CONVERGENCE OF THE FINITE ELEMENT METHOD FOR THE POROUS MEDIA EQUATION WITH VARIABLE EXPONENT∗

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Abstract. In this work, we study the convergence of the finite element method when applied to the following parabolic equation: 
\[ u_t = \text{div}(|u|^{\gamma(x)} \nabla u) + f(x, t), \quad x \in \Omega \subset \mathbb{R}^m, \; t \in [0, T]. \]
Since the problem may be of degenerate type, we utilize an approximate problem, regularized by introducing a parameter \( \varepsilon \). We prove, under certain conditions on \( \gamma \) and \( f \), that the weak solution of the approximate problem converges to the weak solution of the initial problem, when the parameter \( \varepsilon \) tends to zero. Discrete solutions are built using the finite element method and the convergence of these for the weak solution of the approximate problem is proved. Finally, we present some numerical results of a MATLAB implementation of the method.

Key words. finite element method, porous media equation, variable exponent

AMS subject classifications. 76S05, 65N30, 65N12

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1. Introduction. We shall study the Dirichlet problem for a class of semilinear parabolic equations with variable exponent of nonlinearity. Let \( \Omega \subset \mathbb{R}^m, \; m \geq 1 \), be a domain with Lipschitz-continuous boundary \( \partial \Omega \) and \( \Omega_T = \Omega \times [0, T] \) a cylinder of height \( T < \infty \). We consider the following problem:
\[
\begin{aligned}
&u_t = \text{div}(|u|^{\gamma(x)} \nabla u) + f(x, t) \quad \text{in } \Omega_T, \\
&u(x) = 0 \quad \text{on } \Gamma_T = \partial \Omega \times [0, T], \\
&u(x, 0) = u_0(x) \quad \text{in } \Omega,
\end{aligned}
\]
where \( \gamma \) is a bounded function defined on \( \Omega_T \) such that
\[
1 < \gamma^- \leq \gamma(x) \leq \gamma^+ < \infty \quad \forall x \in \bar{\Omega}.
\]
Problems of type (1.1) appear in continuum mechanics [1] to model the motion of an ideal barotropic gas through a porous medium, where pressure is assumed to depend explicitly on the density and on the temperature. In the last decades, a large variety of methods to approximate degenerate parabolic equations have been proposed. For example the finite difference method has been studied by Di Benedetto and Hoff [5] and Karlsen, Risebro, and Towers [13], the finite volume method by Baughman and Walkington [3] and Eymard et al. [9], the mixed finite element method by Arbogast and Wheeler [2], Radu, Pop, and Knabner [17], Woodward and Dawson [22], and Yotov [23], the relaxation scheme was proposed by Jäger and Kačur [11], the characteristic method by Chen et al. [4], and combinations of these methods by Kacur [12] and Eymard, Hilhorst, and Vohralík [10].

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If $\gamma$ does not depend on $x$, we have the classical porous media equation. In this case, numerical approximations using the finite element method in space and Euler schemes in time have been studied previously by other authors. The first error estimates can be found in Rose [19]. Later Nochetto and Verdi [16] improved these results. Using a regularization procedure, they were able to prove $L_2$ error estimates. Assuming initial data in $L_1$, Rulla and Walkington [20] derived optimal convergence rates of order $h + \delta$ (for space mesh width $h$ and time step $\delta$) in the space $L_\infty(0,T;H^{-1}(\Omega))$. By a nondegeneracy property, Ebmeyer [7] improved the results of Nochetto and Verdi. Recently, Ebmeyer and Liu [6] used new results on regularity and proved optimal order of convergence in a seminorm. More recently, Emmrich and Siska [8] using the monotone operators theory proved the convergence for very weak solutions. In this work we prove similar results using the time discontinuous Galerkin finite element method proposed by Rivière and Wheeler [18] (see also Thomée [21]).

Problem (1.1) does not, in general, admit classical solutions. A weak solution of problem (1.1) is understood as follows.

**Definition 1.1.** A locally integrable bounded function $u(x,t)$ is said to be a weak solution of problem (1.1) if

(i) $u \in L_\infty(0,T;L_\infty(\Omega))$, $|u|^2 \nabla u \in L_2(0,T;L_2(\Omega))$, $u_t \in L_2(0,T;H^{-1}(\Omega))$;

(ii) $u = 0$ on $\Gamma_T$;

(iii) for any test function $\chi(x,t)$ satisfying the conditions

$\chi \in L_2(0,T;H_0^1(\Omega)) \cap L_\infty(0,T;L_\infty(\Omega))$, $\chi_t \in L_2(0,T;L_2(\Omega))$, and every $0 \leq t_1 \leq t_2 \leq T$, the following integral identity holds:

$$
\int_{t_1}^{t_2} \int_{\Omega} \left( -u \chi_t + |u|^2 \nabla u \cdot \nabla \chi - f \chi \right) \, dx \, dt = -\int_{\Omega} u \chi \, dx \bigg|_{t_1}^{t_2};
$$

(iv) and $u(x,0) = u_0(x)$ in $\Omega$.

The existence and uniqueness of weak solutions, in the sense of this definition, was proved in [1]. To the best of the authors’ knowledge, there are no results concerning the convergence of the finite element method when applied to problems of this type with variable exponent. Our main result is the derivation of error estimates for numerical approximations to solutions of problem (1.1).

The paper is organized as follows: in section 2, we define an approximate regularized problem and obtain some bounds for its solutions and their derivatives and we also prove the convergence of the approximate problem to the original problem; in section 3, we discretize the approximate problem using the Galerkin method and derive some estimates for the error of the discrete solution; section 4 is devoted to the numerical analysis of the discrete problem and in section 5 we present two examples; finally, a summary of the results and outlooks for future research are presented in section 6.

### 2. Regularization of the problem

One source of difficulty in deriving error estimates for degenerate parabolic problems is the roughness of their solutions. In order to obtain a parabolic boundary value problem with a smooth solution, we must perturb problem (1.1) and this can be done in several ways. We shall utilize the following approximate problem, regularized by introducing a parameter $\varepsilon$ in the
If the solution of problem (2.1) represents a constant, but not always the same value. The definition of a weak solution to this problem is similar to the previous one.

\[ a(x, v) = (x^2 + \varepsilon^2)^{\frac{3}{2}} \quad 0 < \varepsilon < 1. \]

It is evident that if \( \|v\| L_{\infty} < M \) and \( \gamma \) satisfies (1.2), then

\[ 0 < \varepsilon^{\gamma^+} \leq a(x, v) \leq (M^2 + 1)^{\frac{1}{\gamma^+}} < \infty. \]

The definition of a weak solution to this problem is similar to the previous one.

**Definition 2.1.** A locally integrable bounded function \( v(x, t) \) is said to be a weak solution of problem (2.1) if

(i) \( \forall \varepsilon > 0 \), the regularized problem has a weak solution \( v(x, t) \) which satisfies this definition. Moreover, if \( \gamma \in H^2(\Omega) \), then \( v_t, \Delta v \in L_2(\Omega_T) \) and the weak solution also satisfies (2.1). If \( \gamma \in C^{2+\alpha}(\Omega) \), then the solution is classical, in the sense that \( v_t, D^2v \in C^\alpha(\Omega_T) \). In what follows, we will only refer to the conditions on the regularity of \( \gamma \) needed to prove each estimate. Since the influence of the parameter \( \varepsilon \) on the regularity of the solutions of problem (2.1) is not clear, we begin with a collection of basic regularity results. In what follows, \( C \) will represent a constant, but not always the same value.

**Theorem 2.2.** Let \( \gamma \) be a measurable function in \( \Omega \) which satisfies condition (1.2). If

\[ \|u_0\| L_{\infty}(\Omega) + \int_0^T \|f\| L_{\infty}(\Omega) \, dt < C, \]

then the solution \( v \) of problem (2.1) satisfies

\[ \|v(x, t)\| L_{\infty}(\Omega) \leq \|v(x, 0)\| L_{\infty}(\Omega) + \int_0^T \|f\| L_{\infty}(\Omega) \, dt \leq C, \quad t \in [0, T], \]

where \( C \) does not depend on \( \varepsilon \).

**Proof.** Multiplying the first equation of (2.1) by \( v^{2k-1} \) and integrating over \( \Omega \), we arrive at the relation

\[ \int_{\Omega} v^{2k-1} \, dx + \int_{\Omega} \text{div}(a(x, v) \nabla v) v^{2k-1} \, dx = \int_{\Omega} f v^{2k-1} \, dx, \]
whence, by Holder’s inequality and Green’s theorem,

\[ \frac{1}{2k} \frac{d}{dt} \|v\|_{L_{2k}}^2 + (2k - 1) \int_{\Omega} v^{2k-2} a_\varepsilon(x, v) |\nabla v|^2 \, dx \leq \|f\|_{L_{2k}} \|v\|_{L_{2k}}^{2k-1}. \]

If we ignore the middle term (since it is nonnegative) and simplify the factor \(\|v\|_{L_{2k}}^{2k-1}\), we get

\[ \frac{d}{dt} \|v\|_{L_{2k}} \leq \|f\|_{L_{2k}}. \]

Integrating this relation in \(t\), we obtain the following estimates:

\[ \|v(x, t)\|_{L_{2k}} \leq \|v(x, 0)\|_{L_{2k}} + \int_0^T \|f\|_{L_{2k}} \, dt. \]

Passing to the limit when \(k \to \infty\) and using (2.3), we obtain (2.4).

