Abstract

This survey contains both old and very recent results in non-quantitative aspects of inductive inference of total recursive functions. The survey is not complete. The paper was written to stress some of the main results in selected directions of research performed at the University of Latvia rather than to exhaust all of the obtained results. We concentrated on the more explored areas such as the inference of indices in non-Gödel computable numberings, the inference of minimal Gödel numbers, and the specifics of inference of minimal indices in Kolmogorov numberings.

Gödel numberings have many specific properties which influence the inference process very much. On the other hand, when discussing the desirability inductive inference we usually do not mention these properties. Hence the motivation is valid for inference of indices in non-Gödel computable numberings as well. Section 2 contains several results showing that the inference of indices in computable numberings can differ very much. For instance, there are computable numberings which are difficult for the inference, and only finite classes of total recursive functions can be identified. This shows that computable numberings can be very much removed from Gödel numberings.

We get rather similar results and even very similar methods of proofs when we consider the identification of minimal indices in Gödel numberings. It is difficult to express this similarity explicitly but many proofs can be expressed in parallel. Criteria for the identifiability of the minimal numbers, lattice-theoretical properties of the partial ordering of Gödel numberings with respect to identifiability of the minimal numbers, and identifiable classes with extremal characteristics are considered. Kolmogorov numberings which have special status in defining Kolmogorov complexity turn out to have special properties in inference of minimal numbers as well.

Section 9 presents results in abstract theory of identification types. It turns out that there are "typical" or "complete" classes of functions in many identification types \( \mathcal{M} \) such that to decide whether or not \( \text{EX}_\mathcal{M} \subseteq \text{EX}_L \) it suffices to check whether or not the "\( \mathcal{M} \)-complete" class \( \mathcal{V}_\mathcal{M} \) is in \( \text{EX}_L \). Unfortunately, the proposed technique for constructing these classes does not work for those types with "identification of minimal indices in specific Gödel numberings".

INDUCTIVE INFERENCE OF RECURSIVE FUNCTIONS: QUALITATIVE THEORY
1. INTRODUCTION

Recursion-theoretical concepts not explained below are treated in [Rog 67].
In textbooks motivating theory of numberings for programmers Goedel numberings
usually are explained as follows [Ersh 77].

Compiling of computer programs may be quite a job. Sometimes the reason of this is
not just technical but rather essential. By "a universal programming language" or "a
universal computer" we will understand a 2-argument partial recursive function \( \psi \)
universal for the class of all 1-argument partial recursive functions. By "program to
compute a 1-argument partial recursive function" we will understand its index, i.e. a
number \( n \in \mathbb{N} \) such that \( \varphi(x) = \pi x \cdot \psi(n,x) \).

Hence if we have two universal programming languages (i.e. two universal
functions \( \psi_0, \psi_1 \)) then the translation problem is the problem how to obtain any \( \psi_1 \)-index
for the function whose \( \psi_0 \)-index is the given \( n \).

Unfortunately, not always the translation can be performed by an algorithm, i.e. by a
total recursive function. Moreover, there are "universal programming languages" \( \psi_0 \) and
\( \psi_1 \) such that the translation is impossible whichever direction.

On the other hand, there is a "distinguished" universal programming language which
is "more universal" than the others. It allows translations from all universal programming
languages to it. Clearly, only it deserves to be called "universal". It is the Goedel
numbering. (Formally, there are many Goedel numberings but H.Rogers [Rog 58]
proved that they all are recursively isomorphic.)

We are to feel the difference between the two subsequent definitions 1.1 and 1.3.

**DEFINITION 1.1.** A numbering \( \psi \) of partial recursive functions is called
computable if:

1) for arbitrary \( n \), \( \psi_n(x) \) is a partial recursive function of the variable \( x \),
2) for arbitrary partial recursive function \( \chi \) of the variable \( x \), there is an \( n \) such that
\( \psi_n(x) = \chi(x) \),
3) there is a uniform algorithm which computes the values \( \psi_n(x) \) for all pairs \( (n,x) \).

**DEFINITION 1.2.** We say that the numbering \( \psi \) is reducible to the
numbering \( \varphi \) (\( \psi \leq \varphi \)) if there is an algorithm \( g \) such that for arbitrary \( n \), \( \psi_n(x) = \varphi_{g(n)}(x) \) for
all \( x \).

**DEFINITION 1.3.** A computable numbering \( \varphi \) is a Goedel numbering if
every computable numbering \( \psi \) of partial recursive functions is reducible to \( \varphi \).

There exist Goedel numberings. More than that. All "natural" (i.e. not specially
designed) computable numberings usually turn out to be Goedel ones. It is rather difficult
to construct a non-Goedel computable numbering.

Section 2 shows that non-Goedel computable numberings may be very difficult for
inductive inference. There is a computable numbering \( \psi \) such that \( \psi \)-indices can be
identified in the limit only for finite classes of total recursive functions.

Section 3 contains a series of results showing that computable numberings can differ very much with respect to inferrability. We investigate the partial ordering of computable numberings with respect to the possibility of inductive inference. The structure turns out to be very regular.

Goedel numberings are very much different for identification of minimal indices. (We remind these results in much detail in Section 4.) There are Goedel numberings where only finite sets of functions are identifiable and there are Goedel numberings where rather rich classes can be identified. In [Fre 75] a criterion of identifiability in at least one Goedel numbering was proved (it is repeated as our Theorem 4.2). The lattice of equivalence classes of Goedel numberings was studied in a paper [FK77] apparently unknown in the West. These results are presented in Section 6 of this paper. E.Kinber [Kin77] proved a series of theorems (with very complicated ideas of proofs, by the way) on mindchange complexity for identification of minimal Goedel numbers. Another paper by Kinber [Kin 83] considers a modification of our notion of nearly minimal Goedel numbers. He considers what does it mean "the second minimal" (as opposite to "not much larger than the minimal" in [Fre 75]). The main result in [Kin 83] shows that c-minimal Goedel numbers can be identified in at least one Goedel numbering if and only if strictly minimal Goedel numbers can be identified in another Goedel numbering.

Unfortunately, it is still an open problem (since 1977!) whether or not for any two Goedel numberings $\varphi_1$ and $\varphi_2$ there is a third one where minimal numbers can be identified for every class such that for it the minimal numbers are identifiable in at least one numbering $\varphi_1$ or $\varphi_2$. We do not know even whether or not there are two classes $U_1$ and $U_2$ for which the minimal Goedel numbers are identifiable but never in the same numbering.

Our Section 8 shows that progress is made recently for Kolmogorov numberings. Kolmogorov numberings are "the most informative" ones. By the definition, every Goedel numbering can be reduced to the Kolmogorov numbering via a recursive function with no more rapid than linear growth.

Section 8 contains Theorem 8.3 solving a long standing open problem. (Unfortunately, the proof turned out to be not very much complicated). It was proved, in contrast to Theorem 1.1 , that for arbitrary Kolmogorov numbering there always is an infinite identifiable class of total recursive functions.

Let $\mathcal{G}$ denote the family of all Goedel numberings. Let $\varphi \in \mathcal{G}$. By $\mathcal{B}$ we denote the class of all total recursive functions of one argument.

We fix a Cantor (i.e. computable one-one) numbering $<x_1,x_2,...,x_n>$ of all n-tuples $\{x_1,x_2,...,x_n\}$ of nonnegative integers as n varies. We fix also a Cantor numbering $c(x,y)$ of all pairs of nonnegative integers (c(x,y)=z, l(z)=x, r(z)=y).

By $f^{x+1}$ we denote the (x+1)-tuple $\{f(0),f(1),...,f(x)\}$.

Strategy is an arbitrary partial recursive function.

**Definition 1.4** A strategy $F$ identifies in the limit a $\varphi$-number for a total
recursive function $F(f \in EX_{\varphi}(F))$ if:

1) for every $n$ the value $F(<f[n]>)$ is defined, and

2) the limit $\lim_{n \to \infty} F(<f[n]>) = a$ exists and equals to a $\varphi$-index of the function $f$ (i.e. $\varphi_a = f$).

There is a "folk lemma" used by nearly all authors in papers on inductive inference.

**Lemma 1.1.** For arbitrary Gödel numbering $\{F_i\}$ of all partial recursive strategies there is a family $\{F'_i\}$ of total recursive strategies such that for arbitrary $i$ and total recursive $f$, if $\lim_{n \to \infty} F_i(<f[n]>)$ exists then $\lim_{n \to \infty} F'_i(<f[n]>)$ exists as well and the limits are equal.

**Definition 1.5.** A class $U$ of total recursive functions is $\varphi$-identifiable in the limit ($U \in EX_{\varphi}$) if there is a strategy $F$ which identifies in the limit a $\varphi$-index for every function in the class $U$.

It is easy to see that if $\varphi$ and $\psi$ are two Gödel numberings then $EX_{\varphi} = EX_{\psi}$. Hence it is natural to write just $U \in EX$ instead of $U \in EX_{\varphi}$.

To strengthen some of our results we use a modification of our Definitions 1.4 and 1.5.

**Definition 1.6.** A strategy $F$ identifies the class $U$ of total recursive functions in the sense $BC_{\varphi}$ if for every $f \in U$:

1) for every $n$ the value $F(<f[n]>)$ is defined,

2) there is an $n_0$ such that for every $n > n_0$ the value $F(<f[n]>)$ equals a $\varphi$-index of the function $f$.

Like $EX$-identification, $BC_{\varphi} = BC_{\psi}$ for all Gödel numberings $\varphi$ and $\psi$. Hence we use the notation $BC$.

J.M. Barzdin [Bar 74] proved that $EX \subset BC$ and $EX \neq BC$.

The minimal $\varphi$-index of a function $f$ is denoted by $\min_{\varphi}(f)$.

