Analytic parametric equations of log-aesthetic curves in
terms of incomplete gamma functions

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Abstract

Log-aesthetic curves (LACs) have recently been developed to meet the requirements of industrial design for visually pleasing shapes. LACs are defined in terms of definite integrals, and adaptive Gaussian quadrature can be used to obtain curve segments. To date, these integrals have only been evaluated analytically for restricted values (0, 1, 2) of the shape parameter $\alpha$.

We present parametric equations expressed in terms of incomplete gamma functions, which allow us to find an exact analytic representation of a curve segment for any real value of $\alpha$. The computation time for generating a LAC segment using the incomplete gamma functions is up to 13 times faster than using direct numerical integration. Our equations are generalizations of the well-known Cornu, Nielsen, and logarithmic spirals, and involutes of a circle.

Keywords: log-aesthetic curve, spiral, linear logarithmic curvature graph, log-aesthetic spline, fair curve

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1. Introduction

In designing shapes, such as the exterior surfaces of automobiles, which are subject to very significant aesthetic considerations, the quality of the surfaces is often assessed in terms of reflections of a linear light source. Convoluted reflection lines usually are taken to indicate that the corresponding part of the shape not acceptable. Since variation in curvature determines the pattern of the reflections, a lot of work has been done to generate curves with monotonically varying curvature. Such curves are generally assumed to be fair [3].

Plane curves with monotone curvature were studied by Mineur et al. [17], and this research has been extended by [6], who introduced Class A 3D Bézier curves with monotone curvature and torsion. Meek et al. [15] showed how to construct a curve from arcs of circles and Cornu spirals with continuous curvature, which is greatest for one of the circular arcs. The Pythagorean-hodograph curves introduced by Farouki et al. [7] have been used to construct transition curves of monotone curvature. Frey et al. [8] analysed the curvature distributions of segments of conic sections represented as rational quadratic Bézier curves in standard form. The conditions sufficient for planar Bézier and B-spline curves to have monotone curvature have been described by Wang et al. [27]. Sapidis et al. [22] described a simple geometric condition that indicates when a quadratic Bézier curve segment has monotone curvature. An interesting idea was presented by Xu et al. [29], who used a particle-tracing method to create curves that simulate the orienting effect of a magnetic field on iron filings. These curves are known to be circular or helical.

Recently, Harada et al. [10, 34] introduced log-aesthetic curves, which exhibit monotonically varying curvature because they have linear logarithmic curvature graphs (LCGs). Harada et al. [10, 34] noted that many attractive curves in both natural and artificial objects have approximately linear LCGs. LCGs together with logarithmic torsion graphs (LGTs) for analyzing planar and space curves were studied in [31]. Curves with LCGs which are straight lines were called log-aesthetic curves by Yoshida et al. [32], and they called curves with nearly straight LCGs quasi-log-aesthetic curves [33]. Both of these types of curve can be used for aesthetic shape modelling, and are likely to be an important component of next-generation CAD systems. Log-aesthetic curves can be also considered in the context of computer-aided aesthetic design (CAAD) [4], in which designers evaluate the quality of a
curve by looking at plots of curvature or radius of curvature. Fig. 1 shows an example of a log-aesthetic curve segment together with its smooth evolute, which means that radius of curvature is changing monotonically.

Figure 1: An example of a LAC segment (red line), and its evolute (purple line). Like a quadratic Bézier curve, a LAC segment can be controlled by three control points and specifying $\alpha$.

Log-aesthetic splines, which consist of many LAC segments connected with tangent or curvature continuity, can be associated with fair curves [14]. They are actually non-linear splines, the theory of which arising from a variational criterion of the type $\int \kappa^2 ds \rightarrow \min$ has been briefly described in [16]. Moreover, one of the LAC cases, Cornu spiral, were used for “staircase” approximation in [16]. Fig. 2, 3 exhibit the usage of $G^1$ log-aesthetic spline with shape parameter $\alpha = 3/2$ in car body and Japanese characters design respectively.
Figure 2: Aesthetic design of a car body by means of log-aesthetic splines: (a) with control polygon, (b) without control polygon (some points intentionally satisfy only $G^0$ continuity).

