Improved Parameterized Algorithms for Constraint Satisfaction

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Abstract

Results from inapproximability provide several sharp thresholds on the approximability of important optimization problems. We give several improved parameterized algorithms for solving constraint satisfaction problems above a tight threshold. Our results include the following:

- **Improved algorithms for any Constraint Satisfaction Problem.** Take any boolean Max-CSP with at most $c$ variables per constraint such that a random assignment satisfies a constraint with probability $p$. There is an algorithm such that for every instance of the problem with $m$ constraints, the algorithm decides whether at least $pm + k$ constraints can be satisfied in $O(2^{(c(c+1)/2)k}m)$ time. This improves on results of [Alon et al., SODA 2010] who gave a $2^{O(k^2)} + m^{O(1)}$ algorithm, and [Crowston et al., SWAT 2010] who gave a $2^{O(k \log k)} + m^{O(1)}$ algorithm. We observe that an $O(2^{\varepsilon k + \varepsilon m})$ time algorithm for every $\varepsilon > 0$ would imply that 3SAT is in subexponential time, so it seems unlikely that our runtime dependence on $k$ can be significantly improved. Our proof also shows that every Max-$c$-CSP has a linear kernel (of at most $c(c+1)k/2$ variables) under this parameterization, and that every Max-$c$-CSP admits a nontrivial hybrid algorithm.

- **Better algorithms for problems approximable by SDP.** There is an algorithm for Max Cut such that for every graph with $m$ edges, the algorithm outputs a cut with value at least $\alpha OPT + k$ in $2^{O(k)}m$ time (provided such a cut exists), where $\alpha = .878\ldots$ is the Goemans-Williamson constant. Also there is an algorithm for Max-2-SAT such that for 2CNF with $m$ clauses, the algorithm outputs an assignment with value at least $\alpha OPT + k$ in $2^{O(k)}m$ time (provided such an assignment exists), where $\alpha = .940\ldots$ is the Lewin et al. constant. We present these results based on a more general claim for Max-2-CSP.

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1 Introduction

Many interesting optimization problems exhibit an hardness threshold, where it is relatively easy to obtain a feasible solution with a value that is a certain fraction of the optimum, yet it is difficult to find a feasible solution that is even slightly better. Such threshold phenomena have been exhibited for many problems (examples include \[8\] [13] [21] [22] [25]). These thresholds are naturally quantified using the language of approximability: for many optimization problems, it is known that there is some \(\alpha\) (possibly a function of the input) for which an \(\alpha\)-approximation can be found in polynomial time, but the problem is \(\alpha(1 - \varepsilon)\)-inapproximable in polynomial time (for all \(\varepsilon > 0\)) unless \(\text{NP}\) has more efficient algorithms.\(^1\) Perhaps the most well-known example is \(\text{MAX-E3-SAT}\), where we are given a \(\text{CNF}\) formula in which all clauses have exactly three literals, and wish to find a variable assignment satisfying a maximum number of clauses. It is easy to satisfy \(7/8\) of the clauses of any such instance (this is the expected fraction satisfied by a uniform random assignment), and Hästad \([21]\) proved that it is \(\text{NP}\)-hard to satisfy \(7/8 + \varepsilon\) for every \(\varepsilon > 0\). These threshold phenomena are fascinating in that they provide a sharp boundary between feasibility and infeasibility: while the average is easy to obtain, satisfying an “above average” fraction is intractable.

While the identification of such thresholds is extremely interesting, it does not end the story but rather initiates another one. As it is likely that practitioners will require feasible solutions that exceed these easy thresholds, it is important to understand how much computational effort is required to solve a problem beyond its threshold. One way of cleanly formalizing this question uses parameterized complexity.

Suppose you have a favorite \(\text{poly}(n)\)-time algorithm \(A\) for an optimization problem which on every instance \(I\) can output feasible solutions with a high value \(v(I)\). The general question we ask is: under what conditions can \(A\) be converted into an “anytime algorithm”? When can we modify \(A\) so that, after running for \(f(k)\text{poly}(n)\) time, \(A(I)\) either reports a feasible solution of value at least \(v(I) + k\), or concludes that none exist? That is, as one increases the time budget indefinitely, one obtains progressively better solutions in a parameterized way. We say one can parameterize above \(A\) when such a modification exists.

Observe that this formulation captures problems with a known hardness threshold: for \(\text{MAX-E3-SAT}\), we are asking if we can parameterize above the random assignment algorithm. The question is also sensible for problem areas where tight hardness thresholds are still open. Currently the richest such domain in algorithms is semi-definite programming, where many good approximation algorithms are known but tight hardness results remain elusive. Parameterizing above \(\text{SDP}\) rounding is a good way to measure the time-solution tradeoffs that are possible in this paradigm. In general, the question is most interesting when we have a favorite hammer \(A\) and want to test how hard it can really hit.

A related but less general approach to coping with problem hardness was introduced by Mahajan and Raman \([28]\) in 1997. Motivated by the fact that every graph has a cut of size at least \(m/2\) and every \(\text{CNF}\) formula has an assignment satisfying \(m/2\) clauses, they showed one can determine if a graph has a cut of size \(m/2 + k\) in \(2^{O(k)} m^{O(1)}\) time, and similarly one can determine if a \(\text{CNF}\) formula has an assignment satisfying \(m/2 + k\) clauses. In later work \([29]\), Mahajan, Raman and Sikdar observed several problems whose ‘easy’ value \(v\) is tight in the sense that for infinitely many instances the fraction of \(v\) in the instance size is optimal. For example, the \(\text{MAX-E3-SAT}\) problem exhibits an ‘easy’ value \(7m/8\), which is also optimal for infinitely many instances. Only a few nontrivial results were known about parameterization of this kind until recently \([19] [1] [9] [20]\).

In last year’s SODA, Alon et al. \([1]\) finally showed that \(\text{MAX-c-SAT}\) parameterized above \((1 - 1/2^c)m\) is fixed-parameter tractable, giving a \(2^{O(k^2)} + O(m)\) time algorithm. This was subsequently improved by Crowston et al. \([9]\) who gave a \(2^{O(k \log k)} + O(m)\) algorithm.

\(^1\)Here we are using the performance ratio notation for approximation, where approximation ratios are always at least 1. From here on, we will use the more typical ratio of Solution Value Provided by Best Algorithm over Optimal Solution Value, which means that hard maximization problems have approximation ratios less than 1.
1.1 Our Results

We improve on several results from prior work, and introduce a new type of parameterization for constraint satisfaction problems. For Boolean $c$-CSPs, we show that one can always parameterize above the random assignment algorithm in only $2^{O(k)}m$ time. For Boolean 2-CSPs, we show that one can efficiently parameterize in $2^{O(k)}\text{poly}(n)$ time above the worst-case guarantees of semi-definite programming algorithms. Both results can be formulated as parameterization above $A$: In the first case $A$ is a uniform random assignment algorithm, and in the second case $A$ is an appropriate SDP rounding algorithm.

**Efficient Parameterization Above Average for Every Boolean $c$-CSP.** We show that every $c$-CSP admits an $2^{O(k)}m$ time algorithm for finding a solution that satisfies more than $k/2^c$ constraints above the average. This dependence on $k$ improves over the main results of Alon et al. [1] and Crowston et al. [9].