**Definition 1.7.** Let $h(x)$ be a total recursive function. We say that $h$-minimal $\varphi$-indices are identifiable in the limit for a class $U$ ($U \in EX_{\varphi}^{h\text{-min}}$) if there is a strategy $F$ such that:

1) $U \subseteq EX_{\varphi}(F)$,

2) for arbitrary $f \in U$, $\lim_{n \to \infty} F(<f[n]>) \leq h(\min_{\varphi}(f))$.

We pay somewhat more attention to the particular case $h(x) = x$. We use $EX_{\varphi}^{\text{min}}$ instead of $EX_{\varphi}^{h\text{-min}}$ for this function.

Now we introduce some more pieces of notation. Let $\alpha$ be an arbitrary string of integers. By $\alpha 0^k$ we denote a string the first elements of which are taken from $\alpha$ and the last $k$ elements are zeros. Similarly, by $\alpha 0^\infty$ we denote a total function the first values of which correspond to the string $\alpha$, and all the other values equal 0.
**DEFINITION 1.8.** A Gödel numbering $\varphi$ is called Kolmogorov numbering if every computable numbering $\psi$ is reducible to $\varphi$ via total recursive function which never exceeds a linear function.

**2. INFERENCE IN COMPUTABLE NUMBERINGS**

It is natural to question which classes of total recursive functions are identifiable in the limit in all computable numberings. Theorem 2.1 in this Section shows that finite classes and only they are identifiable in all computable numberings. Afterwards we try to understand what properties of the numberings make them difficult for the inference.

**LEMMA 2.1.** There is a total function $f$, graph of which is in $\Pi_1$, such that:

1) for arbitrary partial recursive function $\psi$ the equality $\psi(x) = f(x)$ can hold for no more than finite number of different $x$,
2) $(\forall x) (\exists p) (f(x) = c(x, p)).$

**PROOF.** To prove that the graph of a function $f$ is in the class $\Pi_1$ of Kleene-Mostowski [Rog 67], we need to prove that there is an algorithm $M$ uniform in $x$ and $y$ such that it stops for all pairs $(x, y)$ such that $f(x) \neq y$ and it does not stop for the pairs $(x, y)$ such that $f(x) = y$.

We consider an auxiliary algorithm $M$ which has only one argument $x$ and outputs an infinite sequence of all possible true inequalities $f(x) \neq y$. Since for arbitrary $x$ inequalities $f(x) \neq y$ are produced for all nonnegative integers $y$ but one, the function $f$ will be defined unambiguously.

The algorithm $M$ works as follows. It computes in parallel $\varphi_0(x), \varphi_1(x), \ldots, \varphi_x(x)$ (where $\varphi$ is a fixed Gödel numbering) during 1, 2, 3, ... steps. In parallel with these computations the following inequalities are output: $f(x) \neq 0$, $f(x) \neq 1$, $f(x) \neq 2$, ..., $f(x) \neq c(x,0) - 1$, $f(x) \neq c(x,0) + 1$, $f(x) \neq c(x,0) + 2$, ...

If one of the computations $\varphi_0(x), \varphi_1(x), \ldots, \varphi_x(x)$ stops then $M$ compares whether the result equals $c(x,0)$. If not then $M$ goes on to output the same sequence. If yes then $M$ outputs $f(x) \neq c(x,0)$, searches for the least $p_1$ such that $f(x) \neq c(x, p_1)$ has not yet been output and $c(x, p_1)$ does not equal the values $\varphi_0(x), \varphi_1(x), \ldots, \varphi_x(x)$ computed up to this moment, and $M$ outputs inequalities ..., $f(x) \neq c(x, p_1) - 2$, $f(x) \neq c(x, p_1) - 1$, $f(x) \neq c(x, p_1) + 1$, $f(x) \neq c(x, p_1) + 2$, ...

In parallel, the computation of $\varphi_0(x), \varphi_1(x), \ldots, \varphi_x(x)$ is continued, and in the case of new results again it is checked whether the new result equals $c(x, p_1)$, etc.

There is a bounded number of computations $\varphi_0(x), \varphi_1(x), \ldots, \varphi_x(x)$ that can change the current $c(x, p_1)$. Inevitably, there will be the last $c(x, p_1)$ and this value will become $f(x).$

Relativizing this proof to a creative oracle, we get a proof for
LEMMA 2.2. There is a total function $f$, graph of which is in $\Pi_2$, such that:
1) for arbitrary partial function $\psi$, graph of which is in $\Sigma_2$, the equality $\psi(x) = f(x)$ can hold for no more than finite number of different $x$,
2) $(\forall x) (\exists p) (f(x) = c(x,p))$.

The graph of $f$ being in $\Pi_2$ can be represented by an algorithm $M$ uniform in $x$ and $y$ such that for all pairs $(x,y)$ it produces a sequence of zeros and ones, and if $f(x) \neq y$ then the sequence contains no more than a finite number of ones.

R.M. Friedberg [Fri 58] proved that there is a computable one-to-one numbering $v'$ of all 1-argument partial recursive functions. We use $v'$ and the algorithm $M$ to construct a computable numbering $v$. The function $v_n(t)$ is defined as follows. It equals $v'_{l(n)}(t)$ if $M$ for $x = l(n)$ and $y = r(n)$ outputs a sequence containing no less than $t$ ones, and it is undefined, otherwise.

LEMMA 2.3. $v$ is a computable numbering of all 1-argument partial recursive functions, and every total recursive function has only one index in $v$.

PROOF. If a function $\psi$ has a $v'$-index $n$ then its $v$-index is $c(n,f(n))$, and all the numbers of type $c(n,p)$ where $p \neq f(n)$ are $v$-indices of finite functions. Hence, if $\psi$ has an infinite domain then it has no $v$-indices differing from $c(n,f(n))$. □

THEOREM 2.1. There is a computable numbering $v$ of all 1-argument partial recursive functions such that no infinite class of total recursive functions is in $\text{EX}_v$.

PROOF. Assume from the contrary that an infinite class of total recursive functions is $v$-identifiable in the limit by a strategy $F$. We define an auxiliary function $\eta(x,t)$:

$$
\eta(x,t) = \begin{cases} 
0, & \text{if } t = 0, \\
F(<v'_{x[t-1]}>,) & \text{if } t > 0.
\end{cases}
$$

Unfortunately, the function $\eta(x,t)$ is only partial recursive. We define another auxiliary function, a total one. To compute $g(x,t)$, compute sequentially $\eta(x,0), \eta(x,1), \eta(x,2), \ldots$ using $t$ steps of computation in total. Then $g(x,t)$ equals the last computed value in the abovementioned sequence (and it equals 0 if $t$ steps do not suffice to compute $\eta(x,0)$). We define $\psi(x) = \lim_{t \to \infty} r(g(x,t))$. Since $\psi$ is a limit of a total recursive function, its graph is in $\Sigma_2$.

Recall that every total recursive function has only one index in $v'$ and one index in $v$. If the $v'$-index of a total function is $n$ then its $v$-index is $c(n,f(n))$. If $F$ identifies the $v$-index of $v'_n$ then $\lim_{t \to \infty} \eta(n,t) = c(n,f(n))$. Hence $\lim_{t \to \infty} g(n,t) = c(n,f(n))$ and $\psi(n) = \lim_{t \to \infty} r(g(n,t)) = f(n)$. 

Lemma 2.2 implies that the values of $\psi$ and $f$ can equal only for finite number of values of the argument. Contradiction.

3. PARTIAL ORDERING OF COMPUTABLE NUMBERINGS

The reducibility of computable numberings $\psi \leq \phi$ means the existence of an algorithm which, for arbitrary function $\psi_i$ finds its $\phi$-index $j$ ($\phi_j = \psi_i$). There are equivalent numberings which are reducible two ways. For instance, all Goedel numberings are equivalent. The partial ordering $P_1$ of the equivalence classes of the computable numberings of all 1-argument partial recursive functions is studied intensively (see [Ersh 77]). It is known that $P_1$ is upper semilattice but it is not lower semilattice.

We could consider "limiting reducibility" and the corresponding partial order of equivalence classes of computable numberings. It would produce rather similar results. Instead, we considered another reducibility of the same computable numberings. We use $\psi \leq_{EX} \phi$ to denote that for arbitrary class $U$ of total recursive functions $U \in EX_{\psi}$ implies $U \in EX_{\phi}$. It is easy to see that if $\psi \leq \phi$ (and even if $\psi \leq \phi$ in the limit) then $\psi \leq_{EX} \phi$.

Let $P_2$ be the partial ordering of the equivalence classes with respect to $\leq_{EX}$. We will see that $P_1$ and $P_2$ are very much different.

The numbering considered in Theorem 2.1 is in the minimum element of $P_2$. On the other hand, there is no minimum element in $P_1$.

The ordering $P_1$ and $P_2$ have maximum elements, namely, the classes of computable numberings equivalent to Goedel numberings.

It is known (see [Ersh 77]) that $P_1$ is not a lower semilattice.

THEOREM 3.1. $P_2$ is a distributive lattice.

PROOF. The greatest lower bound for the equivalence classes containing the numberings $\psi'$ and $\psi''$, respectively, is the equivalence class containing the numbering.

$$(\psi' \otimes \psi'')_{c(i,j)}(x) = \begin{cases} 
\psi'_i(x), & \text{if } \psi'_i(x) = \psi''_j, \\
\text{undefined, if otherwise.} & \end{cases}$$

To prove that $(\psi' \otimes \psi'')$ is contained in the greatest lower bound, we are to prove that if a computable numbering $\psi$ is such that $\psi \leq_{EX} \psi'$ and $\psi \leq_{EX} \psi''$ then $\psi \leq_{EX} (\psi' \otimes \psi'')$. Indeed, let $\psi \leq_{EX} \psi'$, and $U \in EX_{\phi'}$. Then $U \in EX_{\psi'}$. We denote the strategy identifying in the limit $\psi'$-indices for $U$ by $F$. Let $\psi \leq_{EX} \psi''$. Then $U \in EX_{\psi''}$ as well. We denote the strategy identifying in the limit $\psi''$-indices for $U$ by $G$. We consider the strategy $H(<f[x]>)=c(F(<f[x]>),G(<f[x]>))$. It is easy to see that $H$ identifies in the limit the $(\psi' \otimes \psi'')$-indices for $U$.