**Theorem 2.3.** Suppose that \(v\) is a solution of (2.1) with \(\gamma \in H^1(\Omega)\) and that the conditions of the previous theorem are satisfied. Then

\[ \int_0^T \|\nabla v\|_{L_{2k}(\Omega)}^2 \, dt \leq C \varepsilon^{-\gamma^+}, \]

where \(C\) does not depend on \(\varepsilon\).

**Proof.** As in the previous proof, multiplying the first equation of (2.1) by \(v\) and integrating over \(\Omega\), we arrive at

\[ \frac{1}{2} \frac{d}{dt} \|v\|_{L_2}^2 + \int_{\Omega} a_\varepsilon(x, v) |\nabla v|^2 \, dx \leq \|f\|_{L_2} \|v\|_{L_2}. \]

Integrating in \(t\), we obtain

\[ \frac{1}{2} \|v(T)\|_{L_2}^2 + \int_{\Omega_T} a_\varepsilon(x, v) |\nabla v|^2 \, dx \, dt \leq \frac{1}{2} \|v(0)\|_{L_2}^2 + \int_0^T \|f\|_{L_2} \|v\|_{L_2} \, dt \]

and by Theorem 2.2,

\[ \int_{\Omega_T} a_\varepsilon(x, v) |\nabla v|^2 \, dx \, dt \leq C. \]

Using the lower bound of \(a_\varepsilon(x, v)\), we obtain the result. \(\Box\)

**Theorem 2.4.** Suppose that \(\gamma \in H^2(\Omega)\) and \(\nabla u_0 \in L_2(\Omega)\). If \(v\) is a solution of (2.1) and the conditions of Theorem 2.2 are satisfied, then

\[ \int_0^T \|\Delta v\|_{L_{2k}(\Omega)}^2 \, dt \leq C \varepsilon^{-8\gamma^+}, \]

where \(C\) does not depend on \(\varepsilon\).

**Proof.** Considering the function

\[ w(x, t) = \int_0^v (x^2 + \varepsilon^2)^{\frac{1}{2}} \, d\xi = \int_0^v a_\varepsilon(x, \xi) \, d\xi, \]

and using the lower bound of \(a_\varepsilon(x, v)\), we obtain the result. \(\Box\)
we obtain \( w_t = a_\varepsilon(x, v)v_t \) and
\[
(2.6) \quad \nabla w = (v^2 + \varepsilon^2) \nabla v + \int_0^v \nabla \gamma \left( \frac{\Delta \gamma}{2} a_\varepsilon(x, \xi) \ln(\xi^2 + \varepsilon^2) \right) d\xi = a_\varepsilon(x, v) \nabla v + \int_0^v \nabla a_\varepsilon(x, \xi) d\xi.
\]

Hence the relations
\[
v_t = \frac{1}{a_\varepsilon(x, v)} w_t \quad \text{and} \quad a_\varepsilon(x, v) \nabla v = \nabla w - \int_0^v \nabla a_\varepsilon(x, \xi) d\xi
\]
are true. Consequently, \( w \) satisfies the equation
\[
\frac{1}{a_\varepsilon(x, v)} w_t = \text{div} \left( \nabla w - \int_0^v \nabla a_\varepsilon(x, \xi) d\xi \right) + f
\]
and therefore we can rewrite the previous equation in the form
\[
w_t = a_\varepsilon(x, v) \Delta w - a_\varepsilon(x, v) g,
\]
where
\[
g = \nabla a_\varepsilon(x, v) \cdot \nabla v + \int_0^v \left( \frac{\Delta \gamma}{2} a_\varepsilon(x, \xi) \ln(\xi^2 + \varepsilon^2) + \frac{|\nabla \gamma|^2}{4} a_\varepsilon(x, \xi) \ln^2(\xi^2 + \varepsilon^2) \right) d\xi - f.
\]

Multiplying the last equation by \( \Delta w \), integrating over \( \Omega \), and applying Green’s theorem to the first term on the right-hand side, we obtain the relation
\[
\int_\Omega \nabla w_t \cdot \nabla w \, dx + \int_\Omega a_\varepsilon(x, v)(\Delta w)^2 \, dx = \int_\Omega a_\varepsilon(x, v) g \Delta w \, dx.
\]

Integrating in \( t \) and noting that the first term on the right-hand side is a derivative, we arrive at
\[
\frac{1}{2} \int_\Omega |\nabla w(x, T)|^2 \, dx + \int_{\Omega_T} a_\varepsilon(\Delta w)^2 \, dx dt = \frac{1}{2} \int_\Omega |\nabla w(x, 0)|^2 \, dx + \int_{\Omega_T} a_\varepsilon g \Delta w \, dx dt.
\]

By Holder’s inequality, it follows that
\[
\int_\Omega |\nabla w(x, T)|^2 \, dx + \int_{\Omega_T} a_\varepsilon(\Delta w)^2 \, dx dt \leq \int_\Omega |\nabla w(x, 0)|^2 \, dx + \int_{\Omega_T} a_\varepsilon^2 g^2 \, dx dt.
\]

Using the estimates for \( v \) and \( \nabla v \) of Theorems 2.2 and 2.3, we get
\[
\int_\Omega |\nabla w(x, T)|^2 \, dx + \int_{\Omega_T} a_\varepsilon(\Delta w)^2 \, dx dt \leq C_0 + C_1 \varepsilon^{-\gamma^+} \leq C \varepsilon^{-\gamma^+}.
\]

In particular,
\[
\int_{\Omega_T} (\Delta w)^2 \, dx dt \leq C \varepsilon^{-2\gamma^+}.
\]

On the other hand, according to [15], we have
\[
\int_{\Omega_T} |\nabla w(x, t)|^4 \, dx dt \leq C \sup_{(x, t) \in \Omega_T} |w|^2 \int_{\Omega} \int_{\Omega_T} (\Delta w)^2 \, dx dt \leq C \varepsilon^{-2\gamma^+}.
\]
Now we derive estimates for the function $v$. Since
\[ a_\varepsilon(x, v) \nabla v = \nabla w - \int_0^v \nabla a_\varepsilon(x, \xi) \, d\xi, \]
we have
\[ \int_{\Omega} \lvert \nabla v \rvert^4 \, dx dt \leq C \varepsilon^{-6\gamma^+}. \]

In addition, differentiating (2.6), we obtain
\[ \Delta w = \text{div}(\nabla w) = a_\varepsilon \Delta v + a_\varepsilon \ln(v^2 + \varepsilon^2) \nabla (\varepsilon v) \cdot \nabla (\varepsilon v) + \gamma(v^2 + \varepsilon^2) \frac{\varepsilon^2}{2} v |\nabla v|^2 + r, \]
where
\[ r = \int_0^v \frac{\Delta \gamma}{2} a_\varepsilon(x, \xi) \ln(\xi^2 + \varepsilon^2) \, d\xi + \int_0^v \frac{|\nabla \gamma|^2}{4} a_\varepsilon(x, \xi) \ln^2(\xi^2 + \varepsilon^2) \, d\xi. \]

Consequently,
\[ \int_{\Omega} (a_\varepsilon \Delta v)^2 \, dx dt \leq \int_{\Omega} (\Delta w)^2 \, dx dt + \int_{\Omega} \left( \gamma(v^2 + \varepsilon^2) \frac{\varepsilon^2}{2} v |\nabla v|^2 \right)^2 \, dx dt \]
\[ + \int_{\Omega} a_\varepsilon^2 \ln^2(v^2 + \varepsilon^2)(\nabla (\varepsilon v) \cdot \nabla (\varepsilon v))^2 \, dx dt + \int_{\Omega} r^2 \, dx dt \]
\[ \leq C \varepsilon^{-2\gamma^+} + C \varepsilon^{-6\gamma^+} + C \varepsilon^{-\gamma^+} + C \leq C \varepsilon^{-6\gamma^+}. \]

Hence, by the lower bound of $a_\varepsilon$ the claim is proved.

**Theorem 2.5.** Suppose that $\gamma \in H^2(\Omega)$, $v$ is a solution of (2.1), condition (2.3) is satisfied, and $\nabla u_0 \in L_2(\Omega)$. Then
\[ \int_0^T \|v_t\|^2 \, dx \leq C \varepsilon^{-2\gamma^+}, \]
where $C$ does not depend on $\varepsilon$.