The key to our results is an algorithm for the following problem:

**MAX-$c$-LIN-2 Above Average:** We are given a parameter $k$ and a system of linear equations over $\mathbb{F}_2$ with at most $c$ variables per equation. Each equation $e$ has a positive integer weight, and the weight of an assignment in the system is defined to be the total sum of weights of equations satisfied by the assignment. Determine if there is an assignment with weight at least $W/2 + k/2$, where $W$ is the total weight of all equations.

Note the random assignment algorithm yields $W/2$ weight, and it is NP-hard to attain $W/2 + \varepsilon W$ for every $\varepsilon > 0$ [21].

**Theorem 1.** For every $c \geq 2$, MAX-$c$-LIN-2 Above Average can be solved in $O(2^{(c\varepsilon +1)/2}k \cdot m)$ time.

We observe that a running time of $2^{O(k)}m$ is optimal up to constants in the exponent, unless there are subexponential time algorithms for many NP-complete problems:

**Theorem 2.** If MAX-3-LIN-2 Above Average can be solved in $O(2^k \cdot 2^m)$ time for every $\varepsilon > 0$, then 3SAT can be solved in $O(2^{3n})$ time for every $\delta > 0$, where $n$ is the number of variables.

That is, if there is an algorithm with only a subexponential dependence in $k$, then the Exponential Time Hypothesis is false. Hence a significantly faster algorithm would entail a breakthrough.

Using the generality of MAX-$c$-LIN-2 Above Average, the algorithm of Theorem 1 can be used to solve any constraint satisfaction problem efficiently in the above-average setting.

**MAX-$c$-CSP Above Average:** We are given a parameter $k$ and a set of arbitrary constraints with at most $c$ variables per constraint. Each constraint has a positive integer weight. Determine if there is an assignment with weight at least $\rho \cdot W + k/2^c$, where $W$ is the total weight of all equations and $\rho$ is the expected fraction of weighted constraints satisfied by a uniform random assignment.

**Theorem 3.** For every $c \geq 2$, MAX-$c$-CSP Above Average can be solved in $O(2^{(c\varepsilon +1)/2}k \cdot m)$ time.

As a byproduct, we show MAX-$c$-LIN-2 Above Average and MAX-$c$-CSP Above Average admit a polynomial-time preprocessing algorithm into a problem kernel with $O(k)$ variables. Our proofs imply the stronger result that every constraint satisfaction problem admits a hybrid algorithm, in the following sense:

**Corollary 1.** For every Boolean MAX-$c$-CSP, there is an algorithm with the property that, for every $\varepsilon > 0$, on any instance $I$, the algorithm outputs either:

- an optimal solution to $I$ within $O^*(2^{(c\varepsilon +1)/2}m)$ time, or
- a $(\rho + O(\varepsilon^2))$-approximation to $I$ within polynomial time, where $\rho$ is the expected fraction of weighted constraints satisfied by a uniform random assignment to the CSP.

This resolves an open problem of Vassilevska, Williams, and Woo [32], who asked if MAX-3-SAT had an algorithm of this form.
Parameterizing Above Worst-Case Guarantees for Max-2-CSP. As mentioned before, Mahajan and Raman [28] showed one can determine if there is a cut with at least \( m/2 + k \) edges in \( 2^{O(k)}m \) time. This algorithm is useful when the maximum cut is close to half the edges, but it becomes quite costly for graphs with larger cuts. It is well-known that often one can efficiently find cuts much larger than \( m/2 \), for Goemans and Williamson [17] showed that Max Cut can be approximated within .878 of the optimum in polynomial time using semi-definite programming. In a technical tour de force, this .878 factor was proven to be the best possible for general graphs, assuming the Unique Games Conjecture (UGC) from approximation complexity [25]. It is not known whether .878 · \( \text{OPT} \) is truly a tight hardness threshold for approximating the maximum cut (where \( \text{OPT} \) is the optimal cut value), but the prevailing wisdom is that it looks very tough to improve upon. Indeed, the analyses of the Goemans-Williamson algorithm and variants are known to be quite tight for infinitely many graphs [24, 2, 15], and it is even known that one cannot improve its performance by choosing many random hyperplanes and taking the best [25].

Therefore it is of interest to study the following parameterized problem:

\[
\text{Max Cut Above .878}: \text{Given a graph and parameter } k, \text{ find a cut in the graph for which at least } .878 \cdot \text{OPT} + k \text{ edges cross, where OPT is the optimal cut value. (Alternatively, conclude that no such cut exists.)}
\]

We prove that finding a cut with \( k \) edges above the .878-approximation is fixed-parameter tractable:

**Theorem 4.** Max Cut Above .878 can be solved in \( O(2^{8.24k}\text{poly}(n)) \) time. More precisely, there is a randomized parameterized algorithm with the following properties on every graph \( G \):

- For all \( k \in [1, 122 \cdot \text{OPT}(G)] \), the algorithm outputs a cut of \( G \) with value at least .878 · \( \text{OPT}(G) + k \), with high probability.
- For all \( k > 122 \cdot \text{OPT}(G) \), the algorithm always outputs NO.

Moreover, there is a deterministic algorithm which runs in \( O(2^{16.4k}\text{poly}(n)) \) time.

In fact, a version of Theorem 4 is possible for every Max-2-CSP. Define:

**Max-2-CSP Above \( \alpha \): Given a Max-2-CSP instance \( F \) and parameter \( k \), find an assignment of \( F \) for which at least \( \alpha \cdot \text{OPT} + k \) constraints are satisfied, where \( \text{OPT} \) is the optimal value of an assignment. (Alternatively, conclude that no such assignment exists.)**

In fact, Theorem 4 can be derived from a more general result for Max-2-CSP Above \( \alpha \).

**Theorem 5.** Suppose that for a given constraint family \( \Lambda \), there is a polynomial-time algorithm \( A \) which is a local \( \alpha \)-approximation for Max-2-CSP(\( \Lambda \)). Then Max-2-CSP Above \( \alpha \) can be solved in \( O(r^k_{\rho, \alpha} \text{poly}(n)) \leq 2^{O(k)} \text{poly}(n) \) time. More precisely, there is a randomized parameterized algorithm with the following properties on each instance \( F \):

- For all \( k \in [1, (1 - \alpha) \cdot \text{OPT}(F)] \), the algorithm outputs an assignment satisfying at least \( \alpha \cdot \text{OPT}(F) + k \) constraints, with high probability.
- For all \( k > (1 - \alpha) \cdot \text{OPT}(F) \), the algorithm always outputs NO.

\(^2\)The precise approximation factor is the minimum of \( \frac{\pi}{\theta} \) over all \( \theta \in [0, \pi] \). Throughout the paper, we use .878 to represent this constant succinctly.

\(^3\)Note there is some recent algorithmic evidence [3, 26] that the Unique Games Conjecture may not be true. Regardless of the truth of UGC, it may still be that a .878-approximation for Max Cut is the best we can hope for in polynomial time: the falsity of UGC is not known to imply a better Max Cut algorithm.

\(^4\)Note that here, .122 is a placeholder for \( 1 - \alpha \) where \( \alpha = .878567 \ldots \) is the Goemans-Williamson constant.
Here $\rho$ is the expected fraction of weighted constraints satisfied by a uniform random assignment to $F$, and $r_{\rho, \alpha} = \min_{d \geq 1} \{ (\rho^{-1}-\alpha)^{-d} \cdot 2^{d/(1-\alpha)^2} \}$. Moreover, there is a deterministic algorithm with overhead of $(\rho^{-r} \cdot (1-\rho)^{-1} \cdot \rho^{-\alpha})^{\frac{k}{n-\alpha}}$ in running time.

