NEW OSTROWSKI TYPE INEQUALITIES FOR CO-ORDINATED $s$-CONVEX FUNCTIONS IN THE SECOND SENSE

M. A. LATIF AND S. S. DRAGOMIR

ABSTRACT. In this paper some new Ostrowski type inequalities for co-ordinated $s$-convex functions in the second sense are obtained.

1. Introduction

In 1938, A. Ostrowski proved the following interesting inequality [21]:

**Theorem 1.** Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on $(a, b)$ whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on $(a, b)$, i.e., $\|f'\|_\infty := \sup_{t \in (a, b)} |f'(t)| < \infty$. Then we have the inequality

$$
|f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt| \leq \frac{1}{4} \left( 1 + \frac{(x-a+b)^2}{(b-a)^2} \right) (b-a) \|f'\|_\infty ,
$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

The inequality (1.1) can be rewritten in equivalent form as:

$$
|f(x) - \frac{1}{b-a} \int_a^b f(t) \, dt| \leq \frac{1}{4} \left( 1 + \frac{(x-a)^2 + (b-x)^2}{2(b-a)} \right) \|f'\|_\infty .
$$

Since 1938 when A. Ostrowski proved his famous inequality, many mathematicians have been working about and around it, in many different directions and with a lot of applications in Numerical Analysis and Probability, etc.

Several generalizations of the Ostrowski integral inequality for mappings of bounded variation, Lipschitzian, monotonic, absolutely continuous, convex mappings, $s$-convex mappings and $n$-times differentiable mappings with error estimates for some special means and for some numerical quadrature rules are considered by many authors. For recent results and generalizations concerning Ostrowski’s inequality see [4, 5, 7, 8, 11, 12, 20, 23, 24, 25, 26] and the references therein.

Let us consider now a bidimensional interval $\Delta := [a, b] \times [c, d] \subset \mathbb{R}^2$ with $a < b$ and $c < d$, a mapping $f : \Delta \to \mathbb{R}$ is said to be convex on $\Delta$ if the inequality

$$
f(\lambda x + (1 - \lambda) z, \lambda y + (1 - \lambda) w) \leq \lambda f(x, y) + (1 - \lambda) f(z, w),
$$

holds for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$. The mapping $f$ is said to be concave on the co-ordinates on $\Delta$ if the above inequality holds in reversed direction, for all $(x, y), (z, w) \in \Delta$ and $\lambda \in [0, 1]$. 

\textit{Date:} October 12, 2011.

2000 \textit{Mathematics Subject Classification.} 26D07, 26D15.

\textit{Key words and phrases.} co-ordinated $s$-convex function, Hermite-Hadamard type inequalities, Ostrowski inequality.

This paper is in final form and no version of it will be submitted for publication elsewhere.
A modification for convex (concave) functions on $\Delta$, which is also known as co-ordinated convex (concave) functions, was introduced by S. S. Dragomir [9, 13] as follows:

A function $f : \Delta \rightarrow \mathbb{R}$ is said to be convex (concave) on the co-ordinates on $\Delta$ if the partial mappings $f_y : [a,b] \rightarrow \mathbb{R}, f_y(u) = f(u, y)$ and $f_x : [c,d] \rightarrow \mathbb{R}, f_x(v) = f(x, v)$ are convex (concave) where defined for all $x \in [a, b], y \in [c, d]$.

A formal definition for co-ordinated convex (concave) functions may be stated in:

**Definition 1.** [18] A mapping $f : \Delta \rightarrow \mathbb{R}$ is said to be convex on the co-ordinates on $\Delta$ if the inequality

$$f(tx + (1-t)y, ru + (1-r)w) \leq tf(x,u) + t(1-r)f(x,w) + r(1-t)f(y,u) + (1-t)(1-r)f(y,w), \tag{1.2}$$

holds for all $t, r \in [0,1]$ and $(x, u), (y, w) \in \Delta$. The mapping $f$ is concave on the co-ordinates on $\Delta$ if the inequality (1.2) holds in reversed direction for all $t, r \in [0,1]$ and $(x, u), (y, w) \in \Delta$.