Figure 3: Aesthetic design of Japanese word “shape” by means of log-aesthetic splines: (a) with control polygon, (b) without control polygon, and colored in black. (some points intentionally satisfy only $G^0$ continuity).
Main results

We show how to derive analytic parametric equations of log-aesthetic curves in terms of tangent angle efficiently and accurately. Yoshida et al. [32] used numerical integration based on adaptive Gaussian quadrature [12, 13] to evaluate log-aesthetic curves. Representing them in analytic form in some cases avoids numerical integration and makes them more suitable for interactive applications, which is specially important if we generate surfaces containing many log-aesthetic curve segments. Furthermore, analytic equations will facilitate research on log-aesthetic curves. Table 1 compares previous results with ours. Our work makes the following contributions:

- We obtain analytic parametric equations of log-aesthetic curves in terms of tangent angle, from which we can obtain exact representations of any real value of shape parameter;
- Because our obtained parametric equations consist of incomplete gamma functions, for which good approximation methods exist, we can compute log-aesthetic curve segments accurately;
- Our analytic formulation allows log-aesthetic curve segment to be computed up to 13 times faster than using the Gauss-Kronrod or Newton-Cotes methods of numerical integration;
- Results obtained using our equations have been shown to agree with numerical results obtained using CAS Mathematica and Maple.

Organization

The rest of this paper is organized as follows. In Section 2 we briefly review the basic mathematical concepts of log-aesthetic curves. In Section 3 we derive the general analytic equations of log-aesthetic curves and discuss particular cases, illustrated with the shapes of different spirals. In Section 4 we compare the computation time of the curve segment using the analytical equations with the computation time using numerical integration. In Section 5, we conclude our paper and suggest future work.

2. Preliminaries

2.1. Nomenclature

We are going to use the notation presented in Table 2.
It was shown that many of the aesthetic curves in artificial objects and the natural world have LCGs that can be approximated by straight lines.

The first step towards a mathematical theory of log-aesthetic curves.

Log-aesthetic curves was shown to exhibit self-affinity.

Numerical integration using adaptive Gaussian quadrature was used to evaluate log-aesthetic curves. A new method of using a log-aesthetic curve segment to perform Hermite interpolation was proposed.

Log-aesthetic curve segments can be computed accurately, up to 13 times faster than by direct numerical integration [32].

Table 1: Comparison of the present study with previous work on log-aesthetic curves.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<th>Description</th>
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<tbody>
<tr>
<td>$\rho$</td>
<td>radius of curvature</td>
<td>$\Delta \rho$</td>
<td>change in radius of curvature</td>
</tr>
<tr>
<td>$\alpha$</td>
<td>first parameter of an LAC is the slope of a line in the LCG (shape parameter)</td>
<td>$s$</td>
<td>arc length of a curve</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>second parameter of an LAC</td>
<td>$\kappa$</td>
<td>curvature of a curve</td>
</tr>
<tr>
<td>$c$</td>
<td>a constant, log $\lambda$</td>
<td>$\theta, \psi$</td>
<td>tangent angle (angle between a tangent line and the $x$-axis)</td>
</tr>
<tr>
<td>$\Delta s$</td>
<td>change in arc length</td>
<td>$[x(\psi), y(\psi)]$</td>
<td>parametric equation of a log-aesthetic curve in terms of tangent angle</td>
</tr>
<tr>
<td>$\Gamma(a, z)$</td>
<td>incomplete gamma function</td>
<td>$P_n(u)$</td>
<td>polynomial of degree $n$</td>
</tr>
<tr>
<td>$E(n)$</td>
<td>integral part of a real number</td>
<td>$P_n^{(k)}(u)$</td>
<td>derivative of $P_n(u)$ of order $k$</td>
</tr>
<tr>
<td>$C(t)$ and $S(t)$</td>
<td>Fresnel integrals</td>
<td>$\gamma$</td>
<td>any natural number except zero</td>
</tr>
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</table>