Informally, a local $\alpha$-approximation is an SDP-based randomized rounding $\alpha$-approximation algorithm that admits a worst-case analysis of the expected number of constraints satisfied based on linearity of expectation. Theorem 5 shows that for every such rounding algorithm, we can satisfy $k$ constraints beyond the worst-case approximation ratio in $2^{O(k)} \text{poly}(n)$ time.

1.2 Related Work

Besides the related work mentioned in the introduction, there has been also work in developing subexponential approximation algorithms. The idea of accepting subexponential running time in exchange for better approximation guarantees already appeared in the long history of approximating the permanent [23]. In recent years, the study of subexponential approximation to circumvent the limits of polynomial approximation has received active attention, [11, 5, 10] for example.

The direction of this paper (and in general, the prior work on parameterized algorithms above tight lower bounds) is rather different in spirit from subexponential approximations, and arguably much stronger. When our algorithms return an answer, they determine whether or not there is a feasible solution with a value that is a positive integer, every $S_i$ is an ordered tuple from $[n] \cup [n]^2 \cup \cdots [n]^c$, and $f_i : \{0,1\}^{[S_i]} \rightarrow \{0,1\}$ for all $i$. A variable assignment $A : [n] \rightarrow \{0,1\}$ satisfies a constraint $(f_i, (s_1, \ldots, s_{c'}), w_i)$ provided that $f_i(A(s_1), \ldots, A(s_{c'})) = 1$. The weight of an assignment $A$ is the sum of $w_i$ over all constraints $(f_i, S_i, w_i)$ that are satisfied by $A$. Throughout the paper, the MAX-$c$-CSP problems under consideration take boolean variables.

2 MAX-$c$-CSP Above Average

We now turn to describing improved parameterized algorithms for maximum constraint satisfaction problems with a constant number of variables per constraint, including the problems of satisfying a maximum subset of linear equations and maximum CNF satisfiability. At the heart of our approach is a faster algorithm for Max-$c$-Lin-2 Above Average that can be applied in a general way to solve other CSPs.
Reminder of Theorem 1. For every $c \geq 2$, Max-$c$-LIN-2 Above Average can be solved in $O(2^{(c(c+1)/2)k}.m)$ time.

In [32], the authors gave a “hybrid algorithm” for the unweighted problem Max-E3-Lin-2 (where exactly three variables appear in each equation), with the property that, after a polynomial time test of the instance, the algorithm either

- outputs an assignment satisfying $(1/2 + \varepsilon)m$ equations in polynomial time, or
- outputs the optimal satisfying assignment in $2^{O(em)}$ time.

Their algorithm works by finding a maximal subset of equations such that every pair of equations share no variables; based on the size of this set, the hybrid algorithm decides to either approximately solve the instance or solve it exactly. Our algorithm is in a similar spirit, but requires several modifications to yield a parameterized algorithm for the weighted case, to deal with any $c \geq 2$, and to deal with “mixed” equations that can have different numbers of variables.

Let $F$ be a set of equations over $F_2$, where each equation $e$ contains at most $c$ variables and has a positive integral weight $w(e)$. For a single equation $e \in F$, let $\text{var}(e)$ be the set of all variables appearing in $e$. Let $\text{var}(F) = \bigcup_{e \in F} \text{var}(e)$. For a set of equations $F'$, the weight $w(F')$ is the sum of weights $w(e)$ over $e \in F'$. The weight of an assignment is said to be the total weight of equations the assignment satisfies.

Note that Max-2-Lin-2 Above Average is a generalization of Max Cut Above Average on weighted graphs: by simulating each edge $\{u, v\}$ of weight $w$ with an equation $x_u + x_v = 1$ of weight $w$, the Max-2-Lin-2 problem easily captures Max Cut.

Proof of Theorem 1. We assume that the given instance is reduced in the sense that there is no pair of equations $e, e'$ with $e \equiv e' + 1 \pmod{2}$. (Such an equation $e$ is said to be degenerate in [32].) If such a pair exists, one can remove the equation of lesser weight (call it $e'$) and subtract $w(e')$ from $w(e)$. Note the weight of every variable assignment has now been subtracted by $w(e')$.

It is convenient to view an equation $e$ as a set $\text{var}(e)$. We first find a maximal independent (i.e. disjoint) collection $S_c \subseteq F$ of $c$-sets. All remaining equations in $F$ now have at most $c - 1$ variables if we ignore the variables of $\text{var}(S_c)$. Next, we pick another collection $S_{c-1} \subseteq F$ of sets with the property that, after we remove all variables in $\text{var}(S_c)$ from $F$, $S_{c-1}$ forms a maximally independent collection of $(c - 1)$-sets in the remaining set system. In general, for $j = c - 2$ down to 1, once the variables in $\text{var}(S_c \cup \cdots \cup S_{j+1})$ have been removed from the remaining equations, a maximal independent set of $j$-sets is chosen and we set $S_j$ to be a collection of corresponding original sets in $F$ (with the variables in $\text{var}(S_c \cup \cdots \cup S_{j+1})$ added back). We continue until $S_1$, in which each set in the collection has exactly one variable after those in $\text{var}(S_c \cup \cdots \cup S_2)$ have been removed. For notational convenience, let $S_{c+1} = \text{var}(S_c) = \emptyset$. By properties of maximal disjoint sets, we have:

Observation 1. For every $1 \leq j \leq c$, eliminating the variables appearing in $\text{var}(S_c \cup \cdots \cup S_{j+1})$ leaves at most $j$ variables in every equation.

Now we consider two cases: either (1) $w(S_j) < k$ for every $j = 1, \ldots, c$, or (2) there is a $j$ such that $w(S_j) \geq k$.

Case (1) is easily handled: for every $j$, each equation in $S_j$ contains $j$ variables which do not appear in $S_c \cup \cdots \cup S_{j+1}$. Hence, $|\text{var}(F)| = |\text{var}(\bigcup_{i=1}^{c} S_i)| < ck + (c - 1)k + \cdots + k < (c(c+1)/2) \cdot k$. By trying all possible $2^{(c(c+1)/2)k}$ assignments to $\text{var}(F)$, we can find an optimal assignment for $F$.

Case (2) is more delicate and is handled by the two claims below. Recall an equation $e \in F$ is non-degenerate if there is no $e' \in F$ such that $e \equiv e' + 1 \pmod{2}$. As mentioned earlier, we may assume without loss of generality that every equation in $F$ is non-degenerate.

Claim 1. For every $1 \leq j \leq c$, a random assignment satisfying all equations in $S_j$ will satisfy every non-degenerate equation in $F - S_j$ with probability $1/2$. Moreover, we can output such a random assignment in polynomial time.
Proof. To prove the first part of the claim, it suffices to show that no equation $e \in F - S_j$ (or its negation $e + 1$) can be expressed as a linear combination of one or more equations in $S_j$. Put another way, we will show that every equation in $e \in F - S_j$ is linearly independent of the equations in $S_j$.