**Proof.** We use the technique used in Theorem 2.4.

Multiplying the first equation of (2.1) by $w_t$, where $w$ is defined by (2.5), and integrating over $\Omega$, we obtain the relation
\[ \int_\Omega v_t w_t \, dx + \int_\Omega \nabla w \nabla w_t \, dx = \int_\Omega f \, w_t \, dx + \int_\Omega \int_0^v \nabla a_\varepsilon(x, \xi) \, d\xi \nabla w_t \, dx, \]
which can be rewritten in the form,
\[ \int_\Omega \frac{1}{a_\varepsilon(x, v)} (w_t)^2 \, dx + \int_\Omega \nabla w \cdot \nabla w_t \, dx = \int_\Omega f \, w_t \, dx + \int_\Omega \int_0^v \nabla a_\varepsilon(x, \xi) \, d\xi \nabla w_t \, dx. \]

Applying Green’s formula to the last term on the right-hand side, we arrive at
\[ \int_\Omega \frac{1}{a_\varepsilon} (w_t)^2 \, dx + \int_\Omega \nabla w \cdot \nabla w_t \, dx = \int_\Omega f \, w_t \, dx + \int_\Omega \int_0^v \nabla a_\varepsilon \cdot \nabla v \, d\xi - \int_\Omega w_t r \, dx, \]
where
\[ r = \int_0^v \frac{\Delta \gamma}{2} a_\varepsilon(x, \xi) \ln(\xi^2 + \varepsilon^2) \, d\xi + \int_0^v \frac{|\nabla \gamma|^2}{4} a_\varepsilon(x, \xi) \ln^2(\xi^2 + \varepsilon^2) \, d\xi. \]
Integrating in $t$, this equation becomes
\[
\int_{\Omega_T} \frac{1}{a_\varepsilon} (w_t)^2 \, dx \, dt + \frac{1}{2} \int_{\Omega} |\nabla w(x,T)|^2 \, dx \\
= \frac{1}{2} \int_{\Omega} |\nabla w(x,0)|^2 \, dx + \int_{\Omega_T} f \, w_t \, dx \, dt - \int_{\Omega_T} w_t \nabla a_\varepsilon \cdot \nabla v \, dx \, dt - \int_{\Omega_T} w_t r \, dx \, dt.
\]
By Cauchy’s inequality,
\[
\frac{1}{2} \int_{\Omega_T} \frac{1}{a_\varepsilon} (w_t)^2 \, dx \, dt + \frac{1}{2} \int_{\Omega} |\nabla w(x,T)|^2 \, dx \\
\leq \frac{1}{2} \int_{\Omega} |\nabla w(x,0)|^2 \, dx + C \int_{\Omega_T} a_\varepsilon f^2 \, dx \, dt \\
+ C \int_{\Omega_T} a_\varepsilon |\nabla a_\varepsilon \cdot \nabla v|^2 \, dx \, dt + C \int_{\Omega_T} a_\varepsilon r^2 \, dx \, dt.
\]
Using the estimates for $v$ and $\nabla v$, we obtain
\[
\int_{\Omega_T} \frac{1}{a_\varepsilon} (w_t)^2 \, dx \, dt + \int_{\Omega} |\nabla w|^2 \, dx \, dt \leq C_0 + C_1 \varepsilon^{-\gamma^+} \leq C \varepsilon^{-\gamma^+}.
\]
Since
\[
|v_t| = \frac{1}{a_\varepsilon(x,v)} |w_t|,
\]
it follows that
\[
\int_{\Omega_T} (v_t)^2 \, dx \, dt \leq \varepsilon^{-\gamma^+} \int_{\Omega_T} \frac{1}{a_\varepsilon} (w_t)^2 \, dx \, dt \leq \varepsilon^{-2\gamma^+}
\]
and the result is established. □

Our next task is to show that $v$ is close to $u$ in an appropriate norm. To prove this result, we follow the ideas in the proof of the uniqueness theorem in [1] and the references therein.

**Lemma 2.6.** If $\gamma$ satisfies (1.2) and $v$ is bounded, then, for every $0 < \varepsilon < 1$, we have

\[
(2.7) \quad ||v|^\gamma(x) - (v^2 + \varepsilon^2)^{\frac{\gamma}{2}}|| \leq C \varepsilon.
\]

**Proof.** Applying the mean value theorem to the function $f(y) = (v^2 + y^2)^{\frac{\gamma}{2}}$, we have that, for every $x$, there exists $w \in [0, \varepsilon]$ such that
\[
f(\varepsilon) - f(0) = \gamma w(v^2 + w^2)^{\frac{\gamma-1}{2}} \leq \gamma (v^2 + w^2)^{\frac{\gamma-1}{2}} \varepsilon
\]
and this implies (2.7). □

**Lemma 2.7.** Let $u$ be a weak solution of (1.1) and $v$ a weak solution of (2.1). The function $w = u - v$ satisfies the equation

\[
(2.8) \quad \int_{\Omega_T} w(-\chi_t - A \Delta \chi + B \nabla \chi) \, dx \, dt = \int_{\Omega_T} (F[v] - F_\varepsilon[v]) \Delta \chi + (G[v] - G_\varepsilon[v]) \nabla \chi \, dx \, dt
\]
with
\[
(2.9) \quad A = \frac{1}{\gamma + 1} \frac{u|u|^\gamma - v|v|^\gamma}{u - v} \quad \text{and} \quad B = A \nabla \ln(\gamma + 1) - D,
\]
where
\[
D = \left| \frac{\nabla g}{2(\gamma + 1)} \left( \frac{u|u|^{\gamma} \ln(u^2) - v|v|^{\gamma} \ln(v^2)}{u - v} \right) \right|.
\]

**Proof.** Consider the functions
\[F(x, t) = F[u(x, t)] = \frac{u|u|^{\gamma}}{\gamma + 1} \quad \text{and} \quad F_\varepsilon(x, t) = F_\varepsilon[v(x, t)] = \int_0^t (\tau^2 + \varepsilon^2)^{\frac{\gamma}{2}} \, d\tau \]
which have the properties
\[
\nabla F = \left| u \right|^{\gamma} \nabla u + \frac{\nabla g}{\gamma + 1} \left( \frac{\ln(u^2)}{2} - \frac{1}{\gamma + 1} \right),
\]
\[
\nabla F_\varepsilon = (v^2 + \varepsilon^2)^{\frac{\gamma}{2}} \nabla v + \int_0^t \frac{\nabla g}{2(\tau^2 + \varepsilon^2)^{\frac{\gamma}{2}}} \ln(\tau^2 + \varepsilon^2) \, d\tau.
\]
Then we have
\[
\left| u \right|^{\gamma} \nabla u = \nabla F[u] - G[u]
\]
and
\[
(v^2 + \varepsilon^2)^{\frac{\gamma}{2}} \nabla v = \nabla F_\varepsilon - \int_0^t \frac{\nabla g}{2(\tau^2 + \varepsilon^2)^{\frac{\gamma}{2}}} \ln(\tau^2 + \varepsilon^2) \, d\tau = \nabla F_\varepsilon[v] - G_\varepsilon[v].
\]
The equations of the definitions of a weak solution can be rewritten as follows:
\[
\int_{\Omega_T} (-u\chi_t + \nabla F[u] \nabla \chi - G[u] \nabla \chi - f\chi) \, dx \, dt = - \int_\Omega u \chi \, dx \bigg|_{t=0}^T,
\]
\[
\int_{\Omega_T} (-v\chi_t + \nabla F_\varepsilon[v] \nabla \chi - G_\varepsilon[v] \nabla \chi - f\chi) \, dx \, dt = - \int_\Omega v \chi \, dx \bigg|_{t=0}^T.
\]
Since \( u = v = 0 \) on \( \partial \Omega \), we have that \( F[u] = F_\varepsilon[v] = 0 \) on \( \partial \Omega \). Applying Green’s theorem to the second term on the left-hand side, we obtain
\[
\int_{\Omega_T} (-u\chi_t - F[u] \Delta \chi - G[u] \nabla \chi - f\chi) \, dx \, dt = - \int_\Omega u \chi \, dx \bigg|_{t=0}^T,
\]
\[
\int_{\Omega_T} (-v\chi_t - F_\varepsilon[v] \Delta \chi - G_\varepsilon[v] \nabla \chi - f\chi) \, dx \, dt = - \int_\Omega v \chi \, dx \bigg|_{t=0}^T.
\]
By subtraction, we conclude that \( w = u - v \) satisfies the equation
\[
\int_{\Omega_T} (-w\chi_t - (F[u] - F_\varepsilon[v]) \Delta \chi - (G[u] - G_\varepsilon[v]) \nabla \chi) \, dx \, dt = 0,
\]
assuming that \( \chi(x, T) = 0 \). We now decompose the two subtractions as
\[
F[u] - F_\varepsilon[v] = (F[u] - F[v]) + (F[v] - F_\varepsilon[v]),
\]
\[
G[u] - G_\varepsilon[v] = (G[u] - G[v]) + (G[v] - G_\varepsilon[v]).
\]
Since

$$|F[v] - F_\varepsilon[v]| = \left| \frac{v|v|^\gamma}{\gamma + 1} - \int_0^v (r^2 + \varepsilon^2)^{\frac{\gamma}{2}} \, dr \right| = \left| \int_0^v \left( |r|^\gamma - (r^2 + \varepsilon^2)^{\frac{\gamma}{2}} \right) \, dr \right|$$

$$\leq C \sup |v| \varepsilon,$$

$$|G[v] - G_\varepsilon[v]| = \left| u|u|^\gamma \frac{\nabla \gamma}{\gamma + 1} \left( \frac{\ln(u^2)}{2} - \frac{1}{\gamma + 1} \right) - \int_0^u \frac{\nabla \gamma}{2} (r^2 + \varepsilon^2)^{\frac{\gamma}{2}} \ln(r^2 + \varepsilon^2) \, dr \right|$$

$$= \left| \int_0^u \left( |r|^\gamma \nabla \gamma \ln(|r|) - \frac{\nabla \gamma}{2} (r^2 + \varepsilon^2)^{\frac{\gamma}{2}} \ln(r^2 + \varepsilon^2) \right) \, dr \right|$$

$$\leq C \sup |v| \varepsilon$$

and \(F[u] - F[v] = Aw\) and \(G[u] - G[v] = -Bw\), with \(A\) and \(B\) defined in (2.9) and (2.10), (2.11) becomes (2.8).