Table 2: Notation.
2.2. Fundamentals of log-aesthetic curves

Miura et al. [19] defined log-aesthetic curves as having a radius of curvature which is a function of their arc length $s$ as below:

$$\log \left( \frac{ds}{d\rho} \right) = \alpha \log \rho + c,$$

(1)

where the constant $c = -\log \lambda$, and $(0, c)$ are the coordinates of the intersection of the $y$-axis with a line of a slope $\alpha$ (see Fig.4(b)), which is the shape parameter that determines the type of a log-aesthetic curve.

After simply manipulating Eq. (1), and recollecting that $c$ is a constant we obtain

$$\frac{ds}{d\rho} = \frac{\rho^{\alpha-1}}{\lambda}.$$

(2)

When $\alpha = -1, 0, 1, 2$ or $\infty$, we obtain a clothoid, a Nielsen’s spiral, a logarithmic spiral, the involute of a circle, and a circle respectively.

To derive a formula of a log-aesthetic curve we need to consider a reference point $P_r$ on the curve. The reference point can be any point on the curve except for the point whose radius of curvature is either $0$ or $\infty$. The following constraints are placed at the reference point [32]:

- Scaling: $\rho = 1$ at $P_r$, which means that $s = 0$ and $\theta = 0$ at the reference point;
- Translation: $P_r$ is placed at the origin of Cartesian coordinate system;
- Rotation: the tangent line to curve at $P_r$ is parallel to $x$-axis.
Subsequently, after integrating Eq. (2) with respect to $\rho$ with its upper and lower limits 1 and $\hat{\rho}$ respectively, and then replacing $\hat{\rho}$ with $\rho$, Yoshida et al. [32] found the intrinsic (natural) equation of the log-aesthetic curve, also known as the Cesàro equation [30]:

$$\rho(s) = \begin{cases} e^{\lambda s}, & \alpha = 0 \\ (\lambda \alpha s + 1)^{\frac{1}{\alpha}}, & \text{otherwise} \end{cases},$$

where $\lambda = e^{-c}, 0 < \lambda < \infty$. The following relation, which arises in geometric interpretation of the curvature of a regular curve, is well-known in differential geometry [21, 23]

$$\kappa = \frac{1}{\rho} = \frac{d\theta}{ds}.$$  

If we substitute Eq. (3) into this equation, integrate with respect to $s$ from 0 to $\hat{s}$, and afterwards replace $\hat{s}$ by $s$, and set $\theta = 0$ when $s = 0$ we obtain the Whewell equation [28] that relates the tangent angle $\theta$ with the arc length $s$:
\[ \theta(s) = \begin{cases} \frac{1-e^{-\lambda s}}{\lambda}, & \alpha = 0 \\ \log(\lambda s+1), & \alpha = 1 \\ \frac{(\lambda s s + 1)^{\frac{\alpha - 1}{\alpha}} - 1}{\lambda^\alpha}, & \text{otherwise} \end{cases} \] (5)

From Eqs. (2) and (4) we can further obtain:

\[ \frac{d\theta}{d\rho} = \frac{ds}{d\rho} = \frac{\rho^{\alpha-2}}{\lambda}. \] (6)

Integrating this equation with respect to \( \theta \) from 0 to \( \hat{\theta} \), and then replacing \( \hat{\theta} \) with \( \theta \) yields a formulation of a log-aesthetic curve that relates the radius of curvature \( \rho \) to the tangent angle \( \theta \):

\[ \rho(\theta) = \begin{cases} e^{\lambda\theta}, & \alpha = 1 \\ ((\alpha - 1) \lambda \theta + 1)^{\frac{1}{\alpha-1}}, & \text{otherwise} \end{cases}. \] (7)