The least upper bound for the equivalence classes containing the numberings $\psi'$ and $\psi''$, respectively, is the equivalence class containing the numbering
\[
(v' \oplus v'')_n(x) = \begin{cases} 
v'_i(x), & \text{if } n=2i, \\
v'_i(x), & \text{if } n=2i+1. 
\end{cases}
\]

To prove that \((v' \oplus v'')\) is contained in the least upper bound, we are to prove that if a computable numbering \(\psi\) is such that \(v' \leq_{\text{EX}} \psi\) and \(v'' \leq_{\text{EX}} \psi\) then \((v' \oplus v'') \leq_{\text{EX}} \psi\).

Indeed, let a strategy \(F\) identify in the limit \((v' \oplus v'')\)-indices for functions in \(U\). We divide \(U\) into two subclasses \(U_1\) and \(U_2\). If the strategy \(F\) assigns in the limit to the function \(f \in U\) an even value \(2i\) then \(f \in U_1\) and strategy \(F_1\) assigns in the limit to this function the \(v'\)-index \(i\). If the strategy \(F\) assigns in the limit to the function \(f \in U\) an odd value \(2i+1\) then \(f \in U_2\) and strategy \(F_2\) assigns in the limit to this function the \(v''\)-index \(i\).

Since \(U_1 \subseteq \text{EX}_v\) and \(v' \leq_{\text{EX}} \psi\), there exists a strategy \(G_1\) identifying in the limit correct \(\psi\)-indices for \(f \in U_1\). Since \(U_2 \subseteq \text{EX}_v\) and \(v'' \leq_{\text{EX}} \psi\), there exists a strategy \(G_2\) identifying in the limit correct \(\psi\)-indices for \(f \in U_2\). Hence \(U\) can be \(\text{EX}_\psi\)-identified by a strategy.

\[
G(<f[x]>)= \begin{cases} 
G_1(<f[x]>), & \text{if } F(<f[x]>) \text{ is even,} \\
G_2(<f[x]>), & \text{if } F(<f[x]>) \text{ is odd.} 
\end{cases}
\]

To prove the distributivity of the lattice, we are to prove \(\text{EX}\)-equivalence of the numberings:

\[
v' \oplus (v' \otimes v'') \text{ and } (v' \oplus v'') \otimes (v' \oplus v'''),
v' \otimes (v' \otimes v'') \text{ and } (v' \otimes v'') \otimes (v' \otimes v''').
\]

By our definitions above,

\[
(v' \oplus (v' \otimes v'''))_n(x) = \begin{cases} 
v'_i(x), & \text{if } n=2i, \\
v'''_i(x), & \text{if } n=2c(k,l)+1 \text{ and } v'_k(x)=v'''_l(x), 
\end{cases}
\]

undefined, if otherwise.

\[
((v' \oplus v'') \otimes (v' \otimes v'''))_{c(i,j)}(x) = \begin{cases} 
v'_{k(x)}=v'_1(x), & \text{if } i=2k, j=2l \text{ and } v'_{k(x)}=v'_1(x), \\
v'_{k(x)}=v'_1(x), & \text{if } i=2k+1, j=2l \text{ and } v'_{k(x)}=v'_1(x), \\
v''_{i(x)}=v'''_1(x), & \text{if } i=2k, j=2l+1 \text{ and } v''_{k(x)}=v'''_1(x), \\
v''_{i(x)}=v'''_1(x), & \text{if } i=2k+1, j=2l+1 \text{ and } v''_{k(x)}=v'''_1(x), \\
v'_{k(x)}=v'''_1(x), & \text{if } i=2k, j=2l+1 \text{ and } v'_{k(x)}=v'''_1(x), 
\end{cases}
\]

undefined, if otherwise.

The numberings are \(\text{EX}\)-equivalent if for arbitrary class \(U\) of total recursive functions identifiability in the limit in the other numbering. The equivalence of these two numberings can be proved by simple modification of the outputs of the strategies. This modification relies heavily on our interest in total functions only. Indeed, for every index in one of these two numberings it is possible to find effectively an index in the other numbering such that if the given number is an index of a total function then the corresponding index in the other numbering is an index for the same function. (If the index is defined using a \(v'_i\) in the first numbering then the corresponding index in the
other numbering is defined either by \( v'_i \) or by \( v'_i = v'_i \). If \( v' \)-indices are not involved in definition of the given index in the first numbering then it is defined by \( v'\_k = v'\_i \) and the other numbering contains the same function.)

The other two numberings are:

\[
(v' \oplus (v' \ominus v'''))_n(x) = \begin{cases} 
  v'_i(x) = v'\_k(x), & \text{if } n = c(i, 2k), \text{ and } v'_i(x) = v'\_k(x), \\
  v'_i(x) = v''\_i(x), & \text{if } n = c(i, 2l+1), \text{ and } v'_i(x) = v''\_i(x), \\
  \text{undefined, if otherwise,} & 
\end{cases}
\]

\[
((v' \otimes v') \ominus (v' \otimes v'''))_n(x) = \begin{cases} 
  v'_i(x) = v'\_k(x), & \text{if } n = 2c(i, k), \text{ and } v'_i(x) = v'\_k(x), \\
  v'_i(x) = v''\_i(x), & \text{if } n = 2c(i, k)+1, \text{ and } v'_i(x) = v''\_i(x), \\
  \text{undefined, if otherwise.} & 
\end{cases}
\]

For these numberings the proof is even simpler.

Theorem 3.1 shows that \( P_2 \) is much more regular ordering than \( P_1 \) and hence it deserves some attention.

4. IDENTIFICATION OF MINIMAL INDICES

We try to describe

\[
\varphi \in \Sigma \cap \text{EX}_h^{h-\text{min}} \text{ and } \varphi \in \Sigma \cap \text{EX}_h^{h-\text{min}}
\]

in this Section. The results and the methods of proof remind very much the identification of arbitrary indices in computable (non-Goedel) numberings. However we have no formal linkage between these two areas.

THEOREM 4.1. For arbitrary total recursive function \( h(x) \) there is a Goedel numbering \( \varphi \) such that arbitrary class \( U \) of total recursive function is in \( \text{EX}_\varphi^{h-\text{min}} \) iff card \( (U) \) is finite.

LEMMA. There exists a function \( \chi(x, j) \) such that:

1) the graph of \( \chi \) is in the class \( \Pi_2 \) of Kleene-Mostowski hierarchy,
2) \( (\forall i) (\forall j) (\chi(i, j) \text{ defined } \Rightarrow \chi(i, j) = 1) \),
3) \( (\forall i) (\exists j < i+1) (\chi(i, j) = 1) \),
4) \( (\forall \psi \in \text{PLR})(\exists i_0)(\forall i > i_0) (\psi(i) \text{ defined } \Rightarrow \chi(i, \psi(i)) \text{ not defined}) \).

REMARK. It is convenient to regard \( \chi(i, j) \) as a many-valued (many-many) mapping \( N \rightarrow N \). The statement 3) asserts that \( \chi \) maps every \( i \) to at least one integer not exceeding \( i+1 \). The statement 4) asserts that the mapping \( \chi \) differs almost everywhere from any mapping definable by a function in \( \Sigma_2 \).

PROOF OF LEMMA. Let \( K \) be a creative set [Rog 67]. Let \( \mathcal{M}_m^K \) be a Turing oracle-machine with number \( m \) using the oracle \( K \).

We define the function \( \chi \) as follows. If \( j > i+1 \), then \( \chi(i, j) \) is not defined. If \( j < i+1 \) and
there exists \( m \leq i \) such that \( \mathcal{M}_m^K(i) = j \), then \( \chi(i,j) \) is not defined. Otherwise \( \chi(i,j) = 1 \).

The checking of statements 1), 2), 3), 4) is not complicated.

**PROOF OF THEOREM.** Let \( \chi(i,j) \) be the function defined in Lemma. The statements 1) and 2) imply the existence of a g.r. function \( h(i,j,t) \) such that for every \( i,j \) the following two assertions are equivalent:

a) \( \chi(i,j) \) is defined, b) \( h(i,j,t) = 1 \) for infinitely many \( t \).

We shall regard the case \( \text{EX}_{\varphi} \min \) first. Let \( \varphi \) be an arbitrary Goedel numbering. Let \( Z(y,n) \) be a total recursive function such that

\[
\varphi_z(y,n)(x) = \begin{cases} 
\varphi_y(x), & \text{if there are at least } x \text{ such that } h(y,n,t) = 1 \\
\text{not defined, otherwise}
\end{cases}
\]

We define a numbering \( \varnothing \) as follows.

\[
\varnothing_0 = \varphi_z(0,0), \quad \varnothing_1 = \varphi_z(0,1), \quad \varnothing_2 = \varphi_0, \\
\varnothing_3 = \varphi_z(1,0), \quad \varnothing_4 = \varphi_z(1,1), \quad \varnothing_5 = \varphi_z(1,2), \quad \varnothing_6 = \varphi_1, \\
\varnothing_7 = \varphi_z(2,0), \quad \varnothing_8 = \varphi_z(2,1), \quad \varnothing_9 = \varphi_z(2,2), \quad \varnothing_{10} = \varphi_z(2,3), \quad \varnothing_{11} = \varphi_2.
\]

The numbering \( \varnothing \) is evidently computable. It is a Goedel numbering because \( \varphi \) is reducible to \( \varnothing \) via

\[
w(i) = \frac{i + 7i + 4}{2}
\]

If \( f \) is a total recursive function and \( i = \min_{\varphi}(f) \), then by the statement 3)

\[
\frac{(i-1)^2 + 7(i-1) + 4}{2} < \min_{\varnothing}(f) < \frac{i^2 + 7i + 4}{2}
\]

Let an arbitrary strategy \( F \) identify in the limit the minimal \( \varnothing \)-numbers for a class \( U \) of total recursive functions. We will prove that the cardinality of \( U \) is finite.