Suppose there exist some sets $e_1, \ldots, e_m$ from $S_j$ whose modulo sum is the same as another set $e \in F$. That is, viewing $e_1, \ldots, e_m$ and $e$ as indicator vectors of length $n$ (one bit for each of the $n$ variables, omitting the constant terms in the equations), there is a subset $T \subseteq [m]$ such that $e = \sum_{i \in T} e_i \pmod{2}$. Recall that every equation in $S_j$ has $j$ variables which do not appear in $\text{var}(S_c \cup \cdots \cup S_{j+1})$ and moreover they are all distinct by the construction of $S_j$. Hence, if $m > 1$ then the equation $e$ has more than $j$ variables which do not appear in $\text{var}(S_c \cup \cdots \cup S_{j+1})$, which is impossible by Observation 1. Therefore $m = 1$.

Thus any subset \{e_1, \ldots, e_m\} of equations from $S_j$ whose modulo sum is the same as another equation $e \in F - S_j$ has cardinality 1. Hence equation $e$ is degenerate, which is a contradiction to the non-degeneracy assumption. Therefore no equation (or its negation) can be represented as a linear combination of one or more equations from $S_j$.

Now, given that every equation in $e \in F - S_j$ is linearly independent of the equations in $S_j$, we claim that a random assignment that is consistent with $S_j$ will satisfy $e$ with probability 1/2. This is a simple consequence of linear algebra over $\mathbb{F}_2$. Put the system of equations $S_j$ in the form $Ax = b$, where $A \in \mathbb{F}_2^{[S_j] \times n}$, $x \in \mathbb{F}_2^n$, and $b \in \mathbb{F}_2^{[S_j]}$. Let $e \in F - S_j$. Define $B_e \in \mathbb{F}_2^{[|S_j|+1] \times n}$ to be identical to $A$ in its first $|S_j|$ rows, and in the last row, $B_e$ contains the indicator vector for the variables of $e$. Define $c_e \in \mathbb{F}_2^{[S_j] + 1}$ to be identical to $b$ in its first $|S_j|$ components, and $c$ contains the constant term of $e$ in its last component. Saying that $e \in F - S_j$ is linearly independent of $S_j$ is equivalent to saying $\text{rowrank}(B_e) = \text{rowrank}(A) + 1$, and the set of solutions to $Ax = b$ contains the set of solutions to $B_e x = c_e$. The number of solutions to a system of rank $r$ is $2^{n-r}$. Therefore a uniform random variable assignment that satisfies $Ax = b$ will also satisfy $B_e x = c_e$ with probability 1/2.

Finally, producing a uniform random assignment over all assignments that satisfy $S_j$ is easily done. We first produce a random assignment to the variables in $\text{var}(S_{j+1} \cup \cdots \cup S_{j+1})$, then produce a random assignment to those variables in the maximal independent collection of $j$-sets obtained after removing $\text{var}(S_{j+1} \cup \cdots \cup S_{j+1})$, in such a way that every equation in $S_j$ is satisfied. (Exactly one variable in each equation of $S_j$ will be “forced” to be a certain value, but note that none of these forced variables appear in more than one equation of $S_j$, by construction.) The remaining variables are set to 0 or 1 uniformly at random. Note that if $j = 1$ and some equation $e \in S_1$ has $|\text{var}(e)| = 1$, the assignment to the variable of $e$ is decided uniquely.

Claim 2. If there is a $j$ with $w(S_j) \geq k$, then we can find an assignment with weight at least $W/2 + k/2$ in polynomial time.

Proof. Suppose that $j \geq 1$ is the largest integer with $w(S_j) \geq k$. By Claim 1 a random assignment satisfying all equations in $S_j$ will satisfy every other non-degenerate equation with probability 1/2. Hence the weight of such an assignment is at least $(W - w(S_j))/2 + w(S_j) \geq W/2 + k/2$ in average. An assignment can also be found deterministically using conditional expectation.

This completes the proof of Theorem 1.

Observe that this running time of our algorithm is optimal up to constant factors in the exponent, assuming the Exponential Time Hypothesis:

**Reminder of Theorem** [2] If Max-3-Lin-2 Above Average can be solved in $O(2^{k(2^{\gamma m})})$ time for all $\varepsilon > 0$, then 3SAT can be solved in $O(2^{\delta n})$ time for all $\delta > 0$, where $n$ is the number of variables.

Proof. First, by the improved Sparsification Lemma of [6], for every $\delta > 0$ we can reduce 3SAT on $n$ variables and $m$ clauses in $2^{\delta n}$ time to 3SAT on $n$ variables and $m' = (1/\delta)^\gamma n$ clauses, for some fixed constant $c > 1$. This 3SAT instance on $n$ variables and $m'$ clauses can further be reduced to Max-3-Lin-2 on $n$ variables and $O(m')$ clauses using the reduction of Lemma [1] (proved below). Provided that we can determine whether
m'/2 + k/2 equations can be satisfied in \(2^k 2^{m'}\) time, then by trying each \(k\) in the interval \([1, m']\) we can solve the Max-3-Lin-2 instance exactly in at most \(2^{2m'} \leq O(2^{2e(1/\delta')^c})\) time.

This results in an \(O(2^{5n+2e(1/\delta')^c})\) algorithm for 3SAT. Setting \(\varepsilon = \delta^{c+1}\), we obtain \(O(2^{3\delta n})\) time. As this reduction works for every \(\delta > 0\), the conclusion follows. \(\square\)

To apply our Max-\(c\)-Lin-2 algorithm to general constraint satisfaction problems, we use the following reduction.

**Lemma 1.** There is a polynomial time reduction from Max-\(c\)-Csp Above Average with \(n\) variables and parameter \(k\) to Max-\(c\)-Lin-2 Above Average with \(n\) variables and parameter \(k\).

The proof of the lemma for unweighted \(c\)-CSP is sketched in [1], and a full proof is given in [9]. Here we give an alternative proof which also covers the weighted case. (Although we are confident that the proofs in [1,9] also extend to the weighted case, we include our proof for completeness.)

**Proof of Lemma 1.** Let \(F = \{(f_1, S_1, w_1), \ldots, (f_m, S_m, w_m)\}\) be a given set of constraints and let \(\text{AVG}\) be the expected weight in \(F\) of a uniform random variable assignment. (For definitions, see the Preliminaries.) First, convert each constraint function \(f_i : \{0, 1\}^{\mid S_i\mid} \rightarrow \{0, 1\}\) into the form \(f_i' : \{-1, 1\}^{\mid S_i\mid} \rightarrow \{0, 1\}\), replacing all 0’s with 1’s and all 1’s with \(-1\). This can be easily done via the linear transformation \(\ell(x) = 1 - 2x\). Let \(F'\) be the new set of constraints.

Let \(s_i = \mid S_i\). A well-known fact is that for any function \(f_i : \{-1, 1\}^{s_i} \rightarrow \{0, 1\}\) there is always a unique multivariate polynomial \(p_i(x_1, \ldots, x_{s_i})\) such that for all assignments \(\vec{a} \in \{-1, 1\}^{s_i}\), \(p_i(\vec{a}) = f_i(\vec{a})\). This polynomial has the form

\[
p_i(x_1, \ldots, x_{s_i}) = \sum_{T \subseteq [s_i]} \alpha_T^{(i)} \left( \prod_{i \in T} x_i \right),
\]

where every \(\alpha_T^{(i)} = j/2^{s_i}\) for some integer \(j \in [-2^{s_i}, 2^{s_i}]\). Define \(q_i(x) = p_i(x) - \alpha_\emptyset^{(i)}\). That is, \(q_i\) equals \(p_i\) minus the constant coefficient of \(p_i\).

