Optimal control of a two control input buck-boost converter

S. Mariéthoz, S. Almér and M. Morari
Automatic Control Laboratory, ETH Zurich, Switzerland
[mariethoz,almers,morari]@control.ee.ethz.ch

Abstract—The converter system under consideration is a two-input one-output non-inverting buck-boost converter. The two control inputs introduce a degree of freedom that may be used to optimize dynamic performance and other objectives such as energy efficiency at steady state. Dynamic performance during transient is achieved using an MPC scheme, which drives the system to a steady state toward its energy efficiency optimum using a precomputed target reference. The problem is complicated as state and input are combined bilinearly. Different control models and MPC problem formulations are considered to tackle the problem and performance is compared in simulations.

I. INTRODUCTION

Non-inverting buck-boost converters have recently received increased attention as they constitute the enabling technology for energy management of new batteries in portable power electronic devices [6]. The non-inverting buck-boost topology, which comprises a buck stage and a boost stage (see Fig. 1) poses a number of challenging problems. The switched nature of the converter makes traditional control approaches unsuitable and averaging [7], [17] is often applied to obtain a continuous-time model. However, the averaged model only describes the slow modes of the system and can therefore lead to suboptimal performance. In addition to the hybrid characteristics, the non-inverting buck-boost converter is non-minimum phase with respect to the output voltage and there is an ambiguity in the steady state control input. In fact, there is an infinite number of control input pairs which yield the desired steady state output voltage. This ambiguity is often dealt with by forcing a relation between the control inputs effectively reducing the converter to have one control input. This approach simplifies control design but may reduce dynamic performance or even trigger instabilities [20]. It has been recently shown that there is a potential to improve dynamic performance using this additional degree of freedom at the cost of a reduced energy efficiency [3].

Model predictive control (MPC) [5], [11], [12], [21] has recently been applied successfully in a wide range of power electronics applications, where there is a need to handle state and input constraints while optimizing performance. The applications include electric drives [9], [13], [19], inverters [16] and switched mode power supplies [8], [10]. Explicit model predictive control [2] has been the most applied solution for real time implementations as the required switching and sampling frequencies typically range from a few kHz to a few MHz [1], [13]–[16]. Various investigations have been based on piecewise affine (PWA) system models which capture the hybrid and nonlinear behaviour of switched mode power supplies. In [8] the so-called υ-resolution model is introduced in order to model the hybrid behaviour of the buck converter and accurately enforce the state constraint on the coil current. A scaling of the state is used to remove the supply voltage from the dynamic equations. In [1], the nonlinear boost converter dynamics are approximated as a piecewise affine model obtained using a least square fitting procedure over a partition of the duty cycle domain. A controller valid for a wide range of operating conditions is obtained. The paper [14] deals with the nonlinear behaviour of coils using a PWA model. The drawback of these approaches is that the complexity grows exponentially with the number of partitions and the prediction horizon and that the resulting explicit solution is often discontinuous.

The performance and complexity of the MPC controller largely depend on how the system dynamics are modeled. The potential to reduce the complexity and increase the performance using specific modeling approaches has been largely unexplored in the application of MPC to power electronics so far. In the present paper, we consider a number of different system models and corresponding MPC problems to show the advantages and drawbacks of various modeling approaches. We introduce successively a linear model based on classical assumptions, a piecewise affine model extending the work done in [1] to the non-inverting buck-boost converter and finally we introduce a new model and problem formulation. The latter relies on the use of two auxiliary control inputs and an adaptive term updated at each sampling instant. This allows us to formulate the problem to be solved off-line as a quadratic program resulting in an explicit controller of low-complexity with improved performance. A target reference is computed at each sampling time using an observer. The MPC problems corresponding to the different models are solved and yield explicit solutions. The performance of the controllers is evaluated and compared via simulations.

The main contribution of the paper is the application and comparison of different MPC schemes to the non-inverting buck-boost converter. The approaches under consideration include the extension of an MPC technique previously applied to the boost converter, and the introduction of a new problem formulation dedicated to the high-performance low-complexity control of DC-DC converters.