Clearly, every convex (concave) mapping $f : \Delta \rightarrow \mathbb{R}$ is convex (concave) on the co-ordinates. Furthermore, there exists co-ordinated convex (concave) function which is not convex (concave), (see for instance [9, 13]).

The main result proved concerning the co-ordinated convex function from [9, 13] is given in:

**Theorem 2.** [9] Suppose that $f : \Delta \rightarrow \mathbb{R}$ is co-ordinated convex on $\Delta$. Then one has the inequalities:

$$f \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \leq \frac{1}{2} \left[ \frac{1}{b-a} \int_a^b f \left( x, \frac{c+d}{2} \right) dx + \frac{1}{d-c} \int_c^d f \left( \frac{a+b}{2}, y \right) dy \right]$$

$$\leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x, y) dy dx$$

$$\leq \frac{1}{4} \left[ \frac{1}{b-a} \int_a^b f(x, c) dx + \frac{1}{b-a} \int_a^b f(x, d) dx + \frac{1}{d-c} \int_c^d f(b, y) dy \right]$$

$$\leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}. \tag{1.3}$$

The above inequalities are sharp. The inequalities in (1.3) hold in reverse direction if the mapping $f$ is concave.

The concept of $s$-convex functions on the co-ordinates in the second sense was introduced by Almoari and Darus in [1] as a generalization of the co-ordinated convexity in:

**Definition 2.** [1] Consider the bidimensional interval $\Delta = [a,b] \times [c,d]$ in $[0, \infty)^2$ with $a < b$ and $c < d$. The mapping $f : \Delta \rightarrow \mathbb{R}$ is $s$-convex in the second sense on
are stated as follows: Bakula et. al [6], proved Jensen’s inequality for convex functions on the second sense on the co-ordinates on $a; b$, holds for all $t; r \in [0, 1]$ and $(x, y), (u, w) \in \Delta$ with some fixed $s \in (0, 1)$. The mapping $f$ is concave on the co-ordinates on $\Delta$ if the inequality (1.4) holds in reversed direction for all $t, r \in [0, 1]$ and $(x, y), (u, w) \in \Delta$ with some fixed $s \in (0, 1)$.

In [5], Alomari et al. also proved a variant of inequalities given above by (1.3) for s-convex functions in the second sense on the co-ordinates on a rectangle from the plane $\mathbb{R}^2$.

**Theorem 3.** [1] Suppose $f : \Delta = [a, b] \times [c, d] \subseteq [0, \infty)^2 \rightarrow \mathbb{R}$ is s-convex function in the second sense on the co-ordinates on $\Delta$. Then one has the inequalities:

\[
4^{s-1}f\left(\frac{a+b+1}{2}, \frac{c+d}{2}\right) \\
\leq 2^{s-2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right) dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right) dy\right] \\
\leq \frac{1}{(b-a)(d-c)}\int_a^b \int_c^d f(x, y) dy dx \\
\leq \frac{1}{2(s+1)}\left[\frac{1}{b-a}\int_a^b \left[f(x, c) + f(x, d)\right] dx \\
+ \frac{1}{d-c}\int_a^b \left[f(a, y) + f(b, y)\right] dy\right] \\
\leq \frac{f(a, c) + f(b, c) + f(a, d) + f(b, d)}{(s+1)^2}. \quad (1.5)
\]

