_1, x_2 \lor y_2, \cdots, x_n \lor y_n) \]

and

\[ x \land y = (x_1 \land y_1, x_2 \land y_2, \cdots, x_n \land y_n) \]

3. The poset $(\mathbb{Z}_2^n, \preceq)$ is a lattice where for $1 \leq i \leq n$

\[ x \lor y = c_n c_{n-1} \cdots c_2 c_1 \text{ with } c_i = x_i \lor y_i \]

and

\[ x \land y = c_n c_{n-1} \cdots c_2 c_1 \text{ with } c_i = x_i \land y_i. \]

and $x_i \lor y_i$ is the logical or and $x_i \land y_i$ is the logical and operation. Thus, the poset $(\mathbb{Z}_2^n, \preceq)$ in Figure A1 is a lattice.

4. Every chain is a lattice and direct product of lattices is a lattice.

A lattice $(\chi, \preceq)$ is said to be complete lattice if every subset $A$ of $\chi$ has a lub$_\chi(A)$ and glb$_\chi(A)$ in $\chi$. Thus, $(\mathbb{Z}_2^3, \preceq)$ is a complete lattice. It can be shown that every finite lattice is complete. Let $A \subseteq \chi$, where $(\chi, \preceq)$ is a lattice. Then $(A, \preceq)$ is called a sublattice if the glb and lub for every pair of elements in $A$ are in $A$. A sublattice $A$ of a lattice $\chi$ is said to be closed if the glb and lub of every subset of $A$ is in $A$.

C Functions on a poset

Let $X$ and $Y$ be two posets with their respective binary relations. A function $f : X \rightarrow Y$ is called increasing (decreasing) if $a \preceq b$ in $X$ implies $f(a) \preceq f(b)$ ($f(b) \preceq f(a)$) in $Y$. A function $f$ is called monotone if it is either increasing or decreasing. If $a \prec b$ in $X$ implies $f(a) \prec f(b)$ ($f(b) \prec f(a)$), then $f$ is called strictly
increasing (strictly decreasing). Let $P(X)$ denote the power set of the set $X$. A function $f : X \to P(X)$ is called a correspondence from $X$ to $X$. An element $x \in X$ is called a fixed point of $f$ if $x \in f(x)$. When $f$ is a function $f : X \to X$, then $x \in X$ is a fixed point of $f$ if $x = f(x)$. A correspondence is called ascending if for any $x \preceq y$ and $s \in f(x)$ and $t \in f(y)$, $s \lor t \in f(y)$ and $s \land y \in f(y)$. The following result is proved in Zhou [1994].

**Theorem C.1.** Let $(\chi, \preceq)$ be a complete lattice and $f$ be a correspondence from $\chi \to \chi$. Let $E \subseteq \chi$ be the set of fixed points of $f$. If $f(x)$ is a nonempty closed sublattice of $\chi$ for every $x \in \chi$ and $f$ is ascending in $\chi$, then $E$ is a nonempty complete lattice.

**D Supermodular functions on a lattice**

Let $X$ and $Y$ be two partially ordered sets and $f : X \times Y \to \mathbb{R}$ be a real valued function on $X \times Y$. If $f(x_2, y) - f(x_1, y)$ is increasing (decreasing) in $x \in X$ for all $y_1 \preceq y_2$ in $Y$, then $f(x, y)$ is said to have an increasing (decreasing) differences on $X \times Y$. For $x_1 \preceq x_2$ and $y_1 \preceq y_2$, rewriting

$$f(x_1, y_2) - f(x_1, y_1) \preceq f(x_2, y_2) - f(x_2, y_1)$$

as

$$f(x_2, y_1) - f(x_1, y_1) \preceq f(x_2, y_2) - f(x_1, y_2)$$

it follows that the increasing (decreasing) property does not distinguish between the two variables $x$ and $y$ in $f(x, y)$. Let $f : X \to \mathbb{R}$ be a real-valued function on a lattice $X$. For any pair $x_1, x_2 \in X$, if

$$f(x_1) + f(x_2) \preceq f(x_1 \lor x_2) + f(x_1 \land x_2)$$

then, $f$ is called a supermodular function on $X$. If, for any unordered pair $x_1, x_2 \in X$,

$$f(x_1) + f(x_2) < f(x_1 \lor x_2) + f(x_1 \land x_2)$$
then $f$ is called **strictly supermodular function** on $X$. If $f$ is supermodular, then $-f$ is called a **submodular function**. The following theorems give relation between supermodular functions and functions of increasing differences. Refer to Topkis[1998] for details.

**Theorem D.1.** If $X$ is a lattice and $f$ is a (strictly) supermodular function on $X$, then $f$ has (strictly) increasing differences on $X$.