**Lemma 2.8.** Let \(\eta(x, t)\) be the solution of the following parabolic problem:

\[
\begin{cases}
    \eta_t - (A + \epsilon) \Delta \eta + B \nabla \eta = \phi & \text{in } \Omega_T, \\
    \eta(x, 0) = 0 & \text{in } \Omega, \\
    \eta(x, t) = 0 & \text{on } \partial \Omega,
\end{cases}
\]

(2.12)

where \(\epsilon > 0\) is an arbitrary small parameter, \(\phi \in L_2(\Omega_T)\) is an arbitrary function, and \(A, B\) are defined in (2.9) and (2.10). If

\[
\frac{\phi^2}{A + \epsilon} \leq C,
\]

then

\[
\int_\Omega |\nabla \eta|^2 \, dx + \int_0^t \int_\Omega (A + \epsilon)(\Delta \eta)^2 \, dx \, dt \leq C.
\]

**Proof.** It is easy to verify that

$$0 \leq A \leq C, \quad |D| \leq C, \quad |B| \leq C, \quad \frac{|B|^2}{A} \leq C$$

with \(C\) depending only on \(\gamma^-, \nabla \gamma, \sup u,\) and \(\sup v\). By [14], for every \(\epsilon > 0\) and \(\phi \in L_2(\Omega_T)\), this problem has a unique continuous strong solution \(\eta\) such that \(\eta_t, \Delta \eta \in L_2(\Omega_T)\).

Multiplying (2.12) by \(\Delta \eta\) and integrating over \(\Omega\), we obtain

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla \eta|^2 \, dx + \int_\Omega (A + \epsilon)(\Delta \eta)^2 \, dx = \int_\Omega B \nabla \eta \Delta \eta \, dx - \int_\Omega \phi \Delta \eta \, dx = I_1 + I_2,$$

where we have applied Green’s formula to the first term. By Holder’s inequality,

$$|I_1| \leq \frac{1}{4} \int_\Omega (A + \epsilon)(\Delta \eta)^2 \, dx + \int_\Omega \frac{|B|^2}{A + \epsilon} |\nabla \eta|^2 \, dx,$$

$$|I_2| \leq \frac{1}{4} \int_\Omega (A + \epsilon)(\Delta \eta)^2 \, dx + \int_\Omega \frac{\phi^2}{A + \epsilon} \, dx.$$

From the last three relations, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_\Omega |\nabla \eta|^2 \, dx + \frac{1}{2} \int_\Omega (A + \epsilon)(\Delta \eta)^2 \, dx \leq \int_\Omega \frac{|B|^2}{A + \epsilon} |\nabla \eta|^2 \, dx + \int_\Omega \frac{\phi^2}{A + \epsilon} \, dx.$$
We have proved that
\[
\frac{|B|^2}{A + \epsilon} \leq \frac{|B|^2}{A} \leq C.
\]

Using Gronwall’s theorem, we arrive at the estimate (2.14). □

**Theorem 2.9.** Let \( u \) be a weak solution of (1.1) and \( v \) a weak solution of (2.1).

If

\[
1 < \gamma^- \leq \gamma^+ \text{ in } \overline{\Omega} \quad \text{and} \quad \sup_{x \in \Omega} |\nabla \gamma| < \infty,
\]

then

\[
\|u - v\|_{L^{\frac{\gamma^+ + 1}{\gamma^+}}(\Omega_T)} \leq C\epsilon^\frac{1}{2},
\]

where \( C \) does not depend on \( \epsilon \).

**Proof.** Choosing \( \chi(x, t) = \eta(x, T - t) \) in (2.8), we have

\[
\int_{\Omega_T} w\phi \, dx dt = \int_{\Omega_T} (w \epsilon \Delta \eta + (F[v] - F_{\epsilon}[v])\Delta \eta + (G[v] - G_{\epsilon}[v])\nabla \eta) \, dx dt = I_3 + I_4 + I_5.
\]

By Schwarz’s inequality and the estimates already established, we have the bounds

\[
|I_3| \leq \sup(|w|)\epsilon \frac{1}{2}\sqrt{T \text{meas}(\Omega)} \left( \int_{\Omega_T} \epsilon(\Delta \eta)^2 \, dx dt \right)^{\frac{1}{2}} \leq C\epsilon^{\frac{1}{2}},
\]

\[
|I_4| \leq \sup(|F[v] - F_{\epsilon}[v]|)\epsilon^{-\frac{1}{2}}\sqrt{T \text{meas}(\Omega)} \left( \int_{\Omega_T} \epsilon(\Delta \eta)^2 \, dx dt \right)^{\frac{1}{2}} \leq C\epsilon^{\frac{1}{2}},
\]

\[
|I_5| \leq \sup(|G[v] - G_{\epsilon}[v]|)\sqrt{T \text{meas}(\Omega)} \left( \int_{\Omega_T} |\nabla \eta|^2 \, dx dt \right)^{\frac{1}{2}} \leq C\epsilon.
\]

Then

\[
\left| \int_{\Omega_T} w\phi \, dx dt \right| \leq C(\epsilon^{\frac{1}{2}} + \epsilon^{-\frac{1}{2}} + \epsilon).
\]

If we put \( \epsilon = \frac{\epsilon}{2} \), then

\[
\left| \int_{\Omega_T} w\phi \, dx dt \right| \leq C\epsilon^{\frac{1}{2}}.
\]

Choosing \( \phi = |w|^\alpha \text{sign}(w) \), with \( \alpha \) a positive constant, we proved that

\[
\|w\|_{L^{\alpha+1}(\Omega_T)}^{\alpha+1} \leq C\epsilon^{\frac{1}{2}}, \quad \text{or equivalently,} \quad \|u - v\|_{L^{\alpha+1}(\Omega_T)}^{\alpha+1} \leq C\epsilon^{\frac{1}{2}}.
\]

Actually, if \( \phi = |u - v|^\alpha \text{sign}(u - v) \), then (2.13) is true because

\[
\frac{\phi^2}{A + \epsilon} \leq \frac{\phi^2}{A} = \frac{|u - v|^{2\alpha}}{(u - v)^{\gamma+1}(u - v)} \leq C,
\]

as long as \( \gamma \leq 2\alpha \).

Letting \( \alpha = \frac{\gamma^+}{2} \), the proof is complete. □
3. Space-time discretization. In this section, we follow [21] and apply the continuous Galerkin method in the space variable and the discontinuous Galerkin method in the time variable to the approximate problem. For simplicity we will restrict our study to $\mathbb{R}^2$, but the results can be easily generalized to any space dimension.

Suppose that $h$ is a positive constant and let $T_h$ denote a partition of $\Omega$ into disjoint triangles $T_k$ such that no vertex of any triangle lies on the interior of a side of another triangle. Let now $S_{hr}$ denote the set of continuous functions on the closure of $\Omega$ which are polynomials of degree $r$ in each triangle of $T_h$ and which vanish on $\partial\Omega$, that is,

$$S_{hr} = \{ w = w(x) \in C_0^0(\Omega) | w|_{T_k} \text{ is a polynomial of degree } r \text{ for all } T_k \in T_h \}.$$

In the same way, consider $\delta > 0$, the partition $[0,T] = \cup_{n=0}^{N-1} I_n$, $I_n = [t_n, t_{n+1}]$, $t_{n+1} = t_n + \delta$, and the space

$$S_{hr}^{s+} = \left\{ W = W_{h \delta} : [0, +\infty[ \to S_{hr} | W|_{I_n} = \sum_{n=0}^{s} t^n w_n(x), w_n \in S_{hr} \right\}.$$

We do not impose continuity at the nodal points, $t_n$, for those functions but they must be continuous to the left of these. For $W \in S_{hr}^{s+}$, we denote by $W_n$ and $W_n^+$ the value of $W$ and its limit from above at $t_n$, respectively. By $[W]_n$ we mean the jump of $W$ in $t_n$, defined as $[W]_n = W_n^+ - W_n$. We seek a discrete approximation $v \approx V \in S_{hr}^{s+}$ that satisfies the definition of a weak solution. We have

$$\int_{\Omega_T} (-V \chi_t + a_T(x, V) \nabla \cdot \nabla \chi - f \chi) \, dx \, dt = - \int_{\Omega_T} V \chi \, dx \bigg|_0^T,$$

that is

$$\sum_{n=0}^{N-1} \int_{I_n} \int_{\Omega} (-V \chi_t + a_T(x, V) \nabla \cdot \nabla \chi - f \chi) \, dx \, dt = - \int_{\Omega} V \chi \, dx \bigg|_0^T.$$