We define auxiliary functions.

\[
\eta(i,t) = \begin{cases} 
0, & \text{if } t = 0 \\
F(<\varphi_1(0), \varphi_1(1), \ldots, \varphi_1(t-1)>,) & \text{if } t > 0
\end{cases}
\]

Let a Turing machine \( \mathcal{M} \) computing \( \eta \) be fixed. Let \( g(i,t) \) be the maximal \( S \) not exceeding \( t \), such that all the values \( \eta(i,0), \eta(i,1), \ldots, \eta(i,s) \) are computable by \( \mathcal{M} \) in no more than \( t \) steps. If \( t \) steps do not suffice for computing \( \eta(i,0) \), then \( g(i,t) = 0 \).

Let

\[
a \Downarrow b = \begin{cases} 
a - b, & \text{if } a > b \\
0, & \text{if } a < b
\end{cases}
\]

\[
\psi^1(i) = \lim_{t} g(i,t), \quad \psi(i) = \lim_{t} \left( g(i,t) - \frac{(i-1)^2 + 7(i-1) + 4}{2} \right) - 1.
\]
If i is the minimal φ-number of a total recursive function and φ₁ ∈ U, then ψ₁(i) and ψ(i) are defined, ψ₁(i) is the minimal ω-number of φ₁ and χ₁(ψ₁(i))=1. The statement 4) implies there can be at most a finite number of such i. This concludes the proof for the case h(x)≡x.

For other functions h the definition of ω is to be slightly modified. We put functions into the following sequence to define the numbering ω:

\[ \varphi_0(0,0), \varphi_0(0,1), 0, 0, \ldots, 0, \varphi_0, \]

\[ \varphi_1(0,0), \varphi_1(1,1), 0, 0, \ldots, 0, \varphi_1, \]

\[ \varphi_2(0,0), \varphi_2(1,1), \varphi_2(2,2), 0, 0, \ldots, 0, \varphi_2, \]

where 0 is the empty function and if the "block" \( \varphi_{i+1}(0,0), \varphi_{i+1}(1,1), \ldots, \varphi_{i+1}(i+1,0) \) has got numbers \( n, n+1, \ldots, n+i+1 \) in the numbering ω, then 0 is repeated \( \max \{h(n), h(n+1), \ldots, h(n+i+1)\} \) times before the function \( \varphi_i \). It is easy to see that every h-minimal ω-number of a g.r. function is absolutely minimal.

**COROLLARY.**

\[ \bigcap_{h \in \mathbb{N}} \mathcal{E}_\varphi \min = \bigcap_{h \in \mathbb{N}} \mathcal{E}_\varphi \text{h-min} = \text{the class of all finite sets}. \]

**DEFINITION 4.1.** A class U of total recursive functions is φ-standardizable in the limit with a recursive bound \( (U \in \text{LSR}_\varphi) \) if there are total recursive functions \( G(i,n) \) and \( v(i) \) such that if i is a φ-index of a function \( f \in U \) then:

1) the limit \( \lim_{n \to \infty} G(i,n) \) exists and equals to a φ-index of f;
2) if i and j are φ-indices of the same f ∈ U then \( \lim_{n \to \infty} G(i,n) = \lim_{n \to \infty} G(j,n) \);
3) \( \text{card} \{G(i,0), G(i,1), \ldots\} \leq v(i) \).

It is easy to see that if \( \varphi \) and \( \psi \) are two Gödel numberings then \( \text{LSR}_\varphi = \text{LSR}_\psi \). Hence it is natural to write just \( \text{LSR} \).

**THEOREM 4.2.** For arbitrary class U of total recursive functions \( (U \in \bigcup_{\varphi \in \mathcal{G}} \mathcal{E}_\varphi \min) \iff (U \in \mathcal{E} \text{ & } U \in \text{LSR}). \)

**PROOF.** ⇒ Let \( U \in \mathcal{E}_\varphi \min \). Then evidently \( U \in \mathcal{E} \). To prove \( U \in \text{LSR} \) it suffices to consider the functions \( v(i) \) and \( G(i,n) \), where \( v(i)=i+1 \)

\[ G'(i,n) = \begin{cases} F(\varphi_1(0), \varphi_1(1), \ldots, \varphi_1(n)), & \text{if } F(\varphi_1(0), \varphi_1(1), \ldots, \varphi_1(n)) < i, \\ i, & \text{if } F(\varphi_1(0), \varphi_1(1), \ldots, \varphi_1(n)) = i, \\ G(i,t), & \text{if } t \text{ is the maximal integer not exceeding } n, \text{ such that all values } \end{cases} \]

\[ G(i,n) = \begin{cases} G'(i,0), G'(i,1), \ldots, G'(i,t), & \text{are computable in } n \text{ steps of Turing machine,} \\ i, & \text{if } n \text{ steps do not suffice to compute } G'(i,0). \end{cases} \]

⇐ Let \( \varphi \in \mathcal{G} \), \( U \in \mathcal{E} \), \( U \in \text{LSR} \). Let the φ-identification in the limit of the class \( U \) be
carried out by a strategy F and the ϕ-standardization be carried out by G(i,n) with a recursive estimate v(i).

We define three auxiliary functions. P.r. function ξ(i,j) is equal to G(i,k), where k is the minimal integer such that the set \{G(i,0),G(i,1),...,G(i,k)\} contains more than j different elements.

\[ ϕ_{d(m)}(x) = \begin{cases} ϕ_m(x), & \text{if } (\exists y > x) \ (G(m,y) = m), \\ \text{not defined}, & \text{otherwise} \end{cases} \]

\[ z(i,j) = d(ξ(i,j)). \]

We define a numbering \( Γ \) as follows:

\[ Γ_0 = ϕ_z(0,0), Γ_1 = ϕ_z(0,1),..., Γ_{v(0)-1} = ϕ_z(0,v(0)-1), Γ_v(0) = ϕ_0, \]
\[ Γ_{v(0)+1} = ϕ_z(1,0), Γ_{v(0)+2} = ϕ_z(1,1),..., Γ_{v(0)+v(1)} = ϕ_z(v(0)+v(1)-1), Γ_{v(0)+v(1)+1} = ϕ_1 \]
\[ Γ_{v(0)+...+v(i)-1+i} = ϕ_z(i,0), Γ_{v(0)+...+v(i)+i} = ϕ_z(i,v(i)-1), \]
\[ Γ_{v(0)+...+v(i)+i+i} = ϕ_i. \]

The numbering \( Γ \) is a Goedel numbering because \( ϕ \) is reducible to \( Γ \) via \( w(i) = v(0)+v(1)+...+v(i)+i \).

We shall prove a following characteristics of the function \( d(m) \): if \( ϕ_i \in U \), then for every natural \( m \):

\[ (ϕ_{d(m)} = ϕ_i) \iff (m = \lim_{n} G(i,n)) \]

Indeed, the implication \( \Rightarrow \) follows from \( \lim_{n} G(\lim_{n} G(i,n),n) = \lim_{n} G(i,n) \). The implication \( \Rightarrow \) follows from the definition of the function \( d(m) \) (if \( G(m,y) = m \) for infinitely many \( y \), then \( ϕ_{d(m)} = ϕ_i \) and hence \( ϕ_m = ϕ_i \) and \( \lim_{n} G(m,n) = \lim_{n} G(i,n) \)).

Thus no integer of type \( v(0)+...+v(i)+i \) can be the minimal \( Γ \)-number of a function from the class U.

A strategy \( F' \) which identifies the minimal \( Γ \)-numbers for the class \( U \) can be defined as follows. Let \( m(f,n) \) denote the value \( G(F(<f(0),f(1),...,f(n)>),n) \). To define \( F'(<f(0),f(1),...,f(n)>) \) we compute \( m(f,n) \) and use a fixed Turing machine, computing \( ξ \), to calculate the following values

\[ ξ(0,0), ξ(0,1),..., ξ(0,v(0)-1), \]
\[ ξ(1,0), ξ(1,1),..., ξ(1,v(1)-1), \]
\[ ξ(m(f,n),0), ξ(m(f,n),1),..., ξ(m(f,n),v(m(f,n))-1). \]

Let \( (i,j) \) be the first pair (according to the sequence \( (*) \)), such that the computation of \( ξ(i,j) \) stops in no more than \( n \) steps and \( ξ(i,j) = m) \). Then \( F' \) gets a value equal to the number, which the function \( ϕ_{z(i,j)} \) has in the numbering \( Γ \), i.e.

\[ F'(<f(0),f(1),...,f(n)>) = v(0)+v(1)+...+v(i-1)+i-1+j. \]

Otherwise, \( F'(<f(0),f(1),...,f(n)>) = 0. \)
**COROLLARY 1.** There exist Gödel numberings \( \varphi' \) and \( \varphi'' \), such that 

\[ \text{EX}_{\varphi'} \neq \text{EX}_{\varphi''}. \]

**COROLLARY 2.** For every class \( U \) of g.r. functions and for every total recursive function \( h \), such that \( h(x) \geq x \) for all \( x \), it is true that 

\[ (U \in \bigcup_{\varphi \in \mathcal{G}} \text{EX}^h_{\varphi} \land \text{min}) \iff (U \in \text{EX} \land U \in \text{LSR}). \]

**COROLLARY 3.** 

\[ \bigcup_{\varphi \in \mathcal{G}} \text{EX}^h_{\varphi} \land \text{min} = \bigcup_{\varphi \in \mathcal{G}} \text{EX}^{\text{min}}_{\varphi}. \]

**REMARK.** The class \( U_0 \) consisting of all \( \{0,1\} \)-valued total functions which equal 1 for at most finite number of values of the argument, is in \( \text{EX} \) but not in \( \text{EX}^{\text{min}}_{\varphi} \) for any \( \varphi \in \mathcal{G} \). Thus we have proved \( U_0 \not\in \text{LSR} \) and \( \text{EX} \not\subseteq \text{LSR} \). Using a construction developed by E.B. Kinber it can be proved that \( \text{LSR} \not\subseteq \text{EX} \) as well.