In recent years, many authors have proved several inequalities for co-ordinated convex functions. These studies include, among others, the works in [1, 2, 3, 6, 9, 15, 17, 18, 19, 20, 22, 27] (see also the references therein). Alomari et al. [1]-[3], proved several Hermite-Hadamard type inequalities for co-ordinated s-convex functions. Bakula et. al [6], proved Jensen’s inequality for convex functions on the co-ordinates from the rectangle from the plan $\mathbb{R}^2$. Dragomir [9], proved the Hermite-Hadamard type inequalities for co-ordinated convex functions. Hwang et. al [15],
also proved some Hermite-Hadamard type inequalities for co-ordinated convex function of two variables by considering some mappings directly associated with the Hermite-Hadamard type inequality for co-ordinated convex mappings of two variables. Latif et. al [17]-[20], proved some inequalities of Hermite-Hadamard type for differentiable co-ordinated convex functions, for product of two co-ordinated convex mappings, for co-ordinated h-convex mappings and also proved some Ostrowski type inequalities for co-ordinated convex mappings. Özdemir et. al [22], proved Hadamard’s type inequalities for co-ordinated m-convex and (α, m)-convex functions. Sarikaya, et. al [27] proved Hermite-Hadamard type inequalities for differentiable co-ordinated convex function. For further inequalities on co-ordinated convex functions see also the references in the above cited papers.

In the present paper, we establish new Ostrowski type inequalities for co-ordinated s-convex functions in second sense similar to those from [20].

2. Main Results

To establish our main results we need the following identity:

Lemma 1. [20] Let \( f : Δ \to \mathbb{R} \) be a twice partial differentiable mapping on \( Δ^o \). If \( \frac{\partial^2 f}{\partial r \partial t} \in L(Δ) \), then the following identity holds:

\[
\begin{align*}
\int_a^b \int_c^d f(u,v) \, dv \, du - A &= \int_0^1 \int_0^1 rt \frac{\partial^2 f}{\partial r \partial t} (tx + (1 - t)a, ry + (1 - r)c) \, dr \, dt \\
- \int_0^1 \int_0^1 rt \frac{\partial^2 f}{\partial r \partial t} (tx + (1 - t)a, ry + (1 - r)c) \, dr \, dt \\
- \int_0^1 \int_0^1 rt \frac{\partial^2 f}{\partial r \partial t} (tx + (1 - t)b, ry + (1 - r)c) \, dr \, dt \\
+ \int_0^1 \int_0^1 rt \frac{\partial^2 f}{\partial r \partial t} (tx + (1 - t)b, ry + (1 - r)c) \, dr \, dt,
\end{align*}
\]

for all \((x, y) \in Δ\), where

\[
A = \frac{1}{d - c} \int_c^d f(x, v) \, dv + \frac{1}{b - a} \int_a^b f(u, y) \, du.
\]

We begin with the following result:

Theorem 4. Let \( Δ = [a, b] \times [c, d] \subseteq [0, \infty)^2 \to \mathbb{R} \) be a twice partial differentiable mapping on \( Δ^o \) such that

\[
\frac{\partial^2 f}{\partial r \partial t} \in L(Δ).
\]

If \( \left| \frac{\partial^2 f}{\partial r \partial t} \right| \) is s-convex in the second sense on the co-ordinates on \( Δ \) with \( s \in (0, 1] \) and

\[
\left| \frac{\partial^2 f}{\partial r \partial t} (x, y) \right| \leq M, \quad (x, y) \in Δ,
\]

then the following inequality holds:

\[
\begin{align*}
\int_a^b \int_c^d f(u,v) \, dv \, du - A &\leq M \frac{(x - a)^2 + (b - x)^2}{b - a} \left[ \frac{(y - c)^2 + (d - y)^2}{d - c} \right],
\end{align*}
\]

for all \((x, y) \in Δ\), where \( A \) is defined in Lemma 1.
Proof. By Lemma 1, we have that the following inequality holds:

\[
\left| f(x, y) + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(u, v) \, dv \, du - A \right|
\]

\[
\leq \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 rt \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)c) \right| \, dr \, dt
\]

\[
+ \frac{(x-a)^2(d-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 rt \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)d) \right| dr \, dt
\]

\[
+ \frac{(b-x)^2(y-c)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 rt \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)c) \right| dr \, dt
\]

\[
+ \frac{(b-x)^2(d-y)^2}{(b-a)(d-c)} \int_0^1 \int_0^1 rt \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)d) \right| dr \, dt,
\]

(2.3)

for all \((x, y) \in \Delta\).

