STRUCTURE OF CONCURRENCY

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**Introduction**

- What is **concurrency**?
- How it can be **modelled**?
- What are the basic tools for **modelling concurrency**?
- Classical, or the most popular (top down) approaches to modelling concurrency:
  1. **Find a formal model for a system** (automata, process algebra, Petri nets, etc.).
  2. **Define a semantics for the chosen model** (sequences, partial orders, steps, temporal logic, etc.).
- Can we discuss concurrency **without** introducing the system level?
Vocabulary

- An **OBSERVATION**: a system/program run
  a sequence: \[ a \rightarrow b \rightarrow c \rightarrow d \]
  a step sequence: \[ a \rightarrow b \rightarrow c \rightarrow d \]

- an **INTERVAL** order:
  \[ \begin{array}{ccc}
  a & c & b \\
  \hline
  b & \text{beginning of } b & a \\
  \end{array} \]

- An **CONCURRENT BEHAVIOUR/CONCURRENT HISTORY**: a set of **EQUIVALENT** observations.
• **SIMULTANEITY:**
  if two event occurrences \( a \) and \( b \) are allowed, they can be executed either **IN SEQUENCE** or **SIMULTANEOUSLY**

• A formal definition of simultaneity is not obvious and depends on the meaning of events. We assume that it is somehow correctly defined or simply disallowed (impossible to observe).
CONCURRENCY PARADIGMS:
A superposition or a statement about the structure of a concurrent behaviour/history involving a treatment of SIMULTANEITY.

We have 8 paradigms \( \pi_1, \ldots, \pi_8 \), however from those eight only \( \pi_1 \), \( \pi_3 \), and \( \pi_8 \) are important.

\( \pi_8 \): the most restrictive one:

\[
(\exists a \in \Delta. x \overset{o}{\leftrightarrow} y) \iff [(\exists a \in \Delta. x \overset{o}{\rightarrow} y) \land (\exists a \in \Delta. y \overset{o}{\rightarrow} x)]
\]

where: \( \Delta \) - a concurrent behaviour/concurrent history (i.e. set of equivalent observations),

\( x \overset{o}{\leftrightarrow} y \) - in the observation \( o \), \( x \) and \( y \) are simultaneous,

\( x \overset{o}{\rightarrow} y \) - in the observation \( o \), \( x \) preceeds \( y \).
• $\pi_3$: covers most cases that are not covered by $\pi_8$:

\[
[ (\exists a \in \Delta. x \xrightarrow{o} y) \land (\exists a \in \Delta. y \xrightarrow{o} x)] \implies (\exists a \in \Delta. x \leftrightarrow y)
\]

where: $\Delta$ - a concurrent behaviour/concurrent history (i.e. set of equivalent observations),

$x \leftrightarrow y$ - in the observation $o$, $x$ and $y$ are simultaneous,

$x \xrightarrow{o} y$ - in the observation $o$, $x$ proceeds $y$.

• $\pi_1$: no restrictions at all.

\[ \pi_8 \implies \pi_3 \implies \pi_1 \]
• **Remark 1**: If simultaneity is disallowed or unobservable, the concept of concurrency paradigms **disappears**, only sequential observations do exist.

• **Remark 2**: If time is **continuous** and events are **instantaneous**, simultaneity **does not exist** as one of the fundamental principles of modern physics states: no two instantaneous events can be observed exactly at the same time. Such observation would require infinite energy.

• **Remark 3**: Each non instantaneous event \( a \), can be modelled by two instantaneous events \( Ba \) and \( Ea \), with \( Ba \nrightarrow Ea \) for each observation, where \( Ba \) is the beginning of \( a \) and \( Ea \) is the end of \( a \).

• **Do we need a theory for ono instantaneous events?**

• **YES.** Complex numbers can be modelled by pairs of reals, fractions by pairs of integers, etc., but this is not very practical way of using them.
SETS OF PARTIAL ORDERS AND THEIR INVARIANTS

- Let $\triangle$ be a set of partial orders with common domain, and let $\text{invariants}(\triangle)$ be the sets of all INVARAINTS generated by $\triangle$, where an invariant of $\triangle$ is a property (i.e. a relation) shared by all members of $\triangle$.

- All invariants can be derived from the three basic ones: $<$, $\sqsubseteq$ and $\triangleright$, interpreted as CAUSALITY, WEAK CAUSALITY and COMMUTATIVITY.

- Let $inv$, be an invariant, and let $\text{extensions}(inv)$ be the set of all partial orders that are extensions of $inv$, i.e. they satisfy all the properties of the relation $inv$. 
BASIC INVARIANTS

- **CAUSALITY RELATION:** \( a < b \)
  For each observation, \( a \) always PRECEDES \( b \).