Integrating by parts the first term, using the continuity of $\chi(\cdot, t)$, and assuming that $\chi(x, T) = 0$, we obtain

$$\sum_{n=0}^{N-1} \int_{I_n} \int_{\Omega} V T \chi \, dx \, dt - \sum_{n=0}^{N-1} \int_{I_n} \int_{\Omega} V \chi \, dx \bigg|_{t_n}^{t_{n+1}} + \sum_{n=0}^{N-1} \int_{I_n} \int_{\Omega} a_T(x, V) \nabla \cdot \nabla \chi \, dx \, dt$$

$$= \sum_{n=0}^{N-1} \int_{I_n} \int_{\Omega} f \chi \, dx \, dt + \int_{\Omega} V_n^+ \chi_0 \, dx,$$

where $V_t$ is the piecewise polynomial of degree $s - 1$ which interpolates $\frac{\partial V}{\partial t}$ in each $I_n$ and $t_n^+$ is the limit when $t$ tends to $t_n$ from above. In particular, if $s = 0$, then $V_t = 0$. Using the notation $[V]_n = V_n^+ - V_n$, we have

$$\sum_{n=0}^{N-1} \int_{I_n} \int_{\Omega} V T \chi \, dx \, dt - \sum_{n=1}^{N-1} \int_{I_n} [V]_n \chi \, dx + \sum_{n=0}^{N-1} \int_{I_n} \int_{\Omega} a_T(x, V) \nabla \cdot \nabla \chi \, dx \, dt$$

$$= \sum_{n=0}^{N-1} \int_{I_n} \int_{\Omega} f \chi \, dx \, dt.$$
Choosing $S_{hr}^{\delta_s}$ for the test functions space we conclude that

$$\sum_{n=0}^{N-1} \int_{I_n} \int_{\Omega} V_n W \, dx \, dt + \sum_{n=0}^{N-1} \int_{I_n} [V_n]_w \, dx$$

$$+ \sum_{n=0}^{N-1} \int_{I_n} \int_{\Omega} a_\varepsilon(x, V) \nabla V \cdot \nabla W \, dx \, dt$$

$$= \sum_{n=0}^{N-1} \int_{I_n} \int_{\Omega} f W \, dx \, dt. \tag{3.1}$$

**Definition 3.1.** A function $V \in S_{hr}^{\delta_s}$ is said to be a discrete solution of the approximate problem if $V = 0$ on $\partial \Omega$ and satisfies (3.1), for all $W \in S_{hr}^{\delta_s}$.

Since $W$ is not required to be continuous at $t_n$, we may choose $W$ to vanish outside $I_n$. In that case, (3.1) reduces to $N$ equations, one for each $I_n$. Then the discrete problem consists in finding $V \in S_{hr}^{\delta_s}$ such that

$$\int_{I_n} \int_{\Omega} V W \, dx \, dt + \int_{I_n} V_{n-1}^+ W_{n-1}^+ \, dx + \int_{I_n} \int_{\Omega} a_\varepsilon(x, V) \nabla V \cdot \nabla W \, dx \, dt$$

$$= \int_{I_n} \int_{\Omega} f W \, dx \, dt + \int_{I_n} V_{n-1}^+ W_{n-1}^+ \, dx \quad \forall W \in S_{hr}^{\delta_s}, \forall n \in \{0, \ldots, N\}. \tag{3.2}$$

**Theorem 3.2.** If $V_{n-1} \in L_2(\Omega)$ and $f \in L_2(\Omega \times I_n)$, then problem (3.2) has a solution $V$.

**Proof.** Let $n \geq 1$ be fixed. For each $h, \delta > 0$, we define the continuous mapping $F : S_{hr}^{\delta_s} \to S_{hr}^{\delta_s}$ by

$$\int_{I_n} \int_{\Omega} F(V) W \, dx \, dt = \int_{I_n} \int_{\Omega} V W \, dx \, dt + \int_{I_n} V_{n-1}^+ W_{n-1}^+ \, dx$$

$$+ \int_{I_n} \int_{\Omega} a_\varepsilon(x, V) \nabla V \cdot \nabla W \, dx \, dt - \int_{I_n} \int_{\Omega} f W \, dx \, dt$$

$$- \int_{\Omega} V_{n-1}^+ W_{n-1}^+ \, dx \quad \forall W \in S_{hr}^{\delta_s}.$$

Choosing $W = V$,

$$\int_{I_n} \int_{\Omega} F(V) V \, dx \, dt = \int_{I_n} \int_{\Omega} V V \, dx \, dt + \int_{I_n} [V_{n-1}^+]^2 \, dx$$

$$+ \int_{I_n} \int_{\Omega} a_\varepsilon(x, V) [\nabla V]^2 \, dx \, dt - \int_{I_n} \int_{\Omega} f V \, dx \, dt$$

$$- \int_{\Omega} V_{n-1}^+ V_{n-1}^+ \, dx.$$

Using the lower bound of $a$ and applying the Holder inequality, we have

$$\int_{I_n} \int_{\Omega} F(V) V \, dx \, dt \geq \frac{1}{2} \int_{I_n} \frac{d}{dt} \|V\|_{L_2(\Omega)}^2 \, dt + \|V_{n-1}^+\|_{L_2(\Omega)}^2$$

$$+ \varepsilon \int_{I_n} \|\nabla V\|_{L_2(\Omega)}^2 \, dt - \int_{I_n} \|f\|_{L_2(\Omega)} \|V\|_{L_2(\Omega)} \, dt$$

$$- \|V_{n-1}\|_{L_2(\Omega)} \|V_{n-1}^+\|_{L_2(\Omega)}.$$
Hence, integrating the first term in \( t \), using the Cauchy inequality and the Poincaré inequality, we arrive at

\[
\int_{I_n} \int_{\Omega} F(V) \, V \, dx dt \geq \frac{1}{2} \| V_n \|^2_{L^2(\Omega)} + C \varepsilon \gamma^+ \int_{I_n} \| V \|^2_{L^2(\Omega)} \, dt
\]

\[
- \int_{I_n} \| f \|_{L^2(\Omega)} \| V \|_{L^2(\Omega)} \, dt - \frac{1}{2} \| V_{n-1} \|^2_{L^2(\Omega)}.
\]

Applying again the Cauchy inequality, we conclude that

\[
\int_{I_n} \int_{\Omega} F(V) \, V \, dx dt \geq \frac{1}{2} \| V_n \|^2_{L^2(\Omega)} + \frac{C \varepsilon \gamma^+}{2} \int_{I_n} \| V \|^2_{L^2(\Omega)} \, dt
\]

\[
- \frac{1}{2C \varepsilon \gamma^+} \int_{I_n} \| f \|^2_{L^2(\Omega)} \, dt - \frac{1}{2} \| V_{n-1} \|^2_{L^2(\Omega)}.
\]

If \( V \) belongs to

\[
B = \left\{ W \in S^0_{hr} \mid \| W \|_{L^2(\Omega \times I_n)} \leq \epsilon, \epsilon > \frac{1}{C \varepsilon \gamma^+} \| V_{n-1} \|_{L^2(\Omega)} + \frac{1}{(C \varepsilon \gamma^+)^2} + \| f \|_{L^2(\Omega \times I_n)} \right\},
\]

then \( \int_{\Omega} F(V) \, V \, dx \geq 0 \ \forall V \in \partial B \). A corollary to Brouwer’s fixed-point theorem implies the existence of \( V_* \in \partial B \) such that \( F(V_*) = 0 \). The claim is proved with \( V = V_* \).

If the approximation in time is of degree zero, the the definition of \( B \) does not depend on \( \varepsilon \).

Now we are going to prove the uniqueness of the fully discrete solution.