**Theorem D.2.** If $X_i$ is a chain for $1 \leq i \leq n$ and $f(X)$ has (strictly) increasing differences on $X = \prod_{i=1}^{n} X_i$, then $f(X)$ is (strictly) supermodular on $X$. 
APPENDIX B

Derivation of Cost Functions in Interdependent Security Games

A The Interdependent Game Model

There are $n$ players and each player is endowed with two pure strategies or actions – to invest in security measures (denoted by 1) and not to invest (denoted by 0). A play is defined by an $n$-tuple

$$a = (a_1, a_2, \cdots, a_n)$$  (A.1)

where $a_i = 0$ or 1, is the pure strategy chosen by the $i^{th}$ player. Thus, there are a total of $2^n$ distinct plays denoted by the set

$$S = \{a | a = (a_1, a_2, \cdots, a_n), a_i = 0 \text{ or } 1, i = 1 \text{ to } n\}$$  (A.2)

Clearly, the elements of the set $S$ represent the $2^n$ corners of $n$-dimensional binary hypercube.

Let $M : S \rightarrow \mathbb{R}^n$, with

$$M(a) = (M_1(a), M_2(a), \cdots, M_n(a))$$  (A.3)

denote the $n$-tuple of (average) cost/loss associated with the play $a$, where $M_i(a)$ is the cost/loss incurred by player $i$, $1 \leq i \leq n$. The game is completely specified by the set $S$ and the mapping

The cost/loss function $M_i(a)$ has two components – one resulting from self action and the second due to the interdependency of players $j \neq i$.

Average Cost/Loss due to self-action

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Average loss/cost to player $a_i$ due to self-action

\[ a_i = a_i c_i + (1 - a_i) p_i L_i \]

Probability that a catastrophic event affecting player $i$ does not happen is $(1 - p_i)$

Probability that a catastrophic event affecting player $i$ happens is $p_i$

Cost incurred due to investment = $c_i$

Resulting loss $L_i(>> c_i)$

Resulting loss = 0

Prob[Player $i$ does not face a catastrophic event due to self-action] = $1 - (1 - a_i)p_i$

Figure A.1: Effect of self-action on Player $i$
Player $j$

Invest

$\begin{align*}
a_j &= 1 \\
\text{Probability that player } i(\neq j) \text{ will not face a catastrophic event is 1} \\
\text{Resulting loss to player } i &= 0
\end{align*}$

Not invest

$\begin{align*}
a_j &= 0 \\
\text{Probability that player } i(\neq j) \text{ will be facing a catastrophic event is } q_{ji} \\
\text{Resulting loss to player } i &= L_i \\
\text{Probability that player } i(\neq j) \text{ will not be facing a catastrophic event is } (1 - q_{ji}) \\
\text{Resulting loss to player } i &= 0
\end{align*}$

$\text{Prob[Player } j\text{'s action does not cause a catastrophic event on player } i\text{]} = 1 - (1 - a_j)q_{ji}$

Figure A.2: Effect of interdependency of player $j$ on Player $i(\neq j)$
Referring to Figure A.3: Representation of the 2-person interdependency game matrix

<table>
<thead>
<tr>
<th>0</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, (p_1 L_1 + (1 - p_1) q_21 L_1, p_2 L_2 + (1 - p_2) q_12 L_2), (c_1, c_2))</td>
<td>(p_1 L_1, c_2 + q_12 L_2)</td>
</tr>
<tr>
<td>(c_1 + q_21 L_1, p_2 L_2)</td>
<td>(c_1, c_2)</td>
</tr>
</tbody>
</table>

Each entry in the matrix represents (loss of 1, loss of 2)

Figure A.3: Representation of the 2-person interdependency game matrix

Referring to Figure A, player \( i \) has two options – to invest \( (a_i = 1) \) and not to invest \( (a_i = 0) \). The decision to invest is associated with a cost \( c_i \) and it is assumed that with this investment player \( i \) can eradicate the possibility of occurrence of any catastrophic event that could potentially ruin his existence. On the other hand, if player \( i \) decides not to invest, then with probability \( p_i \), he will experience the occurrence of the catastrophic event resulting in a loss \( L_i \), where it is usually the case that \( L_i >> c_i \).

\[
\text{Prob} \begin{bmatrix} 
\text{Player } i \text{ does not face a catastrophic event due to self-action } a_i 
\end{bmatrix} = 1 - (1 - a_i)p_i 
\quad (A.4)
\]

and the average cost/loss to player \( i \) due to self-action \( a_i \) is given by

\[
M_i^{(1)}(a) = a_i c_i + (1 - a_i)p_i L_i 
\quad (A.5)
\]

**Average loss due to interdependency**

Referring to Figure A, if the player \( j \) invests in his own security, then he non only eradicates the possibility of his own ruin, he does not cause any loss to any other player \( i \neq j \). On the other hand, if he decides not to invest, then with a probability \( q_{ji} \), player \( j \)’s action will cause player \( i \) to experience a catastrophic loss.
Thus, 

\[
\text{Prob} \begin{bmatrix}
\text{Player } j \text{'s action } a_j \text{ will } \textbf{not} \text{ cause player } i \\
\text{to face a catastrophic event}
\end{bmatrix} = 1 - (1 - a_j)q_{ji} \quad (A.6)
\]