Using the quadratures by which a plane curve given by its natural equation can be represented [21, 23], we can obtain the parametric equations of a log-aesthetic curve:

\[ x(\psi) = \int_{0}^{\psi} \rho(\theta) \cos \theta d\theta, \] (8)

\[ y(\psi) = \int_{0}^{\psi} \rho(\theta) \sin \theta d\theta, \] (9)
where the upper bound on the tangent angle $\psi$ is $1/(\lambda(1-\alpha))$, $\alpha < 1$, and its lower bound is $1/(\lambda(1-\alpha))$, $\alpha > 1$. If $\alpha = 1$ there are no upper or lower bounds on the tangent angle $\psi$ [32].

Some of the characteristics of log-aesthetic curves are described in details by Yoshida et al. [32]:

- The radius of curvature $\rho$ of log-aesthetic curves can grow from 0 to $\infty$;
- When $\alpha < 0$, a log-aesthetic curves have an inflection points, and the curve is a spiral until the point at which $\rho = 0$;
- When $\alpha = 0$, the curve is also a spiral until $\rho = 0$. The point at which $\rho = \infty$ is at infinity;
- When $0 < \alpha < 1$, the distance to the point at which $\rho = 0$ is finite, and there is an inflection point at infinity;
- When $\alpha = 1$, the curve is a spiral that converges to the point at which $\rho = 0$ with a finite arc length. In the other direction the curve is a spiral that diverges to the point at which $\rho = \infty$;
- When $\alpha > 1$, the point at $\rho = 0$ has a fixed tangent direction;
- The curve is a spiral that diverges to the point at which $\rho = \infty$. 

Figure 5: The geometric meaning of the parameter $\theta$ in Eqs. (8) and (9).
3. General equations and overall shapes of log-aesthetic curves

After integrating in Eqs. (8) and (9), and applying the incomplete gamma function \([\Gamma(a, z)]\)

\[
\Gamma(a, z) = \int_{z}^{\infty} u^{a-1}e^{-u}du,
\]
we can derive the general equations of log-aesthetic curves in terms of the tangent angle \(\psi\):

\[
x(\psi) = \frac{1}{2}(\lambda i(\alpha - 1))^{\frac{1}{\alpha - 1}} \left\{ \Gamma \left( \frac{\alpha}{\alpha - 1}, -\frac{i(1 + (\alpha - 1)\theta\lambda)}{(\alpha - 1)\lambda} \right) \right. \\
\left. \left( \sin \left( \frac{1}{\lambda(1 - \alpha)} \right) - i \cos \left( \frac{1}{\lambda(1 - \alpha)} \right) \right) + (-1)^{\frac{1}{\alpha - 1}} \Gamma \left( \frac{\alpha}{\alpha - 1}, \frac{i(1 + (\alpha - 1)\theta\lambda)}{(\alpha - 1)\lambda} \right) \right. \\
\left. \left( \sin \left( \frac{1}{\lambda(1 - \alpha)} \right) + i \cos \left( \frac{1}{\lambda(1 - \alpha)} \right) \right) \right\} \bigg|_{0}^{\psi},
\]

\[
y(\psi) = \frac{1}{2}(\lambda i(\alpha - 1))^{\frac{1}{\alpha - 1}} \left\{ (-1)^{\frac{1}{\alpha - 1}} \Gamma \left( \frac{\alpha}{\alpha - 1}, \frac{i(1 + (\alpha - 1)\theta\lambda)}{(\alpha - 1)\lambda} \right) \right. \\
\left. \left( \cos \left( \frac{1}{\lambda(1 - \alpha)} \right) - i \sin \left( \frac{1}{\lambda(1 - \alpha)} \right) \right) + \Gamma \left( \frac{\alpha}{\alpha - 1}, -\frac{i(1 + (\alpha - 1)\theta\lambda)}{(\alpha - 1)\lambda} \right) \right. \\
\left. \left( \cos \left( \frac{1}{\lambda(1 - \alpha)} \right) + i \sin \left( \frac{1}{\lambda(1 - \alpha)} \right) \right) \right\} \bigg|_{0}^{\psi}.
\]