**THEOREM 4.3.** There are classes \( U_1 \) and \( U_2 \) of total recursive functions such that 

\[ U_1 \in \text{EX} \setminus \text{LSR}, \]

\[ U_2 \in \text{LSR} \setminus \text{EX}. \]

Theorem 4.2 shows that there are Gödel numberings \( \varphi \) and \( \psi \) such that 

\[ \text{EX}^{\text{min}}_{\varphi} \neq \text{EX}^{\text{min}}_{\psi}. \]

This opens a rich area of investigation, e.g. to characterize the non-equivalent Gödel numberings.

We denote \( \bigcup_{\varphi \in \mathcal{G}} \text{EX}^{\text{min}}_{\varphi} \) by \( \text{EX}^{\text{min}} \). Another corollary of Theorem 4.2 shows that 

\[ \bigcup_{\varphi \in \mathcal{G}} \text{EX}^{h-\text{min}}_{\varphi} = \text{EX}^{h-\text{min}} \]

equals the same \( \text{EX}^{\text{min}} \) for arbitrary \( h(x) \geq x \).

E.B. Kinber [FK 77] proved very difficult theorem on the number of mindchanges in \( \text{EX}^{\text{min}} \)-identification. Mindchanges are characterised by a function whose argument is \( \min_{\varphi}(f) \).

**THEOREM 4.4.** There is a class \( U_1 \in \text{EX}^{\text{min}} \) which cannot be \( \text{EX}^{\text{min}}_{\varphi} \)-identified with a recursive bound on mindchanges for arbitrary Gödel numbering \( \varphi \).

**THEOREM 4.5.** There is a Gödel numbering \( \varphi \) such that:

1) there is an infinite class \( U \) of total recursive functions such that \( U \in \text{EX}^{\text{min}}_{\varphi} \),

2) there is no effectively enumerable class \( U \) of total recursive functions such that \( U \in \text{EX}^{\text{min}}_{\varphi} \).

We consider two lemmas before the proof. It is known that Post’s theorem on the existence of simple sets can be strengthened [Tra 65].

**LEMMA 4.1.** There is a set \( A \) such that:
1) $A$ is recursively enumerable,
2) for arbitrary $m$ there are at most $\log_2 m$ elements in $A \cap \{0,1,2,\ldots,m\}$,
3) $A$ has no infinite recursively enumerable subsets.

**PROOF.** By $W_x$ we denote the domain of $\varphi_x$. Let $B = \{<x,y> | ye W_x \& \& y \geq 2^{x+1}\}$. We fix an effective enumeration of the set $B$, and construct a set $B' = \{<x,y> | \forall z \left( [z \neq y \& <x,z> e B] \Rightarrow <x,y> \text{ succeeds } <x,y> \text{ in the enumeration of } B \right)\}$. We define $A = \{y | (\exists x)(<x,y> e B')\}$. The properties 1), 3) are proved in the same way as in Post’s theorem (see [Rog 67]). The property 2) is implied by $y \geq 2^{x+1}$ in the definition of the set $B$.

By relativization of this proof to a creative oracle we get the proof of

**LEMMA 4.2.** There is a set $A$ such that:
1) $A \in \Sigma_2$ ($A$ is enumerable with a creative oracle),
2) for arbitrary $m$ there are at most $\log_2 m$ elements in $A \cap \{0,1,2,\ldots,m\}$,
3) $A$ has no infinite subsets from $\Sigma_2$.

We use these lemmas to prove the Theorem.

**PROOF OF THEOREM.** Let $\psi$ be a Goedel numbering. We consider two auxiliary numberings of all partial recursive functions of 1 argument.

$$
\beta_i(x) = \begin{cases} 
\psi_j(x), & \text{if } i = 2j, \\
j, & \text{if } i = 2j + 1,
\end{cases}
$$

$$
\gamma_i(x) = \beta_{i(i)}(x)
$$

We use a specific $c(x,y)$, namely, $c(0,0) = 0$, $c(0,1) = 1$, $c(1,0) = 2$, $c(0,2) = 3$, $c(1,1) = 4$, $c(2,0) = 5$, $c(0,3) = 6$, $c(1,2) = 7, \ldots$ Then for arbitrary $n, k$ among the functions $\gamma_0, \gamma_1, \ldots, \gamma_k^2$ the function $\psi_n$ occurs no less than $k-n$ times.

We define a numbering $\nu$ using the numbering $\gamma$ and the set $A$ from Lemma 4.2. $A \in \Sigma_2$. Hence, the partial characteristic function

$$
\chi_A(y) = \begin{cases} 
1, & \text{if } ye A, \\
\text{undefined, if } ye \bar{A}
\end{cases}
$$

can be represented as a limit of a total recursive function $\chi_A(y) = \lim_{s \to \infty} h(y,s)$. For arbitrary $i, x$ the value $\nu_i(x)$ is defined as follows: if in the sequence $h(i,0), h(i,1), h(i,2), \ldots$ there are no less than $x$ values differing from $1$ then $\nu_i(x) = \gamma_i(x)$; otherwise $\nu_i$ is undefined. It is obvious that if $i$ is a $\gamma$-index of a function with infinite domain then $(\nu_i = \gamma_i) \Leftrightarrow (i \in \bar{A})$. 
Now we prove that for every function $\psi_i$ (total or not) $\min_v(\psi_i) \leq 2i^2$. Indeed, for arbitrary $n,k$ among the functions $\gamma_0, \gamma_1, \ldots, \gamma_k^2$ there are no less than $k-n$ occurences of $\beta_n$. For arbitrary $n$ there is $k$ such that $k-n > 1 + \log_2 k^2$. Hence for this $k$ there is an $i \in \{0,1,2,\ldots, k^2\}$ such that $i \in \bar{A}$ and $\gamma_i = \beta_n = \psi_i$.

Now we prove that no infinite effectively enumerable class can be $EX_v^{\min}$-identifiable.

Assume that a strategy $F$ $EX_v^{\min}$-identifies an effectively enumerable class $U = \{\alpha_i(x)\}$. Let $g(i,n) = F(<\alpha_i[n]>)$ and $\eta(i) = \lim_{n \to \infty} g(i,n)$. Since $\eta(i)$ is the limit of a total recursive function, its range is in $\Sigma_2$. Since $F$ identifies $U$ and all the functions in $U$ are total, we conclude that $\{\eta(0), \eta(1), \eta(2), \ldots\} \subseteq \bar{A}$ and $\alpha_i \neq \alpha_j$ imply $\eta(i) \neq \eta(j)$. Then, by the assertion 3) of Lemma 4.2, the range of $\eta$ is finite, and hence $\text{card}(U) < \infty$.

Now we prove that there is an infinite class of total recursive functions in $EX_v^{\min}$.

We consider the class $U_0$ of all constants. We define a strategy $G(<y_0, y_1, \ldots, y_n>) = s(2y_0 + 1, 0)$. It identifies correct $\gamma$-indices for all functions in $U_0$. We will use the property of $\gamma$ mentioned above, namely, if $i$ is a $\gamma$-index of a total function then $(\forall i = \gamma_i) \Leftrightarrow (i \in \bar{A})$.

We introduce a set $c = \{c(2 \cdot 0 + 1, 0), c(2 \cdot 1 + 1, 0), c(2 \cdot 2 + 1, 0), \ldots\}$. For arbitrary $k$ the cardinality of $C \cap \{0, 1, 2, \ldots, 2k^2\}$ is no less than $k$. On the other hand, in $\{0, 1, 2, \ldots, 2k^2 + k\}$ there are no more than $\log_2 (2k^2 + k)$ elements of $A$. Hence $C \cap \bar{A}$ is infinite. Hence $G$ correctly identifies $v$-indices for an infinite class of constant functions.

We will use this strategy $G$ to define new Gödel numberings.

The numberings $\mu = \{\mu_i\}$ is defined by a shuffle of $\{v_i\}$ and $\{\psi_i\}$ which puts $2i^2$ equal functions $\psi_i$ immediately after $v_{2i^2}$. Note that all the minimal $\mu$-indices come from $\{v_i\}$.

We denote by $G'$ the strategy which produces as the results the corresponding $\mu$-indices instead of $v$-indices produced by $G$.

Now we describe the construction of the numbering $\varphi$. First we fix some procedures of parallel computation of functions $\mu_j$, $j = 0, 1, \ldots$. Let $\mu_1(x_1), \mu_2(x_2), \ldots, \mu_j(x_m), \ldots$ be values of $\mu_1$ in the order of their appearance during the parallel computation. Assume that $\mu_1(x_1), \ldots, \mu_j(x_r)$ are already computed and $(x_1, \mu_1(x_1)), \ldots, (x_r, \mu_1(x_r))$ are included in the graph of $\varphi_i$. Carry out the two following procedures simultaneously.