**Theorem 3.3.** Problem (3.2) has a unique solution \( V \).

**Proof.** Let \( V_{n-1} \) and \( f \) be two known functions and \( V_1 \) and \( V_2 \) be two solutions of (3.2). Then, for \( W \in S^0_{hr} \), we have

\[
\int_{I_n} \int_{\Omega} V_i W \, dx dt + \int_{\Omega} V_{n-1}^+ W_{n-1}^+ \, dx
\]

\[
+ \int_{I_n} \int_{\Omega} (a_\varepsilon(x, V_1) \nabla V_1 - a_\varepsilon(x, V_2) \nabla V_2) \cdot \nabla W \, dx dt = 0,
\]

with \( V = V_1 - V_2 \). Writing

\[
a_\varepsilon(x, V_1) \nabla V_1 - a_\varepsilon(x, V_2) \nabla V_2
\]

\[
= \int_0^1 \frac{d}{d\zeta} (a_\varepsilon(x, \zeta V_1 + (1 - \zeta) V_2) \nabla (\zeta V_1 + (1 - \zeta) V_2)) \, d\zeta
\]

\[
= \int_0^1 a_\varepsilon(x, \zeta V_1 + (1 - \zeta) V_2) \nabla V \, d\zeta
\]

\[
+ \int_0^1 \gamma ((\zeta V_1 + (1 - \zeta) V_2)^2 + \varepsilon^2)^{\frac{p-1}{2}} (\zeta V_1 + (1 - \zeta) V_2)(\zeta \nabla V_1 + (1 - \zeta) \nabla V_2) \, d\zeta
\]

\[
= A \nabla V + BV,
\]

with \( A = \int_0^1 a_\varepsilon(x, \zeta V_1 + (1 - \zeta) V_2) \, d\zeta \) and

\[
B = \int_0^1 \gamma ((\zeta V_1 + (1 - \zeta) V_2)^2 + \varepsilon^2)^{\frac{p-1}{2}} (\zeta V_1 + (1 - \zeta) V_2)(\zeta \nabla V_1 + (1 - \zeta) \nabla V_2) \, d\zeta,
\]
the equality (3.3) becomes
\[ \int_{I_n} \int_{\Omega} V_I W \, dx \, dt + \int_{\Omega} V_{n-1}^+ W_{n-1}^{-} \, dx + \int_{I_n} \int_{\Omega} (A \nabla V + BV) \cdot \nabla W \, dx \, dt = 0. \]
Integrating by parts the first term of the last inequality, we arrive at
\[ - \int_{I_n} \int_{\Omega} V W_t \, dx \, dt + \int_{\Omega} V_n W_n \, dx + \int_{I_n} \int_{\Omega} (A \nabla V + BV) \cdot \nabla W \, dx \, dt = 0. \]
Let \( W(x, t) = \eta(x, t_{n-1} + t_n - t) \), with \( \eta \) the discrete solution of the following problem:
\[
\begin{cases}
  \eta_t - div(A \nabla \eta) + B \nabla \eta = \phi & \text{in } \Omega \times int(I_n), \\
  \eta(x, t_{n-1}) = 0 & \text{in } \partial \Omega, \\
  \eta(x, t) = 0 & \text{on } \partial \Omega \times int(I_n).
\end{cases}
\]
In this problem, \( \phi \) is an arbitrary function in \( L_2(\Omega \times int(I_n)) \). Since \( \varepsilon \gamma^+ \leq A \leq C \gamma^+ \) and \( |B| \leq C \varepsilon \gamma^+ \), the problem has a unique weak solution in \( S^{\delta_h}_{\text{reg}} \). Then (3.4) becomes
\[ \int_{I_n} \int_{\Omega} V \phi \, dx \, dt = 0, \]
in virtue of \( W_n = W(x, t_n) = \eta(x, t_{n-1}) = 0 \). Since (3.6) is valid for any \( \phi \in L_2(\Omega \times int(I_n)) \), we conclude that \( V = 0 \) and, consequently, the solution is unique. \( \square \)

The rest of this section is devoted to the proof of an error estimate for the fully discrete scheme (3.1). The proof is very similar to that of Theorem 2.9.

**Lemma 3.4.** Let \( v \) be a solution of problem (2.1) and \( V \) a solution of problem (3.1). The function \( e = V - v \) satisfies the equation
\[ \int_{I_n} \int_{\Omega} (-e W_t + A \nabla e \cdot \nabla W + e B \cdot \nabla W) \, dx \, dt = \sum_{n=1}^{N-1} \int_{\Omega} e_n[W]_n \, dx - \int_{\Omega} e_N W_N \, dx \]
with
\[ A = \int_0^1 a_e(x, \xi V + (1 - \xi)v) \, d\xi \]
and
\[ B = \int_0^1 \frac{\nabla \gamma}{2} ((\xi V + (1 - \xi)v)^2 + \varepsilon^2) \ln((\xi V + (1 - \xi)v)^2 + \varepsilon^2) \, d\xi. \]

**Proof.** First, we rewrite (3.1) as
\[ - \sum_{n=0}^{N-1} \int_{I_n} \int_{\Omega} V W_t \, dx \, dt - \sum_{n=1}^{N-1} \int_{\Omega} V_n[W]_n \, dx + \sum_{n=0}^{N-1} \int_{I_n} a_e(x, V) \nabla V \cdot \nabla W \, dx \, dt \]
\[ = \sum_{n=0}^{N-1} \int_{I_n} \int_{\Omega} f W \, dx \, dt - \int_{\Omega} V_N W_N \, dx. \]
Simplifying, we obtain
\[ - \int_{I} \int_{\Omega} V W_t \, dx \, dt - \sum_{n=1}^{N-1} \int_{\Omega} V_n[W]_n \, dx + \int_{I} \int_{\Omega} a_e(x, V) \nabla V \cdot \nabla W \, dx \, dt \]
\[ = \int_{I} \int_{\Omega} f W \, dx \, dt - \int_{\Omega} V_N W_N \, dx. \]
Noting that the weak solution also satisfies this equation, we have
\[
- \int_J \int_{\Omega} v W_t \, dx \, dt - \frac{1}{N-1} \sum_{n=1}^{N-1} \int_{\Omega} v_n [W_n] \, dx + \int_J \int_{\Omega} a_e(x, v) \nabla v \cdot \nabla W \, dx \, dt
\]
\[
= \int_J \int_{\Omega} f W \, dx \, dt - \int_{\Omega} v_N W_N \, dx.
\]
It follows that
\[
\int_J \int_{\Omega} (\varepsilon W_t + (a_e(x, V) \nabla V - a_e(x, v) \nabla v) \cdot \nabla W) \, dx \, dt
\]
\[
= \sum_{n=1}^{N-1} \int_{\Omega} e_n [W_n] \, dx + \int_{\Omega} e_N W_N \, dx = 0,
\]
where \( e = V - v \) is the error of \( V \). If we write
\[
a_e(x, V) \nabla V - a_e(x, v) \nabla v
\]
\[
= \int_0^1 \frac{d}{d\xi} (a_e(x, \xi V + (1 - \xi)v) \nabla (\xi V + (1 - \xi) v)) \, d\xi
\]
\[
= \int_0^1 \gamma((\xi V + (1 - \xi)v)^2 + \varepsilon^2)^{\frac{1}{2}} (\xi V + (1 - \xi)v)(\xi \nabla V + (1 - \xi) \nabla v) e \, d\xi
\]
\[
+ \int_0^1 a_e(x, \xi V + (1 - \xi)v) \nabla e \, d\xi = Be + A \nabla e,
\]
then we arrive at (3.7).

**Lemma 3.5.** Let \( \eta(x, t) \) be the solution of the following parabolic problem:
\[
\begin{cases}
\eta_t - \text{div}(A \nabla \eta) + B \nabla \eta = \phi & \text{in } \Omega_T, \\
\eta(x, 0) = 0 & \text{in } \Omega, \\
\eta(x, t) = 0 & \text{on } \partial \Omega
\end{cases}
\]
(3.10)

with \( A \) and \( B \) as defined in (3.8) and (3.9) respectively. If
\[
\frac{\phi^2}{A} \leq C,
\]
then
\[
\int_{\Omega} |\nabla \eta|^2 \, dx + \int_0^t \int_{\Omega} A(\Delta \eta)^2 \, dx \, dt \leq C.
\]
(3.12)

**Proof.** We have that \( \varepsilon^{\gamma^+} \leq A \leq C \) and \( |B| \leq C \varepsilon^{\gamma^+} \) and so, by [14], for every \( \phi \in L_2(\Omega_T) \) and \( \varepsilon > 0 \), this problem has a unique continuous weak solution \( \eta \) such that \( \eta_t, \Delta \eta \in L_2(\Omega_T) \). Then we can rewrite the first equation of (3.10) as
\[
\eta_t - A \Delta \eta + \tilde{B} \nabla \eta = \phi \text{ in } \Omega_T
\]
with
\[
\tilde{B} = \int_0^1 \frac{\nabla \gamma}{2} ((\xi V + (1 - \xi)v)^2 + \varepsilon^2)^{\frac{1}{2}} \ln((\xi V + (1 - \xi)v)^2 + \varepsilon^2) \, d\xi.
\]
Using the same arguments as in the proof of Lemma 2.8, we conclude that
\[
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\nabla \eta|^2 \, dx + \frac{1}{2} \int_{\Omega} A(\Delta \eta)^2 \, dx \leq \int_{\Omega} \frac{|\tilde{B}|^2}{A} |\nabla \eta|^2 \, dx + \int_{\Omega} \frac{\phi^2}{A} \, dx.
\]

It is easy to verify that
\[
\frac{|\tilde{B}|^2}{A} \leq \left( \frac{\int_0^1 a_\varepsilon(x, \xi V + (1-\xi)v) \ln((\xi V + (1-\xi)v)^2 + \varepsilon^2) \, d\xi}{\int_0^1 a_\varepsilon(x, \xi V + (1-\xi)v) \, d\xi} \right)^2 \leq C \left( \frac{\int_0^1 a_\varepsilon(x, \xi V + (1-\xi)v) \, d\xi}{\int_0^1 a_\varepsilon(x, \xi V + (1-\xi)v) \, d\xi} \right)^2 \leq C.
\]

By Gronwall’s theorem the result is established. \(\square\)

**Theorem 3.6.** Let \(v\) be a solution of problem (2.1) and \(V\) a solution of problem (3.1). If \(\gamma \in H^2(\Omega)\) and the conditions of Theorem 2.2 are satisfied, then

(3.13)
\[
\|v - V\|_{L^{p+1}_{\text{loc}}(\Omega_T)} \leq C\varepsilon^{-\gamma^+} (h^{r+1}\|v\|_{L^\infty(0,T;H^{r+1}(\Omega))} + \delta^{s+1}\|v\|_{W^{s+1,\infty}(0,T;L^2(\Omega))}),
\]

where \(C\) does not depend on \(\varepsilon, h, r, \delta, \text{ or } s\).