According to [1] an incomplete gamma function can be represented by following series:

\[
\Gamma(a, z) = \Gamma(a) - z^{a} \sum_{k=0}^{\infty} \frac{(-z)^{k}}{(a + k)k!},
\]
and gamma function’s product representation is [1]
\[ \Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right)^{-1} e^{z/k}, \]

where \( \gamma \approx 0.577 \) is the Euler-Mascheroni constant. An asymptotic expansion can be also useful when \( |z| \to \infty \) and \( |\arg z| < \frac{3}{2} \pi \) [2]:

\[ \Gamma(a, z) \sim z^{a-1}e^{-z} \sum_{k=0}^{\infty} \frac{\Gamma(a)}{\Gamma(a-k)} z^{-k}. \]

Now we consider some particular cases of the above equations, using the following well-known formulas [9, 1]:

\[
\int P_n(u) \cos m u \, du = \frac{\sin m u}{m} \sum_{k=0}^{E\left(\frac{n}{2}\right)} (-1)^k \frac{P_n^{(2k)}(u)}{m^{2k}} + \frac{\cos m u}{m} \sum_{k=1}^{E\left(\frac{n+1}{2}\right)} (-1)^k \frac{P_n^{(2k-1)}(u)}{m^{2k-1}},
\]

\[
\int P_n(u) \sin m u \, du = -\frac{\cos m u}{m} \sum_{k=0}^{E\left(\frac{n}{2}\right)} (-1)^k \frac{P_n^{(2k)}(u)}{m^{2k}} + \frac{\sin m u}{m} \sum_{k=1}^{E\left(\frac{n+1}{2}\right)} (-1)^k \frac{P_n^{(2k-1)}(u)}{m^{2k-1}},
\]

where \( P_n(u) \) is a polynomial of degree \( n \), \( P_n^{(k)}(u) \) is a derivative of \( P_n(u) \) of order \( k \), and \( E(n) \) is the integral part of a real number (smallest integer greater than or equal to a number). We can now reduce Eqs. (8) and (9) for \( \alpha \neq 1 \) and \( \frac{1}{\alpha - 1} = \gamma \left( \alpha = 2, 2 \frac{3}{2}, 2 \frac{4}{3}, \ldots, 2 \frac{\gamma + 1}{\gamma} \right) \), where \( \gamma \in \mathbb{N}^* \) (the set of all natural numbers except zero) to a pair of integrals\(^2\):

\[ x(\psi) = \int_{0}^{\psi} \left( (\alpha - 1) \lambda \theta + 1 \right) \frac{1}{\alpha - 1} \cos \theta d\theta = \]

\(^2\)We will now and subsequently use \( (k) \) to signify the \( k^{th} \) derivative with respect to \( \theta \).
\[
\begin{align*}
\sin \theta & \sum_{k=0}^{E\left(\frac{1}{2(\alpha - 1)}\right)} (-1)^k \left[ ((\alpha - 1) \lambda \theta + 1)^{\frac{1}{\alpha - 1}} \right]^{(2k)} + \\
\cos \theta & \sum_{k=1}^{E\left(\frac{1}{2(\alpha - 1)}\right)} (-1)^{k-1} \left[ ((\alpha - 1) \lambda \theta + 1)^{\frac{1}{\alpha - 1}} \right]^{(2k-1)} \bigg|_{0}^{\psi},
\end{align*}
\]

\[y(\psi) = \int_{0}^{\psi} ((\alpha - 1) \lambda \theta + 1)^{\frac{1}{\alpha - 1}} \sin \theta d\theta =
\]

\[
\begin{align*}
- \cos \theta & \sum_{k=0}^{E\left(\frac{1}{2(\alpha - 1)}\right)} (-1)^k \left[ ((\alpha - 1) \lambda \theta + 1)^{\frac{1}{\alpha - 1}} \right]^{(2k)} + \\
\sin \theta & \sum_{k=1}^{E\left(\frac{1}{2(\alpha - 1)}\right)} (-1)^{k-1} \left[ ((\alpha - 1) \lambda \theta + 1)^{\frac{1}{\alpha - 1}} \right]^{(2k-1)} \bigg|_{0}^{\psi},
\end{align*}
\]