Using the co-ordinated \(s\)-convexity of \(\frac{\partial^2 f}{\partial r \partial t}\), we have that the following inequality holds:

\[
\int_0^1 \int_0^1 rt \left| \frac{\partial^2 f}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)c) \right| dr \, dt
\]

\[
\leq \left| \frac{\partial^2 f}{\partial r \partial t} f(x, y) \right| \int_0^1 \int_0^1 t^{s+1} r^{s+1} dr dt + \left| \frac{\partial^2}{\partial r \partial t} f(x, c) \right| \int_0^1 \int_0^1 t^{s+1} r^{(1-r)^s} dr dt
\]

\[
+ \left| \frac{\partial^2}{\partial r \partial t} f(a, y) \right| \int_0^1 \int_0^1 r^{s+1} t^{(1-t)^s} dr dt
\]

\[
+ \left| \frac{\partial^2}{\partial r \partial t} f(a, c) \right| \int_0^1 \int_0^1 rt (1-t)^s (1-r)^s dr dt.
\]

(2.4)

Since

\[
\int_0^1 \int_0^1 t^{s+1} r^{s+1} dr dt = \frac{1}{(s+2)^2},
\]

\[
\int_0^1 \int_0^1 t^{s+1} r^{(1-r)^s} dr dt = \int_0^1 \int_0^1 r^{s+1} t^{(1-t)^s} dr dt = \frac{1}{(s+1)(s+2)^2},
\]

\[
\int_0^1 \int_0^1 rt (1-t)^s (1-r)^s dr dt = \frac{1}{(s+1)^2(s+2)^2}
\]

and

\[
\left| \frac{\partial^2}{\partial r \partial t} f(x, y) \right| \leq M, (x, y) \in \Delta,
\]

where we have used the Euler Beta function defined by

\[
\beta(x, y) = \int_0^1 t^{x-1} (1-t)^{y-1} dt, \ x, y > 0
\]

and the properties

\[
\beta(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} \quad \text{and} \quad \Gamma(x+1) = x \Gamma(x)
\]
of Euler Beta function to evaluate the above integrals. Hence from (2.4), we obtain

\[
\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)c) \right| \, dr dt \\
\leq \frac{M}{(s+2)^2} + \frac{2M}{(s+1)(s+2)^2} + \frac{M}{(s+1)^2(s+2)^2} = \frac{M}{(s+1)^2}
\]

(2.5)

Analogously, we also have

\[
\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)a, ry + (1-r)d) \right| \, dr dt \leq \frac{M}{(s+1)^2},
\]

(2.6)

\[
\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)c) \right| \, dr dt \leq \frac{M}{(s+1)^2}
\]

(2.7)

and

\[
\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f(tx + (1-t)b, ry + (1-r)d) \right| \, dr dt \leq \frac{M}{(s+1)^2}.
\]

(2.8)

Now by making use of the inequalities (2.5)-(2.8) and the fact that

\[
(x-a)^2 (y-c)^2 + (x-a)^2 (d-y)^2 + (b-x)^2 (y-c)^2 + (b-x)^2 (d-y)^2
\]

\[
= \left[ (x-a)^2 + (b-x)^2 \right] \left[ (y-c)^2 + (d-y)^2 \right],
\]

we get the inequality (2.2). This completes the proof. \(\square\)

The corresponding version for powers of the absolute value of the partial derivative is incorporated in the following result:

**Theorem 5.** \(\Delta = [a,b] \times [c,d] \subseteq [0,\infty)^2 \rightarrow \mathbb{R}\) be a twice partial differentiable mapping on \(\Delta^0\) such that \(\frac{\partial^2 f}{\partial r \partial t} \in L(\Delta)\). If \(\left\| \frac{\partial^2 f}{\partial r \partial t} \right\|^q\) is s-convex in the second sense on the co-ordinates on \(\Delta\), \(p, q > 1, \frac{1}{p} + \frac{1}{q} = 1\) and \(\left| \frac{\partial^2}{\partial r \partial t} f(x,y) \right| \leq M, (x,y) \in \Delta\), then the following inequality holds:

\[
\left| f(x,y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v) \, du dv - A \right|
\]

\[
\leq \frac{M}{(1+p)^\frac{2}{q}} \left( \frac{2}{s+1} \right)^\frac{q}{2} \left[ \frac{(x-a)^2 + (b-x)^2}{b-a} \right] \left[ \frac{(y-c)^2 + (d-y)^2}{d-c} \right],
\]

(2.9)

for all \((x,y) \in \Delta\), where \(A\) is defined in Lemma 1.
Proof. By Lemma 1 and using the Hölder inequality for double integrals, we have that inequality holds:

\[
\left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) \, du \, dv \, A - A \right| \leq \left( \int_0^1 \int_0^1 r^p t^p \, dr \, dt \right)^{\frac{1}{p}}
\]

\[
\times \left[ \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial \tau \partial \tau} f(tx + (1-t) a, ry + (1-r) c) \right|^q \, d\tau \, dt \right)^{\frac{1}{q}} + \frac{(x-a)^2(y-d)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial \tau \partial \tau} f(tx + (1-t) a, ry + (1-r) c) \right|^q \, d\tau \, dt \right)^{\frac{1}{q}}
\]

\[
+ \frac{(b-x)^2(y-c)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial \tau \partial \tau} f(tx + (1-t) b, sy + (1-s) c) \right|^q \, d\tau \, dt \right)^{\frac{1}{q}} + \frac{(b-x)^2(y-d)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial \tau \partial \tau} f(tx + (1-t) b, ry + (1-r) d) \right|^q \, d\tau \, dt \right)^{\frac{1}{q}}
\]

\tag{2.10}
\]

for all \((x, y) \in \Delta\).

Since \(\frac{\partial^2 f}{\partial \tau \partial \tau} \) is s-convex in the second sense on the co-ordinates on \(\Delta\) and \(\frac{\partial^2 f}{\partial \tau \partial \tau} (x, y) \leq M, (x, y) \in \Delta\), we have

\[
\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial \tau \partial \tau} f(tx + (1-t) a, ry + (1-r) c) \right|^q \, d\tau \, dt \leq \frac{4M^q}{(s+1)^2}.
\]

Similarly, we also have the following inequalities:

\[
\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial \tau \partial \tau} f(tx + (1-t) a, ry + (1-r) d) \right|^q \, d\tau \, dt \leq \frac{4M^q}{(s+1)^2},
\]

\[
\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial \tau \partial \tau} f(tx + (1-t) b, ry + (1-r) c) \right|^q \, d\tau \, dt \leq \frac{4M^q}{(s+1)^2}
\]

and

\[
\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial \tau \partial \tau} f(tx + (1-t) b, ry + (1-r) d) \right|^q \, d\tau \, dt \leq \frac{4M^q}{(s+1)^2}.
\]

Using the fact

\[
\int_0^1 \int_0^1 r^p t^p \, dr \, dt = \frac{1}{(1+p)^2}
\]

and the above inequalities in (2.10), we get (2.9). This completes the proof of the theorem. \(\square\)

A different approach leads us to the following result:
Theorem 6. Let \( \Delta = [a,b] \times [c,d] \subseteq [0,\infty)^2 \to \mathbb{R} \) be a twice partial differentiable mapping on \( \Delta^o \) such that \( \frac{\partial^2 f}{\partial r \partial t} \in L(\Delta) \). If \( \left| \frac{\partial^2 f}{\partial r \partial t} \right|^q \) is s-convex on the co-ordinates on \( \Delta \), \( q \geq 1 \) and \( \left| \frac{\partial^2 f}{\partial r \partial t} f(x,y) \right| \leq M \), \( (x,y) \in \Delta \), then the following inequality holds:

\[
\left| f(x,y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v) dvdu - A \right| \leq \frac{M}{4} \left( \frac{2}{s+1} \right)^{\frac{q}{2}} \left[ \frac{(x-a)^2}{b-a} \right] \left[ \frac{(y-c)^2}{d-c} \right],
\]

for all \( (x,y) \in \Delta \), where \( A \) is defined in Lemma 1.