**Proof.** First we define \(\tilde{V} \in S^k_{hr}\) in each \(I_n\) by

\[
\int_{\Omega} \tilde{V} w \, dx = \int_{\Omega} \check{v} w \, dx \quad \forall w \in S_{hr}, \ t \in I_n,
\]

where

\[
\check{v}(t_n) = v(t_n), \quad n = 1, \ldots, N
\]

and

\[
\int_{I_n} \int_{\Omega} \check{v} \tilde{W} \, dx dt = \int_{I_n} \int_{\Omega} v \tilde{W} \, dx dt \quad \forall \tilde{W} \in S^k_{hr}.\]

To prove that \(\tilde{V}\) is well defined, see [21].

Next, we write \(e = \theta + \rho\), where \(\theta = V - \tilde{V}\) and \(\rho = \tilde{V} - v\). By standard arguments [21], we can conclude that

\[
\|\rho\|_{L^2(\Omega)} \leq C(h^{r+1}\|v\|_{L^\infty(I_n;H^{r+1}(\Omega))} + \delta^{s+1}\|v\|_{W^{s+1,\infty}(I_n;L^2(\Omega))}) \quad \forall t \in I_n.
\]

It remains to show that \(\theta\) is bounded. Recalling (3.7), \(\theta\) satisfies the relation

\[
\int_{I_n} \int_{\Omega} (-\theta W_t - A \nabla \theta \cdot \nabla W + \theta B \cdot \nabla W) \, dx dt = \sum_{n=1}^{N-1} \int_{\Omega} \theta_n[W]_n \, dx + \int_{\Omega} \theta_N W_N dx
\]

\[
= \int_{I_n} \int_{\Omega} (\rho W_t + A \nabla \rho \cdot \nabla W - \rho B \cdot \nabla W) \, dx dt + \sum_{n=1}^{N-1} \int_{\Omega} \rho_n[W]_n \, dx - \int_{\Omega} \rho_N W_N dx.
\]

Let \(W\) be the discrete solution of problem (3.10). The proof of the existence of such
because, from the definition of \( \tilde{V} \), we have

\[
\int_I \int_\Omega \rho W_t \, dx \, dt = 0 \quad \text{and} \quad \int_\Omega \rho_n W_n \, dx = 0 \quad \text{and} \quad \int_\Omega \rho N W_N \, dx = 0.
\]

Using the estimate (3.12), the properties of the Ritz projection \( \Delta h \), and the definition of the discrete Laplacian \( \Delta_{h} \), (for more details, see [21]) we obtain

\[
|I_1| = \left| \int_I \int_\Omega A \nabla R_h \rho \cdot \nabla W \, dx \, dt \right| = \left| \int_I \int_\Omega A R_h \rho \cdot \Delta h W \, dx \, dt \right|
\]

\[
\leq \left( \int_I \int_\Omega (R_h \rho)^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_I \int_\Omega (A \Delta h W)^2 \, dx \, dt \right)^{\frac{1}{2}}
\]

\[
\leq C \varepsilon^{-\gamma} (h^{r+1} \|v\|_{L^\infty(I_n, H^{r+1}(\Omega))} + \delta^{s+1} |v|_{W^{s+1, \infty}(I_n, L^2(\Omega))})
\]

\[
\times \left( \int_I \int_\Omega A (\Delta h W)^2 \, dx \, dt \right)^{\frac{1}{2}}
\]

\[
\leq C \varepsilon^{-\gamma} (h^{r+1} \|v\|_{L^\infty(I_n, H^{r+1}(\Omega))} + \delta^{s+1} |v|_{W^{s+1, \infty}(I_n, L^2(\Omega))})
\]

\[
|I_2| \leq \left( \int_I \int_\Omega \rho^2 \, dx \, dt \right)^{\frac{1}{2}} \left( \int_I \int_\Omega (B \cdot \nabla W)^2 \, dx \, dt \right)^{\frac{1}{2}}
\]

\[
\leq C (h^{r+1} \|v\|_{L^\infty(I_n, H^{r+1}(\Omega))} + \delta^{s+1} |v|_{W^{s+1, \infty}(I_n, L^2(\Omega))})
\]

\[
\times \varepsilon^{-\gamma} \left( \int_I \int_\Omega |\nabla W|^2 \, dx \, dt \right)^{\frac{1}{2}}
\]

\[
\leq C \varepsilon^{-\gamma} (h^{r+1} \|v\|_{L^\infty(I_n, H^{r+1}(\Omega))} + \delta^{s+1} |v|_{W^{s+1, \infty}(I_n, L^2(\Omega))}).
\]

Letting \( \phi = |\theta|^{\gamma/2} \text{sign}(\theta) \), we have proved the result. \( \Box \)

If we put together the last theorem, Theorem 2.9 and the estimates for the solution \( v \) and its derivatives, then using the triangle inequality, we can estimate the error between \( u \) and \( V \) as

\[
\|V - u\|_{L^{r+1}_{\sigma}^t(\Omega_T)} \leq C \varepsilon^{-\alpha \gamma} h^{r+1} + C \varepsilon^{-\beta \gamma} \delta^{s+1} + C \varepsilon^\frac{\gamma}{2}.
\]

For suitable choices of \( r, s, h, \) and \( \delta \), we can prove that the error vanishes when \( \varepsilon \) tends to zero.

**4. Numerical results.** Finally, we are concerned with the numerical solution of the discretized problem. Obviously, there is no need to solve the problem simultaneously on all time intervals, since the problem on \( I_n \) is only dependent on the information from \( I_{n-1} \), but, for each \( I_n \), we need to solve a nonlinear system of equations.

**4.1. Solution procedure.** Because of limitations concerning the regularity of the solution, in what follows, we will only consider polynomials of degree zero in time. In this case, the equations are simpler since \( V_t = 0 \), \( V_{n-1}^+ = V_n \), and \( W_{n-1}^+ = W_n \).
Denoting \( f_n(x) = \frac{1}{t} \int_0^t f_n(x) \, dt \), we have to solve the equations

\[
\int_{\Omega} V_n W_n \, dx + \delta \int_{\Omega} a_e(x, V_n) \nabla V_n \cdot \nabla W_n \, dx = \int_{\Omega} f_n W_n \, dx + \int_{\Omega} V_{n-1} W_n \, dx
\]

for all \( W_n \in S_{hr}^0 \), \( n \in \{1, \ldots, N\} \).

In (4.1) we have a nonlinear system of equations which must be solved in each time interval. There are several methods we could use such as a Newton-type method or some linearization but we choose the fixed-point method, because it is efficient and easy to analyze and implement. To find \( V_n \), we use the following algorithm.

Given \( V_{n-1} \in S_{hr}^0 \), \( f_n \in L_2(\Omega) \), and \( tol > 0 \), we define \( k = 0 \) and \( W_0 = V_{n-1} \).

1. For \( k \geq 1 \), \( W_k \in S_{hr}^0 \) is calculated such that for all \( W \in S_{hr}^0 \),

\[
\int_{\Omega} W_k W \, dx + \delta \int_{\Omega} a_e(x, W_{k-1}) \nabla W_k \cdot \nabla W \, dx = \int_{\Omega} f_n W_k \, dx + \int_{\Omega} W_0 W \, dx.
\]

2. If \( \|W_k - W_{k-1}\| \geq tol \), set \( k = k + 1 \) and go to (1).

3. If \( \|W_k - W_{k-1}\| < tol \), set \( V_n = W_k \) and terminate.

Before proving the convergence of the algorithm, we need to prove the stability by proving that all the estimates are bounded.

**Lemma 4.1.** Assuming that \( \|V_{n-1}\|_{L_2(\Omega)} \leq C \), where \( C \) does not depend on \( n \) and \( f_n \in L_2(\Omega) \), there exists \( 0 < C' < \infty \) such that

\[
\|W_k\|_{L_2(\Omega)} \leq C' \quad \forall k > 0,
\]

where \( C' \) does not depend on \( n \) or \( k \).