**PROCEDURE 1.** Compute $\mu_i(x_{r+1}), \mu_i(x_{r+2}), \ldots$. After every step $k$ of computation find the maximal $m_k$ such that the result $\mu_i^k(x)$ of computing $k$ step for $\mu_i(x)$ is defined for every $x \leq m_k$, and compute $a_k := G'(<\mu_i^k[m_k]>).$ If $a_k$ appears with the property $a_k \leq i$ then, for every $\mu_i(x_i)$ computed up to the moment $k$, include $(x_i, \mu_i(x_i))$ in the graph of $\varphi_i$. 
PROCEDURE 2. Compute $\mu_1(x_{r+1})$. If $\mu_1(x_{r+1})$ halts in some step $k$, find $m \geq k$ and pairwise distinct numbers $j_1, j_2, \ldots, j_c \leq i$ such that $c > 2(j_1)^2$, and for every $j \in \{j_1, j_2, \ldots, j_c\}$ there holds $\mu_1^{k}[k] \subseteq \mu_1^{m}[m]$. If such $m$ and $j_1, j_2, \ldots, j_c$ have been found, include $(x_{r+1}, \mu_1(x_{r+1}))$ in the graph of $\phi_i$.

After any halting of PROCEDURE 1 or PROCEDURE 2 and after including a new pair in the graph of $\phi_i$ renew PROCEDURE 1 and PROCEDURE 2 for new $\mu_1(x_{r+1})$, $\mu_2(x_{r+2})$, ...

The definition of $\phi_i$ is completed.

Clearly, $\phi$ is a computable numbering of partial recursive functions. To prove that $\phi \in \mathcal{G}$ it suffices to prove that $\mu$ is reducible to $\phi$. We will use the following property of our construction. If $i$ is not one of first $2(\min_{\mu} f)^2$ $\mu$-indices of $\mu_i$ then $\phi_i = \mu_i$. (Indeed, for such an $i$ PROCEDURE 2 halts for every $\mu_i(X_{r+1})$.)

Now we prove that $\phi$ is a G"odel numbering. Let $\mu_1$ be an arbitrary partial recursive function. Given $j$, one can effectively find a $\mu$-index $i > j$ for $\mu_1$ such that at least $c > 2j^2$ $\mu$-indices of $\mu_i$ are between $j$ and $i$. (The well-known padding technique can be used, see [Rog 67]. Define $g(j) = i$. Obviously, $g$ reduces $\mu$ to $\phi$.

Now we prove that $G'$ identifies minimal $\phi$-indices for infinitely many total functions. Indeed, let $U \in \text{Ex}_\mu(f')$, $f \in U$ and $\lim_{k \to \infty} G'(<f^{[k]}>) = i$. Then $\mu_i = f$. We have to prove that $\phi_i = f$ and $\phi_j \neq f$ for all $j < i$. The functions $\phi_i$ and $f$ equal since PROCEDURE 1 halts infinitely often. Further, $i$ is one of the first $2(\min_{\mu} f)^2$ $\mu$-indices of $\mu_i$. Hence if $j < i$ and $\mu_i = f$ then there is no set $\{j_1, j_2, \ldots, j_c\}$ of $\mu$-indices for $f$ such that $j_m < j$, $1 \leq m \leq c$.

Hence we get that there is a step $k_0$ such that for arbitrary $j_1, j_2, \ldots, j_c < j$, $c > 2(j_1)^2$, $m \geq k_0$ there is at least one $j \in \{j_1, j_2, \ldots, j_c\}$ with the property $\mu_i^{k_0}[k_0] \notin \mu_i^{m}[m]$.

Obviously, PROCEDURE 2 never halts after step $k_0$. On the other hand, $f \in \text{Ex}_\mu(G')$. Hence there is a $k_1$ such that $a_k = G'(<\mu_i^{[m]}>) = i$ and $i > j$ for all $k \geq k_1$ and $m_k$ defined in PROCEDURE 1. Hence PROCEDURE 1 also never halts after the moment $k_1$. We see that no pair $(x, \mu_j(x))$ is included in the graph of $\phi_j$ at the steps $k \geq \max(k_0, k_1)$. Hence $\phi_j \neq f$.

THEOREM 4.6. There is a G"odel numbering $\phi$ such that:

1) there is an infinite class $U$ of total recursive functions such that $U \in \text{Ex}_{\phi}^{\min}$,

2) every class $U \in \text{Ex}_{\phi}^{\min}$ is contained in an effectively enumerable class $V$ of total recursive functions such that $V \in \text{Ex}_{\phi}^{\min}$.

PROOF. Let $\psi$ be the G"odel numbering for which $\text{Ex}_{\psi}^{\min}$ consists of finite classes only (see Theorem 4.1). Now we consider the numbering $\phi$: 
\[ \varphi_i(x) = \begin{cases} \psi_j(0), & \text{if } i = 2j, \\ \psi_j(x), & \text{if } i = 2j + 1. \end{cases} \]

The minimal \( \varphi \)-indices for the class of all constants can be identified in the limit by finding the minimal even \( i \) such that \( \varphi_i(0) \) (which equals \( \psi_j(0) \) such that \( i = 2j \)) equals \( f(0) \) (where \( f \) is the function to be identified).

Now we prove the assertion 2). Let \( U \in \text{EX}_q^{\min} \). We consider the class \( U \setminus \text{CONSTANTS} \) and denote it by \( U_1 \). Assume that \( U_1 \) is infinite. It is easy to see that all \( \varphi \)-indices for functions in \( U_1 \) are odd. Moreover, for arbitrary \( f \in U_1 \), \( \min_q(f) = 2\min_p(f) + 1 \). Hence, a strategy exists which identifies in the limit the minimal \( \psi \)-indices for an infinite class \( U_1 \). Contradiction.

**5. UNIONS OF IDENTIFIABLE CLASSES**

J.M. Barzdin [Bar 74] constructed a pair of classes of total recursive functions \( U_1, U_2 \) such that \( U_1 \in \text{EX}, U_2 \in \text{EX}, \) and \( U_1 \cup U_2 \in \text{BC}. \) Since then many similar results have been proved for another types of inductive inference. A similar theorem is valid for \( \text{EX}^{\min} \)-identification as well.

**THEOREM 5.1.** There are classes \( U_1 \) and \( U_2 \) of total recursive functions and a Kolmogorov numbering \( \psi \) such that \( U_1 \in \text{EX}_\psi^{\min}, U_2 \in \text{EX}_\psi^{\min} \) and \( U_1 \cup U_2 \in \text{BC}. \)

**PROOF.** We fix a Kolmogorov numbering \( \varphi \) or all 1-argument partial recursive functions and a Goedel numbering \( \{ F_i \} \) of partial recursive strategies. Let \( \{ F'_i \} \) denote the corresponding numbering of total recursive strategies (see Lemma 1.1.).

The class \( U = U_1 \cup U_2 \) is constructed such that it cannot be BC-identified by any \( F'_i \).

The total recursive function being the counterexample for \( F'_i \) will be connected (but maybe not quite identical) to the following function \( g_i \) constructed as follows.

**STEP 0.** Define \( g_i(0) = i \). Go on to Step 1.

**STEP m (m>0).** Assume \( g_i \) is defined for \( x \in [0,r-1] \). We denote the sequence \((q_i(0),q_i(1),...,q_i(r-1))\) by \( \alpha \). Include the value \( F'_i(<\alpha>) \) in the list \( A \).

**SUBSTEP(m,t) (t=1,2,...)** Define \( g_i(r+t) = 0 \). (Note that \( g_i(r) \) is not defined yet).

Compute \( F'_i(<\alpha^0>) \) and if it is not included in the list \( A \), include it. Then for every \( k \in \alpha \) compute \( t \) steps of \( \varphi_k(r) \). If \( \varphi_k(r) = 0 \) for at least one \( k \in \alpha \) then define \( g_i(r+1) = 1 \) and go to the Step 1. If for every \( k \in \alpha \) either \( \varphi_k(r) \neq 0 \) or \( t \) steps do not suffice for the convergence then go to the Substep \((m,t+1)\).

It is easy to see that either \( g_i \) is total (if infinitely many steps are performed) or \( g_i \) is defined but for one value \( x = r \).

We define
\[ f_i(x) = \begin{cases} 
  g_i(x), & \text{if } g_i(x) \text{ defined,} \\
  0, & \text{otherwise},
\end{cases} \]

and \( U = \{ f_i \mid i = 0, 1, \ldots \} \). Every \( f_i \) is a total recursive function. From the construction, \( U \notin BC \).

We define \( U_1 = \{ f_i \mid g_i \text{ is total} \} \), \( U_2 = \{ f_i \mid g_i \text{ is not total} \} \).

Now we will prove that \( U_1, U_2 \in EX_{\psi}^{\min} \). We construct the following Goedel numbering \( \psi \). For every \( i \), \( \psi_{3i+2} = \varphi_i \) and hence \( \psi \) is a Kolmogorov numbering along with the given \( \varphi \). For every \( i \), the function \( \psi_{3i}(x) = g_i(x) \), and \( \psi_{3i}(x) \) is defined sequentially for \( x = 0, 1, 2, \ldots \). For every \( x \) it is defined by a parallel computation of \( \varphi_i(x) \) and \( g_i(x) \). The value \( \psi_{2i}(x) \) equals that of the two values which is computed first.

Note that \( \psi_{3i} \) is either empty function (if \( \varphi_i(x) \) is not defined) or total one. There is an algorithm \( \mathcal{M} \) uniform in \( i \) and enumerating these \( i \) for which \( \psi_{3i}(0) = \varphi_i(0) \) but \( \psi_{2i} \) is not equal \( f_i \). This algorithm tries to find an \( x \) for which \( \psi_{3i}(x) \) and \( g_i(x) \) are defined but not equal.