Proof. Suppose \( q \geq 1 \). From Lemma 1 and using the power mean inequality for double integrals, we have

\[
\left| f(x,y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u,v) dvdu - A \right| \leq \left( \int_0^1 \int_0^1 t r dt dt \right)^{1-\frac{1}{q}} \times \left[ \frac{(x-a)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 tr \left| \frac{\partial^2 f}{\partial r \partial t} f(tx + (1-t) a, ry + (1-r) c) \right|^q dt dt \right)^{\frac{1}{q}} + \frac{(x-a)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 tr \left| \frac{\partial^2 f}{\partial r \partial t} f(tx + (1-t) a, ry + (1-r) c) \right|^q dt dt \right)^{\frac{1}{q}} + \frac{(b-x)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 tr \left| \frac{\partial^2 f}{\partial r \partial t} f(tx + (1-t) b, ry + (1-r) c) \right|^q dt dt \right)^{\frac{1}{q}} + \frac{(b-x)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 tr \left| \frac{\partial^2 f}{\partial r \partial t} f(tx + (1-t) b, ry + (1-r) c) \right|^q dt dt \right)^{\frac{1}{q}} \right],
\]

for all \( (x,y) \in \Delta \).

By similar argument as in Theorem 5 that \( \left| \frac{\partial^2 f}{\partial r \partial t} \right|^q \) is s-convex on the co-ordinates on \( \Delta \) in the second sense and \( \left| \frac{\partial^2 f}{\partial r \partial t} f(x,y) \right| \leq M \), \( (x,y) \in \Delta \), we have

\[
\int_0^1 \int_0^1 tr \left| \frac{\partial^2 f}{\partial r \partial t} f(tx + (1-t) a, ry + (1-r) c) \right|^q dt dt \leq \left| \frac{\partial^2 f}{\partial r \partial t} f(x,y) \right| \left( \int_0^1 \int_0^1 t^{s+1} r^{s+1} dt dr \right) + \left| \frac{\partial^2 f}{\partial r \partial t} f(x,c) \right| \left( \int_0^1 \int_0^1 t^{s+1} r (1-r)^s dr dt \right) + \left| \frac{\partial^2 f}{\partial r \partial t} f(a,y) \right| \left( \int_0^1 \int_0^1 t (1-t)^s r^{s+1} dr dt \right) + \left| \frac{\partial^2 f}{\partial r \partial t} f(a,c) \right| \left( \int_0^1 \int_0^1 t (1-t)^s r (1+r)^s dr dt \right) = \frac{M^q}{(s+2)^2} + \frac{M^q}{(s+1)(s+2)^2} + \frac{M^q}{(s+1)^2(s+2)^2} + \frac{M^q}{(s+1)^2(s+2)} = \frac{M^q}{(s+1)^2}.
\]
In a similar way, we also have that the following inequalities:
\[
\begin{align*}
\int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial r \partial t} \right| \left( tx + (1 - t) a, ry + (1 - r) d \right) \, dr \, dt & \leq \frac{M^q}{(s + 1)^q} \\
\int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial r \partial t} \right| \left( tx + (1 - t) b, ry + (1 - r) c \right) \, dr \, dt & \leq \frac{M^q}{(s + 1)^q}
\end{align*}
\]
and
\[
\int_0^1 \int_0^1 \left| \frac{\partial^2 f}{\partial r \partial t} \right| \left( tx + (1 - t) b, ry + (1 - r) d \right) \, dr \, dt \leq \frac{M^q}{(s + 1)^q}.
\]
Now using the above inequalities and
\[
\int_0^1 \int_0^1 rtdrdt = \frac{1}{4}
\]
in (2.12), we get the desired inequality (2.11). This completes the proof. \(\square\)