The paper is organized as follows: Section II introduces the buck-boost converter. Section III gives a brief presentation of the MPC framework and in Section IV we derive the different system models used in the MPC problem formulation. Section V compares simulation results obtained
The main control objective is to steer the (DC component of the) output voltage to the desired reference value \( v_{\text{ref}} \). The reference tracking must be achieved under (measured) variations in source voltage and load current and while respecting hard constraints on the control input and state: The duty cycles are per definition confined to the interval \([0,1]\) and the inductor current is constrained to lie below a limit \( i_{\ell,\text{max}} \). This constraint is imposed as a safety measure to avoid entering the saturated region of the coil.

To meet the objectives described above we propose a model predictive control (MPC) approach [5], [11], [12], [21] which provides the ability to handle constraints on state and input without sacrificing performance. Given a discrete time dynamic system, a finite horizon optimal control problem is formulated. At each time step the current state is used as initial condition and the problem is solved to obtain an optimal sequence of control inputs. Only the first control move is implemented. The horizon is then shifted and the procedure is repeated at the next time instant.

Let \( x_{k+l|k} \) be the predicted state at time \( k + l \) given the initial state \( x_k \) at time instant \( k \) and a sequence of control inputs \( u_t \). The performance index of the MPC problem is of the converter model (1) are

\[
A_{c1} = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{r_c}{x_c} \end{bmatrix}, \quad A_{c2} = \begin{bmatrix} 0 & 1 \\ -\frac{1}{x_c} & -\frac{r_c}{x_c} \end{bmatrix}
\]
\[
B_{c1} = \begin{bmatrix} 0 \\ \frac{1}{x_c} \end{bmatrix}, \quad B_{cw1} = \begin{bmatrix} -\frac{1}{x_c} \\ 0 \end{bmatrix}, \quad B_{cw2} = \begin{bmatrix} 0 \\ \frac{r_c}{x_c} \end{bmatrix}.
\]

Neglecting the losses, all the control input pairs that satisfy

\[
\frac{d_1}{d_2} = v_s = v_{\text{ref}}
\]

yield the same steady-state operating point. Furthermore, neglecting the effect of the switching, the obtained average state trajectory does not depend on the phase-shift between the two modulated signals. Therefore the phase-shift angle is not manipulated by the controller but is selected to minimize the steady state losses for the considered operating point. As a consequence, one degree of freedom remains in the selection of the duty cycles. The controller must deal with this ambiguity. A traditional approach consists in imposing a given relation between \( d_1 \) and \( d_2 \) (and the phase), resulting in an unique manipulated variable for the control. The real control signals are then extracted from this unique equivalent control input. This approach is however in general not optimal. During transient, the relation between \( d_1 \) and \( d_2 \) that optimizes the control objective can be found by formulating an optimal control problem. At steady state, duty cycles and the phase are undetermined and we propose to use a target reference for the duty cycles and phase-shift at steady state that minimize some cost function, for instance the converter losses.

III. Control Problem and Approach

The main control objective is to steer the (DC component of the) output voltage to the desired reference value \( v_{\text{ref}} \). The reference tracking must be achieved under (measured) variations in source voltage and load current and while respecting hard constraints on the control input and state: The duty cycles are per definition confined to the interval \([0,1]\) and the inductor current is constrained to lie below a limit \( i_{\ell,\text{max}} \). This constraint is imposed as a safety measure to avoid entering the saturated region of the coil.

To meet the objectives described above we propose a model predictive control (MPC) approach [5], [11], [12], [21] which provides the ability to handle constraints on state and input without sacrificing performance. Given a discrete time dynamic system, a finite horizon optimal control problem is formulated. At each time step the current state is used as initial condition and the problem is solved to obtain an optimal sequence of control inputs. Only the first control move is implemented. The horizon is then shifted and the procedure is repeated at the next time instant.