To identify \( \psi \)-minimal indices for \( U_1 \), the strategy uses \( f(0) \) to find a corresponding \( \psi_{3i+1} \) with \( \psi_{3i+1}(0) = f(0) \) (\( 3i+1 \) is a correct \( \psi \)-index but may be not the minimal one) and then searches among all \( 3j \) and \( 3j+1 \) and where \( j < i \) for the minimal number \( z \) such that \( \psi_2(z) = f(0) \), and either \( z = 3j \) and the algorithm \( \mathcal{M} \) has not yet rejected \( z \) or \( z = 3j+1 \), and \( \mathcal{M} \) has rejected \( z \) because of \( \psi_{3j+1}(a) = \varphi_j(a) \neq g_j(a) \) but \( g_j(a) = f(a) \).

To identify \( \psi \)-minimal indices for \( U_2 \), the strategy uses the initial fragment \((f(0), f(1), \ldots, f(y))\) up to the latest value \( y \) for which \( f(y) = 1 \). It searches among all \( 3j \) for the minimal one with \( \psi_{3j}(0) = f(0), \ldots, \psi_{3j}(0) = f(y) \). Since \( f \in U_2 \), the initial fragment is changed only finite number of times.

On the other hand, we will see from the subsequent Theorem 2.2 that there are some classes \( U \) and some Goedel numberings such that union with such \( U \) does not decrease the identifiability of any class. Such a class \( U \) can be considered as the easiest for the identification of the minimal indices.

We do not know what can be said about existence of such easiest classes for \( EX_{\varphi}^{\min} \)-identification in arbitrary Goedel numberings.

Theorem 1.1 shows that for some Goedel numberings there is no infinite easiest class but every finite class, of course, has the needed property.

Theorem 2.2 serves two purposes in this paper. One purpose is to show that there is no best (for the \( EX_{\varphi}^{\min} \)-identification) Goedel numbering. We shall reconsider this in Section 3. The other purpose is to show that Goedel numberings which allow the easiest classes are arbitrarily high in the partial ordering of Goedel numbering with respect to \( EX_{\varphi}^{\min} \)-identification.
**THEOREM 5.2.** For arbitrary $\varphi \in \mathcal{G}$ there is a $\psi \in \mathcal{G}$ and an effectively enumerable class $U$ such that $U \not\in \text{EX}_\varphi^\text{min}$ and for arbitrary class $V$ either in $\text{EX}_\varphi^\text{min}$ or in $\text{EX}_\psi^\text{min}$ it holds $U \cup V \not\in \text{EX}_\psi^\text{min}$.

**PROOF.** Let $w(i,k)$ be a total recursive function such that

$$
\varphi_{w(i,k)}(x) = \begin{cases} 
\varphi_i(x), & \text{if } x \neq 2, \\
k, & \text{if } x = 2.
\end{cases}
$$

Let $v(i) = \max_{k \leq 2} w(i,k), i)$. Hence $v(i) \geq i$.

For arbitrary pair $(i,j)$ we shall show how one can effectively enumerate a class $U_{ij}$. Then $U$ equals $\bigcup_{i,j} U_{ij}$. Since the enumeration of $U_{ij}$ is uniform in $i$ and $j$, the class $U$ is enumerable as well.

Let $i$ and $j$ be given. We construct simultaneously the class $U_{ij}$ and a partial recursive function $\varphi_{g_j(i)}$. At every stage $m$ of our enumerating procedure we add no more than one new function to the class $U_{ij}$ and the functions already in the class are additionally defined either only for $x = m$ or for all $x \geq m$. The function $\varphi_{g_j(i)}$ is additionally defined no more than for a finite number of values of the argument. Every integer $r \leq v(i)$ may be at some stage be marked. After being marked such an $r$ never becomes unmarked again.

We fix a numbering of all possible strategies $\{F_i\}$.

We define $\varphi_{g_j(i)}(0) = i$, $\varphi_{g_j(i)}(1) = j$. Further on, every new function added to the class $U_{ij}$ at an arbitrary stage will equal $i$ at zero and equal $j$ at one.

**Stage 1.** Go to the Stage 2.

**Stage 2.** The first function $u(x)$ is added to the (empty up to now) class $U_{ij}$ and $u(2) = 1$ is defined. Go to the Stage 3.

**Stage m.** Let $\varphi_{g_j(i)}$ be defined for all $x \leq s$ (maybe but $x = 2$). Let $u(x)$ be the latest function added to $U_{ij}$ up to this step. We assume that $u(x)$ is defined for all $x < m$ and all the other functions already in $U_{ij}$ are total.

(1) We compute $m$ steps for each of $\varphi_0(2), \varphi_1(2), \ldots, \varphi_i(2)$. We look for an $n$ such that $\varphi_n(2)$ is defined and $\varphi_n(2) = u(2)$. If no such $n$ is found, we go to the substage (2). If such an $n$ is found, we mark it, define $u(x) = 0$ for all $x \geq m$ and add the following function $u(1)$ to $U_{ij}$:

$$
u^{(1)}(x) = \begin{cases} 
u(x), & \text{if } x < m \text{ and } x \neq 2, \\
u(x) + 1, & \text{if } x = 2.
\end{cases}$$
Go to the Substage (2).

(2) Let \( u' \) be the latest function in \( U_{ij} \). For all \( x < m \) let

\[
\alpha_x = \begin{cases} 
\varphi_{g_{ij}}(x), & \text{if } \varphi_{g_{ij}}(x) \text{ already defined,} \\
u'(x), & \text{otherwise.}
\end{cases}
\]

We compute \( m \) steps for each of

\[ F_j(<\alpha_0 \alpha_1 \ldots \alpha_s>), F_j(<\alpha_0 \alpha_1 \ldots \alpha_{s+1}>), \ldots, F_j(<\alpha_0 \alpha_1 \ldots \alpha_{m-1}>). \]

(*)

If no of these values is computed then we define \( u'(m) = 1 \) and go to the stage \( m+1 \).

Assume that one abovementioned value is found. Let \( r = F_j(<\alpha_0 \ldots \alpha_k>) \) be the rightmost of them and \( r' \) be the rightmost of the same values (*) computed in \( m-1 \) steps.

If \( r \neq r' \) or \( r > v(i) \) or \( r \) marked then for all \( x \in \{s+1, s+2, \ldots, k\} \) we define

\[
\varphi_{g_{ij}}(x) = u'(x), \quad u'(m) = 1 \quad \text{and go to Stage } m+1.
\]

Otherwise we go to the Substage (3).

(3) We have an unmarked \( r \leq v(i) \). We compute \( m \) steps of \( \varphi_r(k_1) \) where \( k_1 = \max(3, k) \). If \( \varphi_r(k_1) \) does not converge in \( m \) steps or \( \varphi_r(k_1) \neq u'(k_1) \) then we define \( u'(m) = 1 \) and go to the stage \( m+1 \).

If \( \varphi_r(k_1) \) converges and \( \varphi_r(k_1) = u'(k_1) \) then we mark \( r \) and define \( u'(x) = 0 \) for all \( x \geq m \) and add a new function \( u^{(2)} \) to the class \( U_{ij} \):

\[
u^{(2)}(x) = \begin{cases} 
u'(x), & \text{if } x \leq m \text{ and } x \neq k_1 \\
u'(x) + 1, & \text{if } x = k_1.
\end{cases}
\]

Subsequently, for all \( x \) such that \( s \leq x \leq k_1 \) we define \( \varphi_{g_{ij}}(x) = u^{(2)}(x) \) and go to the Stage \( m+1 \).

We have concluded the description of the procedure. It is easy to see that \( U \) is effectively enumerable class of total recursive functions. For arbitrary \( j \) the function \( g_j \) is total recursive. For arbitrary \( i, j \) all the functions \( u^{(1)}, u^{(2)}, \ldots \) are different.

Note the following properties of \( f \in U_{ij} \). If \( f(x) = 0 \) for an \( x \geq 3 \) then \( f(y) = 0 \) for all \( y > x \).

If \( f \) is the last function added to \( U_{ij} \) then \( f(x) \neq 0 \) for all \( x \geq 3 \).

Now we prove that \( U \not\in EX_{\varphi}^{\min} \).

Assume from the contrary that \( F_j \) identifies in the limit the minimal \( \varphi \)-indices of the functions in \( U \) (and hence in \( \bigcup_i U_{ij} \)). Let \( a \) be a fixed point for \( g_j \), i.e. \( \varphi_{g_j(a)} = \varphi_a \).

Note that the class \( U_{aj} \) is finite. Indeed, at substages (1) no more than \( a+2 \) functions can be added to \( U_{aj} \). At substages (2) no functions are added, and at substages (3) at most \( v(a)+2 \) are added to \( U_{aj} \).

Let \( u \) be the last function added to \( U_{aj} \). We shall prove that

\[
\lim_{n \to \infty} F_j(<u^{[n]>}) \neq \min_{\varphi}(\bar{u}).
\]
First we note that \( p = \lim_{n \to \infty} F_t(<\bar{u}^{[n]}>) > v(a) \)

since otherwise \( p \) would be marked at some stage and \( \bar{u} \) would differ from \( q_0^p \). But we see that beginning from a stage \( m \) the integer \( p \) is the rightmost value in \((*)\). It follows from the definition of the substage \((2)\) that because of \( p > v(a) \) the function \( \varphi_{g(a)}(x) \) is defined for all \( x \neq 2 \). Easy to see that \( \varphi_{g(a)} \) equals \( \bar{u} \) for all \( x \neq 2 \) since at every stage \( m \) the value \( \varphi_{g(a)} \) is defined using the corresponding value of the latest function in \( U_{ij} \). It follows from the definition of the substage \((1)\) that \( \bar{u}(2) \leq a + 2 \). Now from the definitions of \( w(a,k) \), \( v \) and

the equality \( \varphi_a = \varphi_{g(a)} \), we get \( p = \lim_{n \to \infty} F_t(<\bar{u}^{[n]}>) \leq v(a) \)

Contradiction. Hence \( U \in EX_{\varphi} \min \).