**Remark 1.** Since \((1 + p)^\frac{q}{p} < 2, p > 1\) and accordingly, we have
\[
\frac{1}{2} < \frac{1}{(1 + p)^\frac{q}{p}}, p > 1
\]
which gives
\[
\frac{1}{4} < \frac{1}{(1 + p)^\frac{q}{p}}, p > 1.
\]
This reveals that the inequality (2.11) gives tighter estimate than that of the inequality (2.9).

**Remark 2.** From the inequalities proved above in Theorem 4-Theorem 6, one can
get several midpoint type inequalities by setting \(x = \frac{a + b}{2}\) and \(y = \frac{c + d}{2}\). However the
details are left to the interested reader.

Now we drive some results with co-ordinated \(s\)-concavity property instead of
co-ordinated \(s\)-convexity.

**Theorem 7.** \(\Delta = [a, b] \times [c, d] \subseteq [0, \infty)^2 \rightarrow \mathbb{R}\) be a twice partial differentiable
mapping on \(\Delta^o\) such that \(\frac{\partial^2 f}{\partial x \partial t} \in L(\Delta)\). If \(\left| \frac{\partial^2 f}{\partial x \partial t} \right|^q\) is concave on the co-ordinates
on \(\Delta\) and \(p, q > 1, \frac{1}{p} + \frac{1}{q} = 1\), then the inequality
\[
\left| f(x, y) + \frac{1}{(b - a)(d - c)} \int_a^b \int_c^d f(u, v) \, dv \, du - A \right| 
\leq \frac{4^{q-1}}{(1 + p)^\frac{q}{p} (b - a)(d - c)} \left[ \begin{array}{c}
(x - a)^2 (y - c)^2 \left| \frac{\partial^2 f}{\partial r \partial t} \right| \left( \frac{x + a}{2}, \frac{d + y}{2} \right) \right]^q \\
+ (x - a)^2 (d - y)^2 \left| \frac{\partial^2 f}{\partial r \partial t} \right| \left( \frac{x + a}{2}, \frac{d + y}{2} \right) \right]^q \\
+ (b - x)^2 (y - c)^2 \left| \frac{\partial^2 f}{\partial r \partial t} \right| \left( \frac{b + a}{2}, \frac{y + c}{2} \right) \right]^q \\
+ (b - x)^2 (d - y)^2 \left| \frac{\partial^2 f}{\partial r \partial t} \right| \left( \frac{b + x}{2}, \frac{d + y}{2} \right) \right]^q \right],
\]
holds for all \((x, y) \in \Delta\), where \(A\) is defined in Lemma 1.
Proof. From Lemma 1 and using the Hölder inequality for double integrals, we have that inequality holds:

\[
\begin{aligned}
&\left| f(x, y) + \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(u, v) \, dv \, du - A \right| \\
\leq & \left( \int_0^1 \int_0^1 r^p t^q \, dr \, dt \right)^{\frac{1}{p}} \\
\times & \left[ \frac{(x-a)^2(y-c)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial x \partial t} f(tx + (1-t) a, ry + (1-r) c) \right|^q \, dr \, dt \right)^{\frac{1}{q}} \\
+ & \frac{(x-a)^2(d-y)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial x \partial t} f(tx + (1-t) a, ry + (1-r) d) \right|^q \, dr \, dt \right)^{\frac{1}{q}} \\
+ & \frac{(b-x)^2(y-c)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial y \partial t} f(tx + (1-t) b, ry + (1-r) c) \right|^q \, dr \, dt \right)^{\frac{1}{q}} \\
+ & \frac{(b-x)^2(d-y)^2}{(b-a)(d-c)} \left( \int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial y \partial t} f(tx + (1-t) b, ry + (1-r) d) \right|^q \, dr \, dt \right)^{\frac{1}{q}} \right],
\end{aligned}
\]

(2.14)

for all \((x, y) \in \Delta\).