We can use Eqs. (15) and (16) to derive exact analytic equations of log-aesthetic curves in terms of trigonometric functions for some special values of \(\alpha\). These and further curves in the present work are drawn using general parametric equations.

* For the case of \(\alpha = 3/2\) we have

\[
x(\psi) = \int_{0}^{\psi} \left( \left( \frac{1}{2} \lambda \theta + 1 \right)^2 \cos \theta \right) d\theta = \sin \theta \sum_{k=0}^{1} (-1)^k \left[ \left( \frac{1}{2} \lambda \theta + 1 \right)^2 \right]^{(2k)} +
\]

\[
\cos \theta \sum_{k=1}^{2} (-1)^{k-1} \left[ \left( \frac{1}{2} \lambda \theta + 1 \right)^2 \right]^{(2k-1)} \bigg|_{0}^{\psi} = \left[ \left( \frac{1}{2} \lambda \psi + 1 \right)^2 - \lambda^2 \right] \sin \theta +
\]

\[
\left[ \lambda \left( \frac{1}{2} \lambda \psi + 1 \right) \cos \theta \right]_{0}^{\psi} = \left( \frac{1}{2} \lambda \psi + 1 \right)^2 - \lambda^2 \right] \sin \psi +
\]

\[
\left[ \lambda \left( \frac{1}{2} \lambda \psi + 1 \right) \right] \cos \psi - \lambda.
\]
\[ y(\psi) = \int_{0}^{\psi} \left( \frac{1}{2} \lambda \theta + 1 \right)^2 \sin \theta d\theta = \left\{ -\cos \theta \sum_{k=0}^{1} (-1)^{k} \left[ \left( \frac{1}{2} \lambda \theta + 1 \right)^2 \right]^{(2k)} \right\} + \]

\[ \sin \theta \sum_{k=1}^{2} (-1)^{k-1} \left[ \left( \frac{1}{2} \lambda \theta + 1 \right)^2 \right]^{(2k-1)} \right\} \bigg|_{0}^{\psi} = \left\{ -\left[ \left( \frac{1}{2} \lambda \theta + 1 \right)^2 - \frac{\lambda^2}{2} \right] \cos \theta + \right\}

\[ \left[ \lambda \left( \frac{1}{2} \lambda \psi + 1 \right) \sin \theta \right] \bigg|_{0}^{\psi} = - \left[ \left( \frac{1}{2} \lambda \psi + 1 \right)^2 - \frac{\lambda^2}{2} \right] \cos \psi + \]

\[ \left[ \lambda \left( \frac{1}{2} \lambda \psi + 1 \right) \right] \sin \psi - \frac{\lambda^2}{2} + 1. \]

The family of LACs with \( \alpha = 3/2 \) is shown in Fig. 6.
Figure 6: Log-aesthetic curves with $\alpha = 3/2$. The value of $\theta$ is changing from its lower bound to 10 radians.

- Applying the same approach for $\alpha = 2$ yields:

$$x(\psi) = \sin \psi - \lambda + \lambda(\cos \psi + \psi \sin \psi),$$
$$y(\psi) = 1 - \cos \psi + \lambda(\sin \psi - \psi \cos \psi),$$

which are the parametric equations of involutes of a circle shown in Fig. 7.
Figure 7: Log-aesthetic curves with \( \alpha = 2 \). The value of \( \theta \) is changing from its lower bound to 10 radians.