Let \( x_{k+l|k} \) be the predicted state at time \( k + l \) given the initial state \( x_k \) at time instant \( k \) and a sequence of control inputs \( u_t \). The performance index of the MPC problem is
defined as
\[ J(x_k, u_l) = \sum_{l=0}^{N-1} \|Qx_{k+l|k}\|^p + \|R_{ul}\|^p \]
where \(N\) is the prediction horizon and \(Q \geq 0, R > 0\) are diagonal matrices. The optimal control input sequence \(\{u_l^*\}_{l=0}^L\), \(L \leq N\) is obtained by minimizing the cost function subject to dynamic equations and constraints on control inputs and state. The complexity of the control problem can grow fast with the horizon length \(N\). To reduce complexity while still capturing the system behaviour over a longer horizon, it is possible to choose the control horizon \(L\) shorter than the prediction horizon \(N\). In this case the constraint \(u_L = u_{L+1} = \cdots = u_N\) is imposed and only the first \(L\) control inputs are variable. This procedure is often referred to as move-blocking.

The solution to the optimal control problem is determined by the initial condition \(x_k\). As shown in [4], the optimal control input \(u_l^*\) is a piecewise affine function of the initial condition \(x_k\). This piecewise affine map can be computed off-line and implemented using a look-up table. This approach is referred to as explicit MPC and it enables the application of MPC also in “fast” systems where the time available for computations is limited.

In the sections below we introduce three different models of the converter dynamics and pose corresponding MPC problems. The problems are solved to obtain explicit solutions and the performance is compared in simulations.

IV. CONTROL MODELS

The control models considered below are all based on the Euler approximation of the continuous time switched dynamics (1). The Euler model is a discrete-time expression on the form

\[ x_{k+1} = (A_1 + d_{2,k}A_2)x_k + d_{1,k}B_1v_s + d_{2,k}B_wi_o + B_{w1}i_o \]

(5)

where \(x_k\) is the approximation of the (continuous) state \(x\) at the \(k\)th sampling instant and where

\[ A_1 = I + T_sA_{c1}, \quad A_2 = T_sA_{c2} \]
\[ B_1 = T_sB_{c1}, \quad B_w = T_sB_{cw} \]
\[ B_{w1} = T_sB_{cw1}, \quad B_{w2} = T_sB_{cw2}. \]

The Euler model is a bilinear system and is therefore not directly tractable in the MPC framework. In the section below we introduce three different approximations of the Euler model which can be handled in MPC. Because the term \(B_{w2}\) is very small it is henceforth neglected.

A. Linear constrained model

The most straightforward way to fit the bilinear dynamics (5) into the MPC framework is to linearize them around some operating point. In the present paper, this classical approach is included to obtain a benchmark to which more sophisticated modeling approaches can be compared.

Consider a (stationary) operating point \(x^0\) with corresponding duty cycles \(d^{[0]}, d^{[2]}\) satisfying

\[ x^0 = (A_1 + d^{[2]}A_2)x^0 + d^{[0]}B_1v_s + B_{w1}i_o \]

for nominal values of supply voltage \(v_s\) and load current \(i_o\). The corresponding linearized model is

\[ z_{k+1} = (A_0 + d^{[2]}A_2)z_k + \delta_{1,k}B_1v_s + \delta_{2,k}A_2x^0 \]

(6)

where \(z_x := x_k - x^0\), \(\delta_{1,k} := d_{1,k} - d^{[0]}\), \(i = 1, 2\) denote the deviation from the operating point.

To obtain a controller for a wide range of supply voltages \(v_s\) and load currents \(i_o\) we extend the state space to include said quantities. The resulting augmented system will thus have dimension four. The linearized extended dynamics were used in an MPC problem formulation with quadratic cost. Only the “original” states \(v_s, i_o\) are included in the cost function with penalty matrices \(Q = \begin{bmatrix} 30 & 0 \\ 0 & 1 \end{bmatrix}\) and \(R = 0.01 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\) for the state and control input respectively.

To capture the inverse step response of the converter, the prediction horizon was chosen as \(N = 10\). To reduce complexity we apply move-blocking and choose the control horizon to be \(L = 4\). The objective function was minimized subject to the dynamic constraints (6) and constraints on the duty cycles and inductor current. An explicit solution was obtained consisting of 233 regions in four dimensions. The performance is evaluated in simulations in Section V below.