Since \(\left| \frac{\partial^2 f}{\partial t^q} \right|^q\) is s-concave on the co-ordinates on \(\Delta\), so an application of (1.3) with inequalities in reversed direction, gives us the following inequalities:

\[
\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial t \partial t} f \left( tx + (1-t) a, ry + (1-r) c \right) \right|^q \, dr \, dt \\
\leq 2^{s-2} \left[ \int_0^1 \left| \frac{\partial^2}{\partial y \partial t} f \left( x + (1-t) a, y + c \right) \right|^q \, dt \right] \\
+ \int_0^1 \left| \frac{\partial^2}{\partial y \partial t} f \left( \frac{x + a}{2}, ry + (1-r) c \right) \right|^q \, dr \right] \\
\leq 4^{s-1} \left| \frac{\partial^2}{\partial t \partial t} f \left( x + a, \frac{y + c}{2} \right) \right|^q,
\]

(2.15)

\[
\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial t \partial t} f \left( tx + (1-t) a, ry + (1-r) d \right) \right|^q \, ds \, dt \\
\leq 2^{s-2} \left[ \int_0^1 \left| \frac{\partial^2}{\partial y \partial t} f \left( x + (1-t) a, \frac{d + y}{2} \right) \right|^q \, dt \right] \\
+ \int_0^1 \left| \frac{\partial^2}{\partial y \partial t} f \left( \frac{x + a}{2}, ry + (1-r) c \right) \right|^q \, dr \right] \\
\leq 4^{s-1} \left| \frac{\partial^2}{\partial t \partial t} f \left( x + a, \frac{d + y}{2} \right) \right|^q,
\]

(2.16)
\[
\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f \left( tx + (1 - t) b, ry + (1 - r) c \right) \right|^q \, dr \, dt \\
\leq 2^{s-2} \left[ \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f \left( tx + (1 - t) a, \frac{y + c}{2} \right) \right|^q \, dt \\
+ \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f \left( \frac{b + x}{2}, sy + (1 - s) c \right) \right|^q \, dr \right] \\
\leq 4^{s-1} \left| \frac{\partial^2}{\partial r \partial t} f \left( \frac{b + a}{2}, \frac{y + c}{2} \right) \right|^q \tag{2.17}
\]

and
\[
\int_0^1 \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f \left( tx + (1 - t) b, ry + (1 - r) d \right) \right|^q \, dr \, dt \\
\leq 2^{s-2} \left[ \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f \left( tx + (1 - t) b, \frac{d + y}{2} \right) \right|^q \, dt \\
+ \int_0^1 \left| \frac{\partial^2}{\partial r \partial t} f \left( \frac{b + x}{2}, ry + (1 - r) d \right) \right|^q \, dr \right] \\
\leq 4^{s-1} \left| \frac{\partial^2}{\partial r \partial t} f \left( \frac{b + a}{2}, \frac{d + y}{2} \right) \right|^q \tag{2.18}
\]

By making use of (2.15)-(2.18) in (2.14), we obtain (2.13). Thus the proof of the theorem is complete. \( \square \)

REFERENCES


[22] M. E. ¨Ozdemir, E. Set and M. Z. Sarıkaya, Some new Hadamard’s type inequalities for co-ordinated m-convex and (α, m)-convex functions, Accepted.


College of Science, Department of Mathematics, University of Hail, Hail 2440, Saudi Arabia.
E-mail address: m.amer.latif@hotmail.com.

School of Engineering & Science, Victoria University, PO Box 14428, Melbourne City, MC 8001, Australia.
E-mail address: sever.dragomir@vu.edu.au