- Setting \( \alpha = -1 \) and integrating Eqs. (8) and (9) yields the following equations:

\[
x(\psi) = \sqrt{\frac{\pi}{\lambda}} \left\{ \cos \left( \frac{1}{2\lambda} \right) C \left( \frac{1}{\sqrt{\pi \lambda}} \right) + \sin \left( \frac{1}{2\lambda} \right) S \left( \frac{1}{\sqrt{\pi \lambda}} \right) \right.
- \left. \cos \left( \frac{1}{2\lambda} \right) C \left( \frac{\sqrt{1 - 2\lambda \psi}}{\sqrt{\pi \lambda}} \right) - \sin \left( \frac{1}{2\lambda} \right) S \left( \frac{\sqrt{1 - 2\lambda \psi}}{\sqrt{\pi \lambda}} \right) \right\},
\]

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\[
y(\psi) = -\sqrt{\frac{\pi}{\lambda}} \left\{ \cos \left( \frac{1}{2\lambda} \right) S \left( \frac{1}{\sqrt{\pi \lambda}} \right) - \sin \left( \frac{1}{2\lambda} \right) C \left( \frac{1}{\sqrt{\pi \lambda}} \right) 
- \cos \left( \frac{1}{2\lambda} \right) S \left( \frac{\sqrt{1-2\lambda\psi}}{\sqrt{\pi \lambda}} \right) + \sin \left( \frac{1}{2\lambda} \right) C \left( \frac{\sqrt{1-2\lambda\psi}}{\sqrt{\pi \lambda}} \right) \right\},
\]
which refer to extended Cornu spiral, the graphs of which are shown on Fig. 8.

Figure 8: Log-aesthetic curves curves (chlotoids) with \( \alpha = -1 \). The value of \( \theta \) is changing from -10 radians to its upper bound.

- When \( \alpha = 3 \) we obtain (Fig. 9):
\[
x(\psi) = \lambda \left\{ \sqrt{\pi} \cos \left( \frac{1}{2\lambda} \right) S \left( \frac{1}{\sqrt{\pi \lambda}} \right) - \sqrt{\pi} \sin \left( \frac{1}{2\lambda} \right) C \left( \frac{1}{\sqrt{\pi \lambda}} \right) \\
+\sqrt{2\lambda \psi + 1} \sin(\psi) \sqrt{\frac{1}{\lambda}} - \sqrt{\pi} \cos \left( \frac{1}{2\lambda} \right) S \left( \frac{\sqrt{2\lambda \psi + 1}}{\sqrt{\pi \lambda}} \right) \\
+\sqrt{\pi} \sin \left( \frac{1}{2\lambda} \right) C \left( \frac{\sqrt{2\lambda \psi + 1}}{\sqrt{\pi \lambda}} \right) \right\},
\]

\[
y(\psi) = \frac{1}{\sqrt{\lambda}} \left\{ \frac{1}{\sqrt{\lambda}} - \sqrt{\pi} \cos \left( \frac{1}{2\lambda} \right) C \left( \frac{1}{\sqrt{\pi \lambda}} \right) - \sqrt{\pi} \sin \left( \frac{1}{2\lambda} \right) S \left( \frac{1}{\sqrt{\pi \lambda}} \right) \\
-\sqrt{2\lambda \psi + 1} \sin(\psi) \sqrt{\frac{1}{\lambda}} + \sqrt{\pi} \cos \left( \frac{1}{2\lambda} \right) C \left( \frac{\sqrt{2\lambda \psi + 1}}{\sqrt{\pi \lambda}} \right) \\
+\sqrt{\pi} \sin \left( \frac{1}{2\lambda} \right) S \left( \frac{\sqrt{2\lambda \psi + 1}}{\sqrt{\pi \lambda}} \right) \right\},
\]

where \( S(x) \) and \( C(x) \) are Fresnel integrals. These are two transcendental functions which commonly occur in the physics of diffraction, and have the following integral representations \([9, 1, 20, 25]\):

\[ S(t) = \int_{0}^{t} \sin(u^2)du, \quad C(t) = \int_{0}^{t} \cos(u^2)du. \]

The simultaneous parametric plot of \( S(t) \) and \( C(t) \) is the Cornu spiral or clothoid.
Figure 9: Log-aesthetic curves with $\alpha = 3$. The value of $\theta$ is changing from its lower bound to 10 radians.