It is tacitly assumed that the effects of all the other players’ actions on player \( i \) are statistically independent. Hence, 

\[
\text{Prob} \begin{bmatrix}
\text{Player } i \text{ will } \textbf{not} \text{ face any catastrophic event} \\
\text{due to actions of all the other players}
\end{bmatrix} = \prod_{j \neq i} [1 - (1 - a_i)q_{ji}] \quad (A.7)
\]

Thus, 

\[
q_i = \text{Prob} \begin{bmatrix}
\text{Player } i \text{ will face a catastrophic event} \\
\text{due to the action of all the other players}
\end{bmatrix} = 1 - \prod_{j \neq i} [1 - (1 - a_j)q_{ji}] \quad (A.8)
\]

and the average loss due to interdependency is 

\[
M_i^{(2)}(a) = q_i L_i \quad (A.9)
\]

Hence, the total average cost/loss to player \( i \) resulting from the play \( a \) is given by 

\[
M_i(a) = M_i^1(a) + M_i^{(2)}(a) \\
= a_i c_i + (1 - a_i) p_i L_i + \\
(1 - (1 - a_i)p_i)[1 - \prod_{j \neq i} [1 - (1 - a_j)q_{ji}]] L_i \quad (A.10)
\]

Tables 1 and 2 provide explicit expressions for \( M_i(a) \), when \( n = 2 \) and 3, respectively. Besides, when \( n = 2 \), \( M(a) \) can be conveniently arranged in the form of \( 2 \times 2 \) matrix, called the game matrix, as shown in Figure 3.
Table 1. Values of $M(a)$ for $n = 2$.

<table>
<thead>
<tr>
<th>Play $a$</th>
<th>$M_1(a)$</th>
<th>$M_2(a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>$p_1L_1 + (1-p_1)q_{21}L_1$</td>
<td>$p_2L_2 + (1-p_2)q_{12}L_2$</td>
</tr>
<tr>
<td>(0, 1)</td>
<td>$p_1L_1$</td>
<td>$c_2 + q_{12}L_2$</td>
</tr>
<tr>
<td>(1, 0)</td>
<td>$c_1 + q_{21}L_1$</td>
<td>$p_2L_2$</td>
</tr>
<tr>
<td>(1, 1)</td>
<td>$c_1$</td>
<td>$c_2$</td>
</tr>
</tbody>
</table>

Values of $M(a)$ for $n = 3$.

<table>
<thead>
<tr>
<th>Play $M_1(a)$</th>
<th>$M_1(a)$</th>
<th>$M_2(a)$</th>
<th>$M_3(a)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0, 0)</td>
<td>$p_1L_1 + (1-p_1)^*$</td>
<td>$p_2L_2 + (1-p_2)^*$</td>
<td>$p_3L_3 + (1-p_3)^*$</td>
</tr>
<tr>
<td></td>
<td>$(q_{21} + q_{31} - q_{21}q_{31})L_1$</td>
<td>$(q_{12} + q_{32} - q_{12}q_{32})L_2$</td>
<td>$(q_{13} + q_{23} - q_{13}q_{23})L_3$</td>
</tr>
<tr>
<td>(0, 0, 1)</td>
<td>$p_1L_1 + (1-p_1)q_{21}L_1$</td>
<td>$p_2L_2 + (1-p_2)q_{12}L_2$</td>
<td>$c_2 + (q_{13} + q_{23} - q_{13}q_{23})L_3$</td>
</tr>
<tr>
<td>(0, 1, 0)</td>
<td>$p_1L_1 + (1-p_1)q_{31}L_1$</td>
<td>$c_2 + (q_{12} + q_{32} - q_{12}q_{32})L_2$</td>
<td>$p_4L_3 + (1-p_3)q_{13}L_2$</td>
</tr>
<tr>
<td>(0, 1, 1)</td>
<td>$p_1L_1$</td>
<td>$c_2 + q_{12}L_2$</td>
<td>$c_3 + q_{13}L_3$</td>
</tr>
<tr>
<td>(1, 0, 0)</td>
<td>$c_1 + (q_{21} + q_{31} - q_{21}q_{31})L_1$</td>
<td>$p_2 + (1-p_2)q_{32}L_2$</td>
<td>$p_3L_3 + (1-p_3)q_{23}L_3$</td>
</tr>
<tr>
<td>(1, 0, 1)</td>
<td>$c_1 + q_{23}L_1$</td>
<td>$p_2L_2$</td>
<td>$c_3 + q_{23}L_3$</td>
</tr>
<tr>
<td>(1, 1, 0)</td>
<td>$c_1 + q_{33}L_1$</td>
<td>$c_2 + p_2L_2$</td>
<td>$p_3L_3$</td>
</tr>
<tr>
<td>(1, 1, 1)</td>
<td>$c_1$</td>
<td>$c_2$</td>
<td>$c_3$</td>
</tr>
</tbody>
</table>