B. Piecewise affine model

A polyhedral piecewise affine (PWA) system [22] is defined on a polyhedral partition of the state space and/or control input space. On each region of the partition the system dynamics are described by an affine equation. The PWA system description provides a systematic tool for approximating nonlinear systems where the number of regions chosen in the partition gives a trade-off between accuracy and complexity of the model.

Before applying the PWA approximation procedure to the buck-boost converter, we introduce an auxiliary control input \(u_{1,k} := d_{1,k}v_s\) that removes the source voltage \(v_s\) from the dynamic equations. We will motivate this substitution and its implications on the optimal control problem in more detail in the next subsection IV-C.

The PWA approximation is defined over a partition of the range of the second duty cycle \(d_{2,k}\) according to

\[ x_{k+1} = A_1x_k + B_1u_{1,k} + B_{2,i}d_{2,k} + B_{w1}i_o + f_i \]

if \(d_{2,k} \in D_i, \quad i = 1, \ldots, \nu \)

\[ 0 \leq u_{1,k} \leq v_s, \quad 0 \leq d_{2,k} \leq 1 \]

(7)

where the sets \(D_i \subset [0, 1]\) are a partition of the interval \([0, 1]\). The matrices \(B_1\) and \(B_{w1}\) of the PWA approximation are defined in (5) and the parameters \(A_i, B_{2,i}, f_i\) are obtained by least squares fitting over a grid of the state space. The
parameters of the $i$th subsystem are chosen to minimize a sum of error terms

$$\|(A_1 + d_{2,j}A_2)x_j - (A_i x_j + B_{2,i} d_{2,j} + f_i)\|$$

where $(x_j, d_{2,j}) \in \mathbb{X} \times \mathbb{D}_i$ is the $j$th grid point where $\mathbb{X}$ is a subset of the state space.

It should be noted that the PWA approximation is not necessarily continuous. By choosing the affine terms $f_i$ appropriately, the system can be made continuous in the control $d_2$ at one single point. However, it will in general not be continuous in $f_2$ over a range of the state. The discontinuity is associated with undesirable phenomena such as chattering.

We consider the PWA approximation (7) of the buck-boost converter defined over the 4 regions $\mathbb{D}_1 = [0.1, 0.2], \mathbb{D}_2 = [0.2, 0.3], \mathbb{D}_3 = [0.3, 0.55], \mathbb{D}_4 = [0.55, 1]$. As in the linearized model, the state is extended with the supply voltage and load current in order to obtain a controller which is valid for a wide range of operating conditions. We pose an MPC problem using the 1-norm with penalty matrices

$$Q = \begin{bmatrix} 30 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad R = 0.01 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$  

The prediction horizon was chosen to be $N = 8$ and the control horizon was $L = 2$ to reduce complexity. The problem was solved and an explicit solution was obtained consisting of 521 regions in four dimensions. The controller is verified in simulations in Section V below.

C. Adaptive linear constrained model

The linear model introduced in subsection IV-A is valid only around the nominal operating point. The performance quickly deteriorates as the operating point gets far from the nominal operating point. The PWA model introduced in subsection IV-B is valid over a wider range of operating points. The complexity of the control problem however grows very quickly with the number of partitions of the model and the length of the prediction horizon. Moreover, as the PWA model is partitioned only with respect to the duty cycle (to keep a problem of reasonable complexity) the continuity cannot be enforced between the partitions. The resulting explicit solution is therefore PWA but not continuous. As a consequence the control inputs often suffer from chattering when close to a transition between two partitions and this also affects the quality of the output.

We propose here a model and a problem formulation that exploit the structure of DC-DC converters in order to obtain a linear controller yielding continuity of the explicit controller, high dynamic performance for a wide range of operating points and with a small number of regions.