4. Computation cost and maximum error estimation

Yoshida et al. [32] observed that the computation time required to evaluate a log-aesthetic curve segment depends on the parameter $\alpha$, the range of integration and number of points needed. Our computations were coded in CAS Mathematica Version 7 [5], and performed on a Pentium Core i7 3.07GHz computer. On every segment we computed 100 points. The tangent angle of every curve segment varies from 0 to 1. We set $\lambda$ such that the curve segment is defined in interval $\theta \in [0, 1]$. If $\lambda$ takes value greater than 1, we set $\lambda = 1$. Our analytic approach is compared with different numerical methods in Table 3. It can be seen that from analytic equations log-aesthetic curve segments can be obtained up to 13 times faster than by means of the
Table 3: The log-aesthetic curve segment computation time (in seconds). N-C is a Newton-Cotes numeric integration method. The last column shows how much faster the analytic equations in comparison with Gauss-Kronrod method. For the cases when $\alpha = \{\frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \frac{6}{5}\}$ Eq. (15), (16) has been used.

Gauss-Kronrod method [12, 13] used by Yoshida et al. [32]; the precise ratio depends on the value of $\alpha$. This is because these curves are formulated as incomplete gamma functions which have good approximation methods [1, 24] and an exact series representation [2]. Other numerical methods are slower; and moreover the Monte-Carlo method may fail for values of $\alpha$ around 1, and we did not consider it to be worth close examination. Since Eqs. (15) and (16) are represented by simple and exact analytic functions, they can be useful for computation of the maximum errors of numerical methods used in previous work [32]. Table 4 includes such a comparison for several values of $\alpha$ and $\lambda$, and it can be seen that the Gauss-Kronrod and the Newton-Cotes methods may have significant errors in the neighbourhood of $\alpha = 1$; in other cases the maximum errors are negligible.
<table>
<thead>
<tr>
<th>Parameters</th>
<th>Maximum error</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Gauss-Kronrod</td>
</tr>
<tr>
<td>( \alpha )</td>
<td>( \lambda )</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>10</td>
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<tr>
<td>( \frac{4}{3} )</td>
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<tr>
<td>( \frac{4}{3} )</td>
<td>100</td>
</tr>
<tr>
<td>( \frac{10}{9} )</td>
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</tr>
<tr>
<td>( \frac{10}{9} )</td>
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<td>( \frac{10}{9} )</td>
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</tbody>
</table>

Table 4: Error estimations for LAC segment computation.

5. Conclusions and future work

We have introduced analytic parametric equations for log-aesthetic curves consisting of trigonometric and incomplete gamma functions. Whereas previous authors [32] formulated parametric equations for particular cases (\( \alpha = 0, 1, 2 \)), our general equations allow an accurate evaluation of log-aesthetic curve segments for \( \forall \alpha \in \mathbb{R} \).

We have simplified the general equations and represented them in terms of trigonometric functions when \( \alpha = 2, \frac{3}{2}, \frac{4}{3}, \ldots, \frac{\gamma+1}{\gamma}, \gamma \in \mathbb{N}^* \), and in terms of Fresnel integrals when \( \alpha = -1, 3 \). Depending on the parameter \( \alpha \), the availability of general parametric equations (11), (12) allows log-aesthetic curve segments to be obtained up to 13 times faster than the Gauss-Kronrod numeric integration used previously. This will be especially significant in the construction of log-aesthetic surfaces [11] containing many curve segments.

An analytic equation of a log-aesthetic curve in terms of arc length is also required, since the equation in terms of tangent angle is unstable when \( \rho \to \infty \), which occurs at inflection points. We are going to examine the possibility of deriving such an equation.

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