We claim that for all assignments \(\vec{a} = (a_1, \ldots, a_n) \in \{0, 1\}^n\), \(\vec{a}\) has weight at least \(\text{AVG} + k\) in \(F\) if and only if \(\sum_{i=1}^m w_i \cdot q_i(\ell(a_1), \ldots, \ell(a_n)) \geq k\). To see this, observe that the sum of all constant coefficients of \(p_i\) multiplied by the weight \(w_i\) is exactly \(\text{AVG}\), since this sum is

\[
\sum_{i=1}^m w_i \cdot \alpha_\emptyset^{(i)} = \sum_{i=1}^m w_i \cdot \mathbb{E}_{x \in \{-1, 1\}^n} [p_i(x) - q_i(x)] = \text{AVG} - \sum_{i=1}^m w_i \cdot \mathbb{E}_{x \in \{-1, 1\}^n} [q_i(x)]
\]

\[
= \text{AVG} - \sum_{i=1}^m w_i \cdot \sum_{T \neq \emptyset} \alpha_T^{(i)} \mathbb{E}_{x \in \{-1, 1\}^n} \left[ \prod_{i \in T} x_i \right] = \text{AVG},
\]

where the last equality follows because any nontrivial product of random variables over \(-1, 1\) has expectation zero. Hence a \(\{0, 1\}\)-assignment with weight \(w\) in \(F\) translates directly to a \(\{-1, 1\}\)-assignment that makes \(\sum_i w_i \cdot q_i = w - \text{AVG}\).

Now we reduce the problem of finding a \(\{-1, 1\}\)-assignment such that \(\sum_i q_i \geq k/2^c\) to finding a \(\{0, 1\}\)-assignment to a Max-\(c\)-Lin-2 instance with weight at least \(W/2 + k/2\). Let \(r(x_1, \ldots, x_n) = \sum_i q_i(x_1, \ldots, x_n)\).

Associate each monomial in \(r(x)\) with a linear equation \(e(T) = 0\) of weight \(2^c \cdot \sum_{i=1}^m w_i \cdot e_T^{(i)}\), if this quantity is positive. If this quantity is negative, then associate with the equation \(e(T) = 1\) of weight \(-2^c \cdot \sum_{i=1}^m w_i \cdot e_T^{(i)}\). Here \(e(T) = \sum_{i \in T} y_i\). Notice that we need the \(2^c\) factor in the weights in order to make them integral. For any \(\{-1, 1\}\)-assignment \(x\), take \(y_i = \ell^{-1}(x_i)\) and note that \(x_i = (-1)^{y_i}\). Hence

\[
2^c \cdot \sum_{i=1}^m w_i \cdot e_T^{(i)} \left( \prod_{i \in T} x_i \right) = 2^c \cdot \sum_{i=1}^m w_i \cdot e_T^{(i)} (-1)^{e(T)}
\]
equals the weight of the corresponding equation \( e(T) \) if \( y \) satisfies it, and equals the negative of the weight of \( e(T) \) if \( y \) falsifies it. Lastly it remains to observe that any \( \{0,1\} \)-assignment \( y \) to a Max-\( c \)-Lin-2 instance has a weight at least \( W/2 + k/2 \) if and only if the weight of satisfied equations minus the weight of falsified equations by \( y \) is at least \( k \). This shows that if there exists a \( \{-1,1\} \)-assignment such that \( \sum_i w_i \cdot q_i \geq k/2^c \), there is a \( \{0,1\} \)-assignment to the corresponding Max-\( c \)-Lin-2 instance with weight at least \( W/2 + k/2 \). The proof of the opposite direction is straightforward.

In the above construction, Max-\( c \)-CSP can be reduced to Max-\( c \)-Lin-2 in \( O(2^c \cdot m) \) time (where \( m \) is the number of constraints) and the number \( m' \) of equations in the transformed instance will be \( O(2^c \cdot m) \) in the worst case.

**Reminder of Theorem** 8. For every \( c \geq 2 \), Max-\( c \)-CSP Above Average can be solved in \( O(2^{(c(c+1)/2)k} \cdot m) \) time.

**Proof of Theorem** 8. Using the reduction of Lemma 1 reduce an instance of Max-\( c \)-CSP Above Average with \( m \) constraints to Max-\( c \)-Lin-2 Above Average with \( O(2^c \cdot m) \) equations. Using the algorithm of Theorem 1 we solve the obtained instance of Max-\( c \)-Lin-2 Above Average. Thus we can determine if the given c-CSP has an assignment with weight at least \( \text{AVG} + k \) in \( O(2^{(c(c+1)/2)k} \cdot 2^c \cdot m) = O(2^{(c(c+1)/2)k} \cdot m) \) time. To finding an actual solution for Max-\( c \)-CSP Above Average, we can simply use the transformation given in the proof of Lemma 1.

The proofs of Theorems 1 and 8 show that every constraint satisfiable problem admits a hybrid algorithm.

**Corollary 2.** For every Boolean Max-\( c \)-CSP, there is a hybrid algorithm with the property that, for every \( \varepsilon > 0 \), on any instance \( I \), the algorithm outputs either:

- an optimal solution to \( I \) within \( O^*(2^{(c+1)\varepsilon m/2}) \) time, or
- a \((\rho + O(\varepsilon/2^c))\)-approximation to \( I \) in polynomial time, where \( \rho \) is the expected fraction of constraints satisfied by a random assignment.

We close this section with showing how our algorithm actually proves that there is a kernel for Max-\( c \)-Lin-2 Above Average and a kernel for Max-\( c \)-CSP Above Average each with a linear number of variables.

**Corollary 3.** For every \( c \geq 3 \), the problem Max-\( c \)-Lin-2 Above Average can be reduced to a problem kernel with at most \((c(c+1)/2)k\) variables in polynomial time.

**Proof.** Consider an execution of the algorithm of Theorem 1 up to the point just before it performs an exhaustive search of assignments. If there is ever a \( j = 1, \ldots, c \) such that \( S_j \) has weight at least \( k \), then the algorithm stops and outputs an assignment with weight at least \( W/2 + k/2 \) in polynomial time. Otherwise, for all \( j = c, c - 1, \ldots, 1 \), the disjoint set \( S_j \) has weight \( w(S_j) < k \). Then it follows (from Case (1) in the proof of Theorem 1) that the total number of variables in the formula \( F \) is at most \( |\text{var}(F)| = |\text{var}(\bigcup_{j=1}^{c} S_j)| < (c(c+1)/2)k \). The running time of the reduction is polynomial by the construction of sets \( S_c, \ldots, S_1 \).

We note that the compactness of Corollary 3 matches the result of 1 for \( c = 2 \), and generalizes to any fixed \( c \geq 3 \).

**Corollary 4.** For every \( c \geq 3 \), the problem Max-\( c \)-CSP Above Average can be reduced to a problem kernel with at most \((c(c+1)/2)k\) variables in polynomial time.

**Proof.** In the proof of Theorem 1 in 1, a procedure \( P \) is given that reduces any instance of Max-\( c \)-Lin-2 Above Average with total sum of weights \( W \) and parameter \( k \) into an instance of Max-\( c \)-CSP Above
with (a multiset of) \(2^{c-1}W\) constraints and parameter \(2^{c-1}k\). To be precise, the procedure \(\mathcal{P}\) considers an instance of \textsc{Max-}c\textsc{-Lin-}2 \textsc{Above Average} in which each equation has weight 1 and the multiplicity of an equation may be larger than one. \(\mathcal{P}\) maps an equation into a set of \(2^{c-1}\) clauses.