One of the issues related to the control of the buck stage is the bilinear product between the first control input $d_{1,k}$ and the supply voltage $v_{s,k}$. For step-down buck DC-DC converters this issue has been successfully solved by scaling the full state over the supply voltage [8]. For boost converters, this approach does not solve the problem. Another more effective way to deal with the nonlinearity is to map the natural control input $d_{1,k}$ by introducing an auxiliary control input

$$u_{1,k} := d_{1,k} \cdot v_{s,k}.$$  

and replacing

$$0 \leq d_{1,k} \leq 1$$  

with the constraint

$$0 \leq u_{1,k} \leq v_{s,k}$$

which is equivalent under the (natural) assumption that $v_{s,k} \geq 0$. Following this substitution, the supply voltage $v_{s,k}$ vanishes from the dynamic equations but it appears in the input constraint. Let us highlight that the constant quantities as the boundaries of (9) are inherently part of the explicit controller structure, while the variable quantities that are required for the optimization, such as the upper boundary $v_{s,k}$ in (10) must be entered as parameter in the optimization by extending the explicit controller state $x_k$. Compared to the scaling of the state used in [8] this approach eliminates the need to perform a number of divisions to scale the state with the supply voltage. This technique can also simultaneously be applied to the boost converter stage by introducing the auxiliary control input

$$u_{2,k} := d_{2,k} \cdot x_{1,k}$$

where $x_{1,k}$ is the capacitor voltage. This new control variable replaces the natural input constraint $0 \leq d_{2,k} \leq 1$ with

$$0 \leq u_{2,k} \leq x_{1,k}.$$  

This inequality links the plant state and the control input. The controller state does not need to be extended as the upper boundary is already part of the controller state. Following these substitutions, one of the two bilinear terms featuring the capacitor voltage and the second duty cycle vanishes from the dynamic equations. The second bilinear term featuring the product between the inductor current $x_{2,k}$ and the duty cycle $d_{2,k}$ is still present in the dynamics. This term is essential as it is responsible for the power transfer between the inductor and the capacitor and it cannot be neglected. To deal with this bilinear term, we approximate the first state dynamic equations

$$x_{1,k+1} = x_{1,k} + x_{2,k} \cdot d^{20} + z_{1,k}$$ \quad (13a)

$$z_{1,k} := v_{s,k} + \left[ x_{2,k} \cdot \left( d_{2}^{2} - d^{20} \right) \right] + \begin{bmatrix} 1 & 0 \end{bmatrix} B_{w1} \cdot v_{s,k}$$ \quad (13b)

where $d^{20}$ is the nominal second duty cycle, $z_{1,k}$ is an auxiliary variable which accounts for the evolution of transition matrix over the time. The evolution of the transition matrix depends on some measurable (or observable) variables, but also on the sought duty cycle which is unknown when $z_{1,k}$ is evaluated. We can however use a target reference for $d_{2,k}$ instead, that corresponds to an equilibrium of the system. This target is computed at each sampling instant solving the steady state considering that the measured output voltage is the desired voltage.
A further refinement of our controller consists in updating the time varying part $z_{1,k}$ with the estimated load current. To this end, we replaced the resistance which is usually used to model the load by a current source considered as a bounded disturbance. The interest of this modeling approach is twofold. This model is closer to the reality as common loads often have some dynamics and are rarely purely resistive. The disturbance model is linear. An effective load current observer can therefore be synthesized and used.

Finally, a reference is computed for the state and input solving the steady state equilibrium for the reference and the disturbance [18]. As mentioned before in the paper, an infinity of pairs $(d_1, d_2)$ satisfy the steady-state equilibrium and a relation between the two control inputs must be enforced to remove this ambiguity. The equilibrium problem is solved for an auxiliary input which enters as parameter with the disturbance in a look-up table computed off-line to optimize the steady-state energy efficiency. The description of the synthesis of the observer and of the off-line computation of the input reference is out of the scope of the current paper. As we derive an explicit controller, the controller state is increased with the state and input reference

$$x_c = \begin{bmatrix} v_c & i_\ell & z_1 & v_s & v_{ref} & i_{ref} & u_{1,ref} & u_{2,ref} \end{bmatrix}^T$$

(14)

It has to be noted that the extension of the state increases the required memory space. It however does not increase the controller complexity as no new constraints are added. The problem was solved and an explicit solution was obtained consisting of 153 regions in nine dimensions. The controller is verified in simulations in Section V below.