- **WEAK CAUSALITY RELATION:** \( a \sqsubseteq b \)
  For each observation, \( a \) is always NOT LATER THAN \( b \).

- **COMMUTATIVITY RELATION:** \( a \leftrightarrow b \)
  For each observation, either \( a \) always PRECEDES \( b \), OR \( b \)
  PRECEDES \( a \), but they are NEVER executed simultaneously.

- **NOTE THAT:** \( a < b \iff a \leftrightarrow b \land a \sqsubseteq b \)

- **CAUSALITY** is always a PARTIAL ORDER, the axioms for WEAK
  CAUSALITY and COMMUTATIVITY are much more complex.

- All invariants can be derived from <, \( \sqsubseteq \) and \( \leftrightarrow \).
CONCURRENT HISTORIES

• Let $\Delta$ be a set of partial orders with common domain. The set of partial orders $\Delta^{cl}$, invariant closure of $\Delta$, is defined as:

$$\Delta^{cl} = \bigcap_{inv \in \text{invariants}(\Delta)} \text{extensions}(inv).$$

• It can be proved that

$$\Delta^{cl} = \text{extensions}(\triangleleft_{\Delta}) \cap \text{extensions}(\sqsubseteq_{\Delta})$$

or, if the paradigms $\pi_3$ holds:

$$\Delta^{cl} = \text{extensions}(\prec_{\Delta}) \cap \text{extensions}(\sqsubseteq_{\Delta})$$

• A set of partial orders with common domain $\Delta$ is called a CONCURRENT HISTORY if

$$\Delta = \Delta^{cl}$$

and the elements of $\Delta$ are interpreted as observations.
**A Simple Example**
Consider two events: $a$ and $b$.
Let \{a, b\} denote a simultaneous execution of $a$ and $b$.
Assume that both $a$ and $b$ must be executed, each of them exactly once.
How many (nonequivalent) transition systems can we construct for this problem?

- Three (or two or only one, dependently on interpretation) sequential models:
• Five (but only four really different) non-sequential models:
CASE 1

Equivalent observations:

\[ o_1 = abc \quad o_2 = acb \quad o_3 = a\{b,c\} \]

\[ o_1 = b \quad o_2 = c \quad o_3 = \]

CAUSALITY: \( < \) is equal to

- \( < \) is the intersection \( o_1 \cap o_2 \cap o_3 \),
- \( \{o_1, o_2, o_3\} \) is the set of all step sequence extensions of \( < \),
- \( \{o_1, o_2, o_3\} \) is also the set of all interval order extensions of \( < \),

CONCLUSION: \( < \) is equivalent to/fully represented by the set \( \{o_1, o_2, o_3\} \)

\( \pi_8 \) is SATISFIED in this case (also \( \pi_3 \) and \( \pi_1 \)).
CASE 2

Equivalent observations:

\[ o_1 = abc \quad \quad \quad o_3 = a\{b,c\} \]

CAUSALITY: \( < \) is equal to

WEAK CAUSALITY: \( \sqsubseteq \) is equal to
For each relation $R$, define $\hat{R}$ as $a\hat{R}b \iff \neg(bRa)$. Note that if $R$ is a total order then $R = \hat{R}$. For the Case 2 we have:

- $o_1 = \hat{o}_1 = \bullet_a \bullet_b \bullet_c$ but $o_3 = \bullet_a \bullet_b \bullet_c \overset{\sim}{\link} \bullet_c \overset{\sim}{\link} \bullet_b$.

- $\circlearrowleft$ is the intersection $o_1 \cap o_3$,
- $\circlearrowright$ is the intersection $\hat{o}_1 \cap \hat{o}_3$,
- $\{o_1, o_3\}$ is the set of extensions of the pair $(\circlearrowleft, \circlearrowright)$ in the sense that $\{o_1, o_3\}$ extends $\circlearrowleft$ and $\{\hat{o}_1, \hat{o}_3\}$ extends $\circlearrowright$.

CONCLUSION: $(\circlearrowleft, \circlearrowright)$ is equivalent to/fully represented by the set $\{o_1, o_3\}$.

$\pi_3$ is satisfied here, but $\pi_8$ is NOT.
CASE 3

Equivalent observations:

\[ o_3 = \{b, c\} \]

\[ o_3 = \]

CAUSALITY: \(<\) is equal to

\[ a \rightarrow b \rightarrow c \]

WEAK CAUSALITY: \(\sqsubseteq\) is equal to

\[ a \leftrightarrow b \leftrightarrow c \]

CONCLUSION: \((<, \sqsubseteq)\) is equivalent to/fully represented by the set \(\{o_3\}\).