Now we construct the needed numbering \( \psi \). Between every two neighbouring functions \( \varphi_{l-1} \) and \( \varphi_l \) we insert \( v(I)+3 \) auxiliary functions. Hence it is easy to reduce \( \varphi \) to \( \psi \). On the other hand, all the functions inserted between \( \varphi_{l-1} \) and \( \varphi_l \) are chosen to make difficult identification of the minimal \( \psi \)-indices.

If \( \varphi_l \) is not defined either on 0, or on 1, or on 2 then all the auxiliary functions between \( \varphi_{l-1} \) and \( \varphi_l \) are nowhere defined.

Let \( \varphi_l(0) = i \), \( \varphi_l(1) = j \) and \( \varphi_l(2) \) be defined. By \( n_k \) we denote the \( \psi \)-index of the \( k \)-th function preceding \( \varphi_l \) in the numbering \( \psi \):

\[
\begin{array}{c}
\varphi_{l-1} \quad n_k \quad v(l)+3 \\
\varphi_l \quad n_1
\end{array}
\]

Fig. 1.

At first we consider the case \( k < v(l)+3 \). To define \( \psi_{n_k} \), we enumerate the class \( U_{ij} \). If this class contains less than \( k \) functions then \( \psi_{n_k} \) is nowhere defined. Let \( U_{ij} \) contain at least \( k \) functions, the \( k \)-th function in the enumeration of \( U_{ij} \) be \( u_k \), and \( m \) be the number of the stage at which \( U_k \) is added to \( U_{ij} \). We compute \( \varphi_l^{[m]} \). If \( \varphi_l^{[m]} = u_k^{[m]} \) then we define \( \psi_{n_k} = u_k \), otherwise \( \psi_{n_k} \) is nowhere defined.

Now we define \( \psi_{n_k} \) for \( k = v(l)+3 \). We compute \( \varphi_l(3), \varphi_l(4), ... \) (in that order). If for an \( r \) we have got \( \varphi_l(r) = 0 \) then \( \psi_{n_k} = \varphi_l^{[r]}0^n \), otherwise \( \psi_{n_k} \) is nowhere defined.

This concludes the definition of the Goedel numbering \( \psi \) (which is computable and \( \varphi \) is reducible to \( \psi \)).

We are going to prove that \( V \in EX_{\varphi} \min \) or \( V \in EX_{\psi} \min \) imply \( U \cup V \in EX_{\psi} \min \). First, we prove the case \( U \in EX_{\psi} \min \).

To this goal, we consider two auxiliary strategies \( G_1 \) and \( G_2 \).

Let \( f \) be arbitrary function in \( U \). For \( r \leq 2 \) we define \( G_1(<f^{[r]}>) = 0 \). Let now be \( r > 2 \). From \( i = f(0) \) and \( j = f(1) \) we find the index \( q \) in the enumeration of \( U_{ij} \) for the first function with the initial fragment \( f^{[r]} \) which is in \( U_{ij} \). Let \( m \) denote the stage at which this function is added to \( U_{ij} \). We compute \( r \) steps of \( \varphi_0^{[m]}, \varphi_1^{[m]}, ... \varphi_r^{[m]} \) and find the least \( l \) such that \( \varphi_l^{[m]} \) is defined, \( \varphi_l^{[m]} = f^{[m]} \) and \( q < v(l)+3 \). (If there is no such \( l \) then \( G_1(<f^{[r]}>) = 0 \).)
For $G_1(<f^{[r]}>)$ we take the $\psi$-index of the k-th function preceding $\phi_1$ in the numbering $\psi$ (see Fig. 1). Thus $G_1$ is a partial recursive strategy defined for all $f \in U$.

Now we define the strategy $G_2$. If all $f(3), f(4), \ldots, f(r)$ differ from 0 then $G_2(<f^{[r]}>) = G_1(<f^{[r]}>)$. Otherwise we find the least $k \geq 3$ such that $f(k) = 0$ and compute $r$ steps of $\phi_0[k], \phi_1[k], \ldots, \phi_r[k]$. We search for the least $p \leq r$ such that $\phi_p[k] = f[k]$ (if there is no such $p$ then $G_2(<f^{[r]}>) = G_1(<f^{[r]}>)$ and we take for $G_2(<f^{[r]}>)$ the $\psi$-index of the $(v(p)+3)$-th function preceding $\phi_p$ in the numbering $\psi$ (see Fig. 1). Again, $G_2$ is a partial recursive strategy defined for all $f \in U$. Now we consider the strategy $F(<f^{[r]}>) = \min(G_1(<f^{[r]}>), G_2(<f^{[r]}>))$. We are going to prove that $\lim F(<f^{[r]}>) = \min_{r \to \infty}(f)$ for all $f \in U$.

Let $f \in U, f(0) = i, f(1) = j$. Let $f$ be the $q$-th function in the enumeration of $U_{ij}$ and $m$ be the number of the stage when $f$ is added to $U_{ij}$.

We consider the case when $f(x) \neq 0$ for all $x \geq 3$. Then $\lim F(<f^{[r]}>) = \lim G_1(<f^{[r]}>)$. It follows from the definition of $G_1$ that the limit $t = \lim G_1(<f^{[r]}>)$ exists and equals a $\psi$-index of $f$. Assume that $t' = \min_{\psi}(f) < t$. Since all functions in $U_{ij}$ are pairwise distinct, either $\psi_t$ is the $q$-th function preceding a $\phi_1$ or $\psi_t$ is certain $\phi_i$ itself. The first possibility is self-contradictory since $G_1$ would replace the hypothesis $t$ by $t'$. The second possibility implies $q > v(i)+2$ since the $q$-th function preceding the function $\phi_1$ in the numbering $\psi$ would be $f$.

We shall get the contradiction by proving $q \leq v(i)+2$.

It follows from the procedure enumerating $U_{ij}$ that the class contains no more than $v(i)+2$ distinct functions. Since $v$ is monotone, it suffices to prove that $i \leq l$. We have $\phi_l(x) \neq 0$ for $x \geq 3$. It follows from the definition of $U_{ij}$ that $\phi_l$ is the last function in $U_{ij}$. Such a function differs from all $\phi_0, \phi_1, \ldots, \phi_i$. Hence $\min_{\phi}(\phi_l) > i$. When $\phi$ is reduced to $\psi$, the index $l$ becomes $t' = \min_{\psi}(\phi_l)$. Hence $\min_{\phi}(\phi_l) = l$ and $l > i$. This concludes the consideration of the case $f(x) \neq 0$ for all $x \geq 3$.

Now we consider the case when there is a $p \geq 3$ such that $f(p) = 0$. Let $p$ be the minimal one. Then the limits $\lim G_1(<f^{[r]}>)$ and $\lim G_2(<f^{[r]}>)$ exist. Hence the limit $t = \lim F(<f^{[r]}>)$ exists as well. First we consider the subcase $t = \lim G_2(<f^{[r]}>)$. Since $f \in U$ the subcase $t = \lim G_2(<f^{[r]}>)$. Since $f \in U$ then $f(p) = 0$ implies $f(x) = 0$ for all $x \geq p$. But then from the definition of $G_2$ we conclude that $t = \min_{\psi}(f)$.

Now we consider the subcase $t = \lim G_1(<f^{[r]}>)$. Then $t = \lim G_2(<f^{[r]}>)$ as well. Assume $t' = \min_{\psi}(f) < t$. The index $t'$ cannot be in the interval between $\phi_{l-1}$ and $\phi_l$ since otherwise $G_1$ would change the hypothesis $t$ into $t'$. Hence $t'$ is an index for a $\phi_l$. But then $\phi_l = f$ and a function preceding $\phi_l$ in $\psi$ would equal $f$. It follows from the definition of $G_2$ that $\lim G_2(<f^{[r]}>) < t' < t$. Contradiction.
Hence $F$ identifies in the limit the minimal $\psi$-indices for $U$. We have either $V \in EX_\varphi^{\min}$ or $V \in EX_\psi^{\min}$. If $V \in EX_\varphi^{\min}$ then we denote by $G$ a strategy identifying in the limit the minimal $\varphi$-indices for the functions in $V$ and by $C$ we denote a strategy which produces hypotheses being the results of reduction $\varphi \rightarrow \psi$ from hypotheses by $G$. If $V \in EX_\psi^{\min}$ then we denote by $C$ a strategy identifying in the limit the minimal $\psi$-indices for the functions in $V$.

Let $\tau$ be a computable one-one numbering of $U$. The minimal $\psi$-indices for $V \cup U$ can be identified by the following strategy $H$. It tries to find an $n$ such that $\tau_n(0)=f(0)$ and $\tau_n(1)=f(1)$. While no such $n$ is found the strategy $H$ outputs the same hypotheses as $C$. When $H$ finds the first $n$ with this property, $H$ starts to output the same hypotheses as $F$ and does this while $f[r]=\tau_n[r]$. When an $r$ is found such that $f[r] \neq \tau_n[r]$ the strategy $H$ again starts to output the same hypotheses as $C$ and, in parallel, tries to find new $n$ such that $\tau_n(0)=f(0)$, $\tau_n(1)=f(1)$. After finding such an $n$ the strategy $H$ copies hypotheses of $F$ while $f[d]=\tau_n[d]$, etc. Only a finite number of functions in $\tau$ have the property $\tau_n(0)=f(0)$ and $\tau_n(1)=f(1)$.

Hence either $\lim H(<f[0]>)=\lim F(<f[0]>)$ or $\lim H(<f[r]>)=\lim C(<f[r]>).$ The first case implies $f \in U$ and $f \in V$. The intervals between any two $\varphi_1$ and $\varphi_1$ in the numbering $\psi$ contain only functions from $U$ (or empty functions). Hence $\min_\varphi(f)$ is the result of the