The kernelization of \textsc{Max-c-Csp} \textsc{Above Average} is as follows. Given a \textsc{Max-c-Csp} \textsc{Above Average} instance on \(n\) variables and \(m\) constraints, we first perform the transformation given by Lemma 3 and obtain a \textsc{Max-c-Lin-2} \textsc{Above Average} instance, with \(O(2^m m)\) equations and \(n\) variables. By applying the kernelization of Lemma 3 we obtain an equivalent instance with at most \((c(c+1)/2)k\) variables and no more than \((c(c+1)/2)^c\) (weighted) equations. Finally, apply the procedure \(\mathcal{P}\) to reduce the problem back into a \textsc{Max-c-Csp} \textsc{Above Average} instance, having \((c(c+1)/2)k\) variables, \(O(2^{c-1} \cdot (c(c+1)/2)^c)\) constraints and parameter \(2^{c-1}k\).

3 \textsc{Max-2-Csp} Above SDP Algorithms

Finally, we consider parameterized algorithms for \textsc{Max-2-Csp} problems with Boolean valued variables. We begin with a generic definition of maximum boolean constraint satisfaction of arity two.

\textbf{Definition 1.} \textsc{Max-2-Csp}(\(\Lambda\)) is specified by a constraint family \(\Lambda = \{f : \{0,1\}^c \to \{0,1\} | c \leq 2\}\). An instance of \textsc{Max-2-Csp}(\(\Lambda\)) with \(n\) variables \(x_1, \ldots, x_n\) and \(m\) constraints is a collection of the form \(\{(f_1, S_1, w_1), \ldots, (f_m, S_m, w_m)\}\), where \(w_i\) is a positive integer, every \(S_i\) is an ordered tuple from \([n] \cup [n]^2\), and \(f_i \in \Lambda\) is applied to the subset \(S_i\) of variables.

\textbf{Reminder of Theorem 5} Suppose that for a given constraint family \(\Lambda\), there is a polynomial-time algorithm \(\mathcal{A}\) which is a local \(\alpha\)-approximation for \textsc{Max-2-Csp}(\(\Lambda\)). Then \textsc{Max-2-Csp} Above \(\alpha\) can be solved in \(O(r_{\rho,\alpha} poly(n))\) \(\leq 2^{O(\alpha)} poly(n)\) time. More precisely, there is a randomized parameterized algorithm with the following properties on each instance \(F\):

- For all \(k \in [1, (1-\alpha) \cdot \text{OPT}(F)]\), the algorithm outputs an assignment satisfying at least \(\alpha \cdot \text{OPT}(F) + k\) constraints, with high probability.
- For all \(k > (1-\alpha) \cdot \text{OPT}(F)\), the algorithm always outputs NO.

Here \(p\) is the expected fraction of constraints satisfied by a uniform random assignment to \(F\), and \(r_{\rho,\alpha} = \min_{d \geq 1} \{(1-\alpha)/\rho \cdot (1-\rho) \cdot 2d/(1-\alpha^2)\}\).

Moreover, there is a deterministic algorithm with overhead of \(O(r_{\rho,\alpha} \cdot \min_{d \geq 1} \{(1-\alpha)/\rho \cdot (1-\rho) \cdot 2d/(1-\alpha^2)\})^k\) in running time.

Fix an instance of \textsc{Max-2-Csp}(\(\Lambda\)). Via the linear transformation \(\ell(x) = 1 - 2x\), we can convert each constraint function \(f_i : \{0,1\}^2 \to \{0,1\}\) into the form \(f_i' : \{-1,1\}^2 \to \{0,1\}\), replacing all 0’s with 1’s and all 1’s with −1’s. We rely on the fact that for any function \(f_i : \{0,1\}^2 \to \{0,1\}\) there is always a unique multivariate polynomial \(p_i(x_1, x_2)\) such that for all assignments \(\bar{a} \in \{-1,1\}^2\), \(p_i(\bar{a}) = f_i'(\bar{a})\). As observed in the previous section, this polynomial has the form:

\[
p_i(x_1, x_2) = \sum_{T \subseteq \{1,2\}} \tilde{p}_T^{(i)} \left( \prod_{i \in T} x_i \right),
\]

where every \(\tilde{p}_T^{(i)} = j/4\) for some integer \(j \in [-4, 4]\).

We can formulate \textsc{Max-2-Csp}(\(\Lambda\)) as an SDP relaxation by introducing a special vector \(v_0\) (representing the value “true”), negating the sign of \(\tilde{p}_T^{(i)}\) if necessary, and replacing the multiplication operation in each polynomial \(p_i\) by inner product on vectors, so that each \(p_i\) now has the form:

\[
\tilde{p}_i(v_1, v_2) = \tilde{p}_T^{(i)} + \tilde{p}_{(1,2)}^{(i)} \cdot \langle v_0, v_1 \rangle + \tilde{p}_{(2)}^{(i)} \cdot \langle v_0, v_2 \rangle + \tilde{p}_{(1,2)}^{(i)} \cdot \langle v_1, v_2 \rangle.
\]

\(\tilde{p}_T^{(i)}\) is as follows. Given a \textsc{Max-c-Sat} instance, with \((\rho, \alpha)\) SDP relaxation by introducing a special vector \(v_0\) (representing the value “true”), negating the sign of \(\tilde{p}_T^{(i)}\) if necessary, and replacing the multiplication operation in each polynomial \(p_i\) by inner product on vectors, so that each \(p_i\) now has the form:

\[
\tilde{p}_i(v_1, v_2) = \tilde{p}_T^{(i)} + \tilde{p}_{(1,2)}^{(i)} \cdot \langle v_0, v_1 \rangle + \tilde{p}_{(2)}^{(i)} \cdot \langle v_0, v_2 \rangle + \tilde{p}_{(1,2)}^{(i)} \cdot \langle v_1, v_2 \rangle.
\]

In \(\Pi\), the transformed instance is in fact a \textsc{Max-c-Sat} instance.
Call $\tilde{p}_i$ the relaxation of $p_i$. The following SDP relaxation captures Max-2-CSP($\Lambda$):

$$\max \sum_{i=1}^{m} w_i \cdot \tilde{p}_i(v_{i_1}, v_{i_2}) \quad \text{subject to} \quad |v_i|^2 = 1 \quad \forall i \tag{1}$$

**Remark 1.** Typically, relaxation (1) is also equipped with other inequalities such as $(v_0 - v_i, v_0 - v_j) \geq 0$, $\forall i, j \in |n|$. Often called “triangle constraints,” these extra inequalities are intended to help the SDP model the original problem which had variables over $\{-1, 1\}$. (Note that $(x_0 - x_i)(x_0 - x_j) \geq 0$ for all $x_0, x_i, x_j \in \{-1, 1\}$.) For simplicity, we omit these inequalities from the discussion as they are not really pertinent to our method.

Let $Z^*$ be the value of relaxation (1). An SDP randomized rounding algorithm takes a solution to the above relaxation and (with randomness) outputs a random function $r : \{v_i\} \rightarrow \{-1, 1\}$, which denotes an assignment to the variables of the original 2-CSP. We now define a generic class of SDP randomized rounding algorithms, argue that most SDP rounding algorithms we are aware of fall in the class, and then show that we can parameterize above the worst-case guarantee of any such algorithm.