\(\pi_3\) is satisfied here, but \(\pi_8\) is NOT.
CASE 4

Equivalent observations:

\[ o_1 = abc \quad o_2 = acb \]

CAUSALITY: \(<\) is equal to

WEAK CAUSALITY: \(\sqsubseteq\) is equal to \(<\)

COMMUTATIVITY: \(\triangleleft\) is equal to
\[ o_1 = \hat{o}_1 = \begin{array}{c} a \\
\downarrow b \\
\uparrow c \end{array} \quad \text{and} \quad o_2 = \hat{o}_2 = \begin{array}{c} a \\
\downarrow c \\
\uparrow b \end{array} \]

- \( \sqsubset \) is the intersection \( \hat{o}_1 \cap \hat{o}_2 \),
- \( \langle \rangle \) is the intersection \((o_1 \cup o_1^{-1}) \cap (o_2 \cup o_2^{-1})\)
- \( \{o_1, o_2\} \) is the set of extensions of the pair \( \langle \rangle, \sqsubset \) in the sense that \( \{\hat{o}_1, \hat{o}_2\} \) extends \( \sqsubset \) and \( \{o_1 \cup o_1^{-1}, o_2 \cup o_2^{-1}\} \) extends \( \langle \rangle \).

**CONCLUSION:** \( \langle \rangle, \sqsubset \) is equivalent to/fully represented by the set \( \{o_1, o_2\} \).

Only \( \pi_1 \) is satisfied, \( \pi_3 \) and \( \pi_8 \) are NOT!
AXIOMS FOR CAUSALITY ($<$), WEAK CAUSALITY ($\sqsubseteq$) and COMMUTATIVITY ($\Leftrightarrow$)

- If the paradigm $\pi_3$ holds then $\Leftrightarrow = < \cup <^{-1}$.

- Observations are SEQUENCES (TOTAL ORDERS), i.e.
  Simultaneity is not allowed:
  
  < is a PARTIAL ORDER
  $\sqsubseteq$ is equal to $<$ and $\Leftrightarrow$ is equal to $< \cup <^{-1}$

- Observations are STEP-SEQUENCES (STRATIFIED ORDERS), i.e.
  Simultaneity is an equivalence relation:

  S1. $a \not\sqsubseteq a$
  S2. $a < b \implies a \sqsubseteq b$
  S3. $a \sqsubseteq b \sqsubseteq c \implies a \sqsubseteq c \lor a = c$
  S4. $a \sqsubseteq b < c \lor a < b \sqsubseteq c \implies a < c$
  S5. $a < b \iff a \Leftrightarrow b \land a \sqsubseteq b$ (needed only if $\pi_3$ does not hold).
• Observations are INTERVAL ORDERS

I1. \( a \not\leq a \)
I2. \( a < b \iff a \sqsubseteq b \)
I3. \( a < b < c \iff a < c \)
I4. \( a \sqsubseteq b < c \vee a < b \sqsubseteq c \iff a \sqsubseteq c \)
I5. \( a < b \sqsubseteq c < d \implies a < d \)
I6. \( a \sqsubseteq b < c \sqsubseteq d \implies a \sqsubseteq d \vee a = d \)
I7. \( a < b \iff a \not\leq b \land a \sqsubseteq b \) (needed only if \( \pi_3 \) does not hold).

• Observations are PARTIAL ORDERS

P1. \( a \not\leq a \)
P2. \( a < b \implies a \sqsubseteq b \)
P3. \( a < b < c \implies a < c \)
P4. \( a \sqsubseteq b < c \vee a < b \sqsubseteq c \implies a \sqsubseteq c \)
P5. \( a < b \iff a \not\leq b \land a \sqsubseteq b \) (needed only if \( \pi_3 \) does not hold).
COCLUSION

- A concurrent history, i.e. a set of partial orders $\Delta$ such that $\Delta = \Delta^{cl}$, is always entirely defined by its invariants, $\{\sqsubset, \sqsupset\}$, i.e. commutativity and weak causality.

- Dependent on the restrictions on the kind of partial orders in $\Delta$, the relations $\prec, \sqsubseteq$ and $\sqsubset$, satisfy axioms S1-S5, I1-I7, or P1-P5.

- If $\Delta$ conforms to the paradigm $\pi_3$, then $\Delta$ is entirely defined by its invariants, $\{\prec, \sqsubseteq\}$, i.e. causality and weak causality.
THANK YOU!