**Proof.** If \( W = W_k \) in (4.2), then

\[
\int_{\Omega} W_k^2 \, dx + \delta \int_{\Omega} a_e(x, W_{k-1})(\nabla W_k)^2 \, dx = \int_{\Omega} f_n W_k \, dx + \int_{\Omega} W_0 W_0 \, dx.
\]

Since the second term is nonnegative and using Holder’s inequality, we have

\[
\|W_k\|_{L_2}^2 \leq \|f_n\|_{L_2} \|W_k\|_{L_2} + \|W_0\|_{L_2} \|W_k\|_{L_2}.
\]

Simplifying, \( \|W_k\|_{L_2} \leq \|f_n\|_{L_2} + \|W_0\|_{L_2} \).

The convergence is proved using the \( L_2 \) contraction and limiting the length of the time intervals.

**Theorem 4.2.** Let \( tol > 0 \). There exists \( C > 0 \) such that if \( \delta < \frac{1}{C(q + e^2 + \gamma^2)} \), then there exists \( k^* \in N \) so that

\[
\|W_k - W_{k-1}\|_{L_2(\Omega)} < tol \quad \forall k > k^*.
\]

**Proof.** To prove the convergence, we establish the \( L_2(\Omega) \) contraction property

\[
\|W_k - W_{k-1}\|_{L_2} < q\|W_{k-1} - W_{k-2}\|_{L_2}, \quad q < 1, \quad \forall k \geq 2.
\]

First, applying (4.2) with \( k \) and \( k - 1 \) and subtracting, we obtain

\[
\int_{\Omega} (W_k - W_{k-1}) W \, dx + \delta \int_{\Omega} ((W_k^2 + \varepsilon^2) \nabla W_k - (W_{k-1}^2 + \varepsilon^2) \nabla W_{k-1}) \cdot \nabla W \, dx = 0.
\]

Let \( E_k = W_k - W_{k-1} \in S_{hr}, W = E_k \),

\[
\int_{\Omega} E_k^2 \, dx + \delta \int_{\Omega} ((W_k^2 + \varepsilon^2) \nabla W_k - (W_{k-1}^2 + \varepsilon^2) \nabla W_{k-1}) \cdot \nabla E_k \, dx = 0.
\]
and we write \( a_ε(x, W_{k-1}) \nabla W_k - a_ε(x, W_{k-2}) \nabla W_{k-1} = a_ε(x, W_{k-1}) \nabla W_{k-1} - a_ε(x, W_{k-1}) \nabla W_k + a_ε(x, W_{k-2}) \nabla W_{k-2} - a_ε(x, W_{k-2}) \nabla W_{k-1} \). Then

\[
\int_Ω E_k^2 \, dx + \delta \int_Ω (a_ε(x, W_{k-1}) \nabla W_k - a_ε(x, W_{k-2}) \nabla W_{k-1}) \cdot \nabla E_k \, dx \\
+ \delta \int_Ω a_ε(x, W_{k-1}) |\nabla E_k|^2 \, dx = \delta \int_Ω a_ε(x, W_{k-2}) \nabla E_{k-1} \cdot \nabla E_k \, dx.
\]

Writing once again

\( a_ε(x, W_{k-1}) \nabla W_k - a_ε(x, W_{k-2}) \nabla W_{k-1} = A \nabla E_{k-1} + B E_{k-1} \),

with \( A = \int_0^1 a_ε(x, \xi) W_{k-1} + (1 - \xi) W_{k-2} \, d\xi \) and

\[
B = \int_0^1 \gamma((\xi W_{k-1} + (1 - \xi) W_{k-2})^2 + \varepsilon^2) \frac{d\gamma}{d\xi} \\
(\xi W_{k-1} + (1 - \xi) W_{k-2})(\xi \nabla W_{k-1} + (1 - \xi) \nabla W_{k-2}) \, d\xi,
\]

we have that \( \varepsilon^{\gamma^+} \leq A \leq C \) and \( |B| \leq C \varepsilon^{\gamma^+} \). Thus

\[
\int_Ω E_k^2 \, dx + \delta \int_Ω a_ε(x, W_{k-1}) |\nabla E_k|^2 \, dx + \delta \int_Ω A \nabla E_{k-1} \cdot \nabla E_k \, dx + \delta \int_Ω B E_{k-1} \cdot \nabla E_k \, dx \\
= \delta \int_Ω a_ε(x, W_{k-2}) \nabla E_{k-1} \cdot \nabla E_k \, dx,
\]

that is,

\[
\int_Ω E_k^2 \, dx + \delta \int_Ω a_ε(x, W_{k-1}) |\nabla E_k|^2 \, dx \\
= \delta \int_Ω (a_ε(x, W_{k-2}) - A) \nabla E_{k-1} \cdot \nabla E_k \, dx - \delta \int_Ω B E_{k-1} \cdot \nabla E_k \, dx.
\]

Using Holder’s inequality, we arrive at

\[
\int_Ω E_k^2 \, dx + \delta \int_Ω \varepsilon^{\gamma^+} |\nabla E_k|^2 \, dx \leq \delta \left( C \frac{\varepsilon^{-\gamma^+}}{2} ||\nabla E_{k-1}||^2 + \frac{\varepsilon^{\gamma^+}}{2} ||\nabla E_k||^2 \right) \\
+ \delta \left( C \frac{\varepsilon^{\gamma^+}}{2} ||\nabla E_{k-1}||^2 + \frac{\varepsilon^{\gamma^+}}{2} ||\nabla E_k||^2 \right).
\]

Simplifying,

\[
\int_Ω E_k^2 \, dx \leq \frac{C \delta \varepsilon^{-\gamma^+}}{2} ||\nabla E_{k-1}||^2 + \frac{C \delta \varepsilon^{\gamma^+}}{2} ||E_{k-1}||^2.
\]

By the inverse estimate \( ||\nabla E_{k-1}|| \leq C h^{-1} ||E_{k-1}|| \), we obtain

\[||E_k||^2 \leq C \delta (h^{-2} \varepsilon^{-\gamma^+} + \varepsilon^{\gamma^+}) ||E_{k-1}||^2 \Leftrightarrow ||E_k|| \leq (C \delta (h^{-2} \varepsilon^{-\gamma^+} + \varepsilon^{\gamma^+}))^{\frac{1}{2}} ||E_{k-1}||.\]

If

\[\delta < \frac{1}{C (h^{-2} \varepsilon^{-\gamma^+} + \varepsilon^{\gamma^+})},\]

the result is established with \( q = (C \delta (h^{-2} \varepsilon^{-\gamma^+} + \varepsilon^{\gamma^+}))^{\frac{1}{2}} < 1. \]
4.2. Example 1. As a first test, we will consider an example where the function $f$ and the initial condition are calculated in such a way that the problem has an exact solution.

Let $\Omega_T = [-1, 1] \times [-1, 1] \times [0, 0.8]$ and consider the problem (1.1) where

$$\gamma = 2 - \left(\frac{x}{2}\right)^2 + \left(\frac{y}{2}\right)^2$$

and the functions $f$ and $u_0$ are calculated such that

$$u(x, y, t) = \max\{(t + 0.1)^2 - x^2 - y^2, 0\}$$

is a solution of the problem. Notice that $\gamma$, $f$, and $u_0$ satisfy all the conditions required. A structured mesh is used and the tolerance is $10^{-10}$. The solution is approximated by Lagrange polynomials of degree 1 and the absolute error is measured in the $L^2(\Omega)$-norm for some values of $\varepsilon$, $h$, and $\delta$. In Figure 4.1, the evolution in $t$ of the discrete solution for $h = 0.1$ and $\delta = 10^{-3}$ is shown. The behavior agrees with the behavior of the exact solution.

We have made several runs with different values of $\varepsilon$ and we have collected the results in the first image in Figure 4.2. In this figure, we have represented epsilon versus the $L^2(\Omega)$-norm of the error on a logarithmic scale. It is clear that the solution converges and that the convergence is of order one.

In the second picture in Figure 4.2, we represent the $L^2(\Omega)$-norm of the solution error versus $h$ for some values of $\delta$ on a logarithmic scale. It is shown that the convergence for $h$ is of order two. The third picture in Figure 4.2 has the representation of the $L^2(\Omega)$-norm of the solution error versus $\delta$ for some values of $h$ on a logarithmic scale. It is shown, in this picture, that the convergence for $\delta$ is of order one, as expected.

4.3. Example 2. In the second example, we have chosen a problem where the initial data have a hole in the support.
Consider $\Omega_T = [-1.5, 1.5] \times [-1.5, 1.5] \times [0, 0.5]$ and the problem (1.1) with

$$\gamma = (x/2)^2 + (y/2)^2 + 1.1,$$

$f$ identically zero, and $u_0$ defined by

$$u_0 = \begin{cases} 
-\sin(2\pi \sqrt{x^2 + y^2}), & 0.5 < \sqrt{x^2 + y^2} < 1, \\
0, & \text{rest of the domain.}
\end{cases}$$

We used the same type of structured mesh as above.

In Figure 4.3, we show the evolution, in time, of the solution. We can see that the hole disappears in a finite time $t_* > 0$.

5. Conclusions and future work. We proved the convergence of a space-time Galerkin finite element method under reasonable conditions. The numerical computations agree with the theoretical results. The case where $\gamma$ depends on $t$ and the case where $\gamma < 1$ are under study.

REFERENCES