V. SIMULATION RESULTS

The controllers derived above were evaluated in Matlab simulations where a number of situations of practical relevance are considered, see Section V-A to V-C below. The simulations were performed using the switched model (1) where the parameter values chosen are representative for a 100 kHz switching frequency buck-boost converter. The parameter values are normalized to obtain a switch period $T_s = 1$ and are expressed in the per unit system. They are $x_c = 7.3$ p.u., $x_\ell = 6.6$ p.u., $r_c = 0.015$ p.u. and $r_\ell = 0.120$ p.u.. The nominal source voltage is $v_s = 1$ p.u. and the nominal load current is $i_o = 0.067$ p.u.. The objective is to keep the output voltage at the reference value $v_{ref} = 2$ p.u. while respecting the current limit $i_{\ell,max} = 1$ p.u..

In order to have a fair comparison, the three proposed approaches are compared assuming the full state and load are measured (columns 1-3 in the plots). The load is however measured with one step delay as a disturbance change cannot be detected faster. To demonstrate the feasibility of the approaches, the third approach is also simulated in a real environment where only the capacitor and supply voltages are measured and where the other variables are estimated using an observer (column 4 in the plots).

A. Startup and load transients

The first scenario shown in Fig. 3 evaluates the performance of the different controllers for a startup at nominal values followed by two load transients. The constraints are
Fig. 4. Scenario 2: start-up transient for $v_s = 10\, \text{V}$, supply step from $v_s = 10\, \text{V}$ down to $v_s = 20\, \text{V}$

Fig. 5. Scenario 3: start-up transient for $v_s = 7\, \text{V}$

Fig. 6. Scenario 4: start-up transient for $v_s = 3\, \text{V}$
respected for the four controllers at startup. The load disturbance rejection is good for all controllers and all differences are mainly caused by different tunings.

B. Startup and supply transients

The second scenario shown in Fig. 4 evaluates the performance of the different controllers for two supply voltage transients: nominal 10 V down to 5 V at 1 ms and then up to 40 V at 1.75 ms. Both the linear and PWA controllers present a static error when the supply voltage get away from the nominal supply voltage. The linear controller presents some instability when the supply voltage is high due to excessive gains.

C. Third and fourth scenarios

The third scenario shown in Fig. 5 and fourth scenario shown in Fig. 6 evaluate the startup transient for supply voltages smaller than the nominal voltage. As the supply voltage becomes smaller, the linear controller remains stable but presents some performance degradations: the static error increases, the step response becomes slower due to a too conservative inductor current limitation. The PWA controller presents some chattering during transient and never reaches the desired operating point: this is due to discontinuities in the control model and to the model mismatches.

D. ALCM evaluation

The ALCM presents an even performance for a wide range of operating conditions. It presents the faster startup response while respecting the current constraint. The static error is small and nearly independent of the operating point. No loss of performance is observed for an observed based implementation requiring fewer measurements and presenting a computation delay.

VI. CONCLUSIONS

Three explicit model predictive control approaches based on different models and problem formulations have been applied to the non-inverting buck-boost converter. One is based on a single linearization, another on a PWA approximation. An adaptive linear constrained model controller is proposed and compared to the other approaches in simulation. The linear constrained model controller performance is very good, but deteriorates as the operating conditions get away from the nominal operating point. The PWA model controller is based on several linearizations around different operating points, thus resulting in a more accurate approximation of the dynamics. It is shown that some chattering might however occur for some operating points. This is due to discontinuities in the model that result in discontinuities in the control law, thus preventing the controller from reaching the steady state. Moreover, it is shown that for some low supply voltages, the controller might not bring the system to the desired reference voltage. This is due to joint effect of discontinuities in the control law and model mismatch. These phenomena cannot be avoided for some operating point when the map from the set of desired steady operating points in terms of reference and disturbance range to the optimal control input is discontinuous, is as the case here. The proposed adaptive linear constrained model controller solves these issues. It performs well for a wide range of operating points and compares favourably to the other approaches.

REFERENCES