**Definition 2.** We say that an SDP randomized rounding algorithm $A$ is a local $\alpha$-approximation if, given any instance $F$ of Max-2-CSP($\Lambda$), there is a constant $\alpha > 0$ such that for every constraint $(f, S, w)$ in $F$ with $S = \{x_i, x_j\}$, the polynomial $p$ corresponding to $f$ satisfies:

$$\min \Pr[p(x_i, x_j) = 1] \geq \alpha \cdot \tilde{p}(v_i, v_j),$$

where $v_0, \ldots, v_n$ is the optimal solution to the SDP relaxation (1) and $x_1, \ldots, x_n$ is the assignment output by $A(v_0, \ldots, v_n)$.

In other words, after solving the relaxation (1), $A$ maps the solution vectors to $\{-1, 1\}$ in such a way that the analysis of $A$ can be done by analyzing the expected behavior on a single constraint of the instance.

By linearity of expectation (and the fact that the value $Z^*$ upper bounds the optimum of the underlying 2-CSP instance), any local $\alpha$-approximation for Max-2-CSP($\Lambda$) is also an $\alpha$-approximation. Most known analyses of SDP rounding algorithms (that we are aware of) prove that a given rounding algorithm is in fact a local $\alpha$-approximation.

**Max Res 2-CSP.** For a given set $M = \{(f_i, S_i, z_i)\}$ of constraints, define an $M$-assignment to be a variable assignment that satisfies every constraint $(f_i, S_i, w)$ in $M$. We call such constraints Max Res 2-CSP constraints. The Max Res 2-CSP problem (a generalization of Max Res Cut [17]) is: given a set $F$ of soft constraints and a set $M$ of hard constraints, satisfy a maximum number of constraints in $F$ with an $M$-assignment. Taking polynomials $p_i$ which represent the constraints in $F$, the following is a relaxation of Max Res 2-CSP:

$$\max \sum_{i \in F - M} w_i \cdot \tilde{p}_i(v_{i_1}, v_{i_2}) \quad \text{subject to} \quad \tilde{p}_i(v_{i_1}, v_{i_2}) = 1 \quad \forall i \in M \quad |v_i|^2 = 1 \quad \forall i \tag{2}$$

The optimal solution $Z(M)^*$ of (2) gives an upper bound on $OPT(M)$, where $OPT(M)$ is the maximum value of an $M$-assignment. The crucial observation is that the performance ratio $\alpha$ given by a local approximation remains safe even if we add some hard constraints:

**Lemma 2.** Suppose there is a polynomial-time algorithm $A$ which is a local $\alpha$-approximation for Max-2-CSP($\Lambda$). Then there is a polynomial time algorithm $A'$ that, given an instance $F$ of Max-2-CSP($\Lambda$) and a subset $M$ of constraints such that $F$ has an $M$-assignment, $A'(F, M)$ produces an $M$-assignment with value at least $\alpha \cdot Z(M)^* + |M| \geq \alpha \cdot (OPT(M) - |M|) + |M|$. 


Proof. (Sketch) It is known that the relaxation $\mathcal{L}$ is powerful enough to exactly solve a boolean 2-CSP, if we force all constraints to be satisfied (if we put all constraints in $M$). More generally, the SDP relaxation produces a collection of vectors $\{v_i\}$ which satisfies a given subset of (feasible) hard constraints, and by locality, the probability that a soft constraint $p$ is satisfied by $A$ remains at least $\alpha \cdot \tilde{p}(v_i, v_j)$. 

We now give the algorithm for Theorem 5. Our discussion will be confined to unweighted 2CSP, but it is straightforward to generalize to the weighted case.

Proof of Theorem 5. Let $F$ be an instance of $\text{MAX-2-CSP}(\Lambda)$ as defined above.

Above $\alpha$-Apx($F$)

If $dk > (1-\alpha)^2m$ then try all assignments in $2^m$ time and return the best.

Compute $\rho$, the expected fraction of satisfied constraints by a random assignment.

Using the algorithm of Theorem 2:

If there is no solution satisfying $\geq \rho m + k$ constraints, then output the optimal solution.

Repeat the following for $(\frac{1}{\rho - (1-\alpha)/d})^{k/(1-\alpha)}$ times:

1. Let $M$ be a subset of $k/(1-\alpha)$ constraints chosen uniformly at random from $F$.
2. Solve the Max Res 2-CSP for $M$. Keep the best solution so far.

If the SDP was infeasible for every $M$, then return NO, else output the best solution found.

Let $OPT$ denote the maximum number of satisfied constraints by any assignment. Let $k' = k/(1-\alpha)$ and let $d$ be a parameter to set later. First, observe that when $k > (1-\alpha) \cdot OPT$, we have $k' > OPT$. Therefore every $M \subseteq F$ with $|M| > OPT$ results in an infeasible SDP, and the algorithm will return NO.

Now suppose $k \leq (1-\alpha) \cdot OPT$. Note we may assume $OPT \geq \rho m$. Suppose $k'(k'-1) > km/d$. Then $m < dk/(1-\alpha)^2$ and we can easily determine the optimum in $O(2^m) \leq O(2^{dk/(1-\alpha)^2})$ time by trying all possible subsets of constraints. Suppose $k'(k'-1) \leq km/d$. By standard inequalities, the probability every constraint in $M$ is satisfied by an optimal assignment is at least

$$\frac{\binom{pm}{k'}}{\binom{m}{k'}} \geq \frac{(\rho m)(\rho m - 1)(\rho m - 2) \cdots (\rho m - (k'-1))}{m^{k'}} = \rho(\rho - 1/m)(\rho - 2/m) \cdots (\rho - (k'-1)/m) \geq (\rho - (k'-1)/m)^k' \geq (\rho - (1-\alpha)/d)^k' ,$$

where the last inequality follows since $(k'-1)/m \leq k/(1-\alpha)m \leq (1-\alpha)/d$. Hence in expected $(\frac{1}{\rho - (1-\alpha)/d})^{k'}$ iterations, we choose an $M \subseteq F$ in which all of $M$ is satisfied by an optimal assignment.

If $M$ is satisfied by an optimal assignment to $F$, then $OPT(M) = OPT$. By Lemma 2 the number of constraints satisfied by the rounding algorithm is at least $\alpha \cdot (OPT - k') + k' = \alpha \cdot OPT + k$. The probability we miss such an $M$ after $(\frac{1}{\rho - (1-\alpha)/d})^{k'}$ experiments while there is an assignment of value at least $\alpha \cdot OPT + k$ is less than $e^{-1}$.

To minimize the running time, we pick $d$ so that $(\frac{1}{\rho - (1-\alpha)/d})^{k'} \approx O(2^{dk/(1-\alpha)^2})$.

Generalizing the algorithm to the weighted case can be done in a straightforward way. The only necessary modification is that the constraints in $M$ are chosen from a biased probability distribution: $\{i, j\}$ is chosen with probability $w_{ij}/(\sum_{\{i', j'\} \in F - M} w_{i', j'})$ (re-adjusting the denominator in each step).

A slight modification of Above $\alpha$-Apx yields a deterministic algorithm. Replace the loop in the procedure with the following: Partition the constraint set into $\rho m/(k')$ parts of $k'/\rho$ constraints each, with possibly fewer constraints in the last part. For each part and for all $\binom{k'}{\rho}$ possible $k'$-subsets $S$ of the part, solve the Max Res 2-CSP SDP for $M$. If no $M$ results in a feasible solution, report NO, otherwise report the best solution found. Since the optimal assignment satisfies at least $\rho m$ constraints, at least one of the parts contains at least $k'$ constraints satisfied by an optimal assignment. Hence at least one $M$ we try will contain only constraints satisfied by an optimal assignment. 

$\square$
Let us consider the most popular special cases of Max-$2$-CSP($\Lambda$), namely Max Cut and Max-$2$-Sat. These problems are among the simplest instantiations of Max-$2$-CSP($\Lambda$), with constraint families $\Lambda = \{\text{not-equal}(x_1, x_2)\}$ and $\Lambda = \{\text{or}(x_1, x_2), \text{or}(\neg x_1, x_2), \text{or}(x_1, \neg x_2), \text{or}(\neg x_1, \neg x_2)\}$ respectively. Theorem 5 can be readily applied to these problems.

**Corollary 5.** Max Cut Above .878 can be solved in $O(2^{8.24k}\text{poly}(n))$ time. More precisely, there is a randomized parameterized algorithm with the following properties on every graph $G$:

- For all $k \in [1,.122 \cdot \text{OPT}(G)]$, the algorithm outputs a cut of $G$ with value at least $.878 \cdot \text{OPT}(G) + k$, with high probability.
- For all $k > .122 \cdot \text{OPT}(G)$, the algorithm always outputs NO.$^6$

Moreover, there is a deterministic algorithm which runs in $O(2^{16.4k}\text{poly}(n))$ time.

**Proof.** We take $\mathcal{A}$ to be the $\alpha$-approximation algorithm by Goemans and Williamson [17] for convenience. It has been known since their initial analysis in [17] that

$$\min_{\{v_i\}} \Pr[p(x_i, x_j) = 1 : \mathcal{A} \text{ produces } x \text{ from } v_i] = \min_{\theta \in [0,\pi]} \frac{2}{\pi} \frac{\theta}{1 - \cos \theta} p(v_i, v_j) \geq \alpha \cdot \tilde{p}(v_i, v_j) \geq 0.878 \cdot \tilde{p}(v_i, v_j).$$

Hence $\mathcal{A}$ is a local .878-approximation and the statement of Lemma 2 holds. On the other hand, it is easy to see that $\rho = 0.5$, where $\rho$ is the expected fraction of constraints satisfied by a uniform random assignment. Now plugging in $\rho = 0.5$ and $\alpha = 0.878$, the application of Theorem 5 yields the claimed result.

**Corollary 6.** Max-$2$-Sat Above .940 can be solved in $O(2^{O(k)}\text{poly}(n))$ time. More precisely, there is a randomized parameterized algorithm with the following properties on every graph $G$:

- For all $k \in [1,.04 \cdot \text{OPT}(G)]$, the algorithm outputs a cut of $G$ with value at least $.940 \cdot \text{OPT}(G) + k$, with high probability.
- For all $k > .04 \cdot \text{OPT}(G)$, the algorithm always outputs NO.$^7$

Moreover, there is a deterministic algorithm which runs in $2^{O(k)}\text{poly}(n)$ time.

**Proof.** We take $\mathcal{A}$ to be the $\alpha$-approximation algorithm by Lewin et al. [27]. It is known in [27] that there is a constant $\alpha$ such that

$$\min_{\{v_i\}} \Pr[p(x_i, x_j) = 1 : \mathcal{A} \text{ produces } x \text{ from } v_i] \geq \alpha \cdot \tilde{p}(v_i, v_j) \approx 0.940 \cdot \tilde{p}(v_i, v_j).$$

It remains to observe that $\rho = 0.75$, $\alpha = 0.940$, and the application of Theorem 5 with $\rho$ and $\alpha$ plugged in yields the claimed result.

We conclude this section with a general remark about SDP rounding. Our approach should be most useful when there exists a rounding algorithm that can achieve the worst-case approximation ratio exactly in polynomial time. Many SDP rounding algorithms only guarantee an $\alpha - \varepsilon$ approximation for every sufficiently small $\varepsilon$, and may have exponential [30] or even doubly-exponential [31] dependence on $1/\varepsilon$ in the running time. (Note that earlier rounding schemes such as Goemans-Williamson have only polylogarithmic dependence on $1/\varepsilon$; for us, polynomial dependence on $1/\varepsilon$ suffices.) Since we are only satisfying $k$ constraints beyond the SDP guarantee, a loss of $\varepsilon \cdot \text{OPT}$ in the overall value is hardly acceptable (unless we are willing to settle for subexponential approximation algorithms).

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$^6$Note that here, .122 is a placeholder for $1 - \alpha$ where $\alpha = .878567 \ldots$ is the Goemans-Williamson constant.

$^7$Note that here, .04 is a placeholder for $1 - \alpha$ where $\alpha = .940 \ldots$ is the lower bound by Lewin et al. [27].

$^8$The minimum is evaluated numerically in [27] and up to the authors' knowledge no rigorous analysis is available.
4 Concluding Remarks

In this paper, we pursued the question of how much one can gain beyond the guarantee of an efficient approximation in the framework of parametrized algorithms. We considered two well-known generic algorithms: a (uniformly) random assignment for Max-$c$-CSP and the generic SDP-based randomized rounding algorithms for Max-2-CSP which guarantee better approximations than the random assignment.

While it is interesting that Max-2-CSP Above $\alpha$ has an efficient FPT algorithm, the perceptive reader will notice that Max Cut Above $0.878$ (and possibly Max-2-CSP Above $\alpha$) is not the best possible formulation one could hope for. The $0.878$ ratio is only tight when the maxcut is about $84.4\%$ of all edges, and for any other percentage, it is known that better approximations are possible. A series of works [34, 7, 14, 30] has focused on calculating the precise tradeoff between the maximum cut value of a graph and the achievable approximation ratio, culminating in an explicit determination of the tradeoff for every possible fraction of the cut value, assuming UGC:

**Theorem 6** (O’Donnell and Wu [30]). For $\frac{1}{2} \leq s \leq c \leq 1$, we call the pair $(c, s)$ an SDP gap if there exists a graph $G$ with the SDP optimal value at least $c$ and the size of a maximum cut at most $s$. The SDP gap curve is defined by $\text{Gap}_{\text{SDP}}(c) = \inf \{ s : (c, s) \text{ is an SDP gap} \}$. Then there is a function $S : [\frac{1}{2}, 1] \to [\frac{1}{2}, 1]$ such that $\text{Gap}_{\text{SDP}}(c) = S(c)$ for all $c$. Here, $c$ and $s$ respectively denote the fraction of SDP optimal value and maximum cut in the sum of edge weights in $G$.

Let $S : [\frac{1}{2}, 1] \to [\frac{1}{2}, 1]$ be any function such that when we are given an $m$-edge graph with an optimal SDP value of $cm$, it is possible to efficiently find a cut of size $S(c)m$ using some SDP rounding. Can we find a cut of size $S(c)m + k$, provided it exists? Let us call this problem Max Cut Above SDP (where the rounding scheme is implicit in the formulation of the problem). We believe that this problem should be fixed-parameter tractable, but so far our best efforts have been stymied.

**Conjecture 1.** For all RPR2 rounding algorithms, Max Cut Above SDP can be solved in $f(k)\text{poly}(n)$ time for some function $f$.

Finally, Raghavendra [31] has shown that every CSP has an optimal SDP rounding scheme (to within $\varepsilon$), assuming the Unique Games Conjecture. Hence it is plausible that every CSP admits an interesting “anytime” algorithm of the kind we have studied in this paper, based on semidefinite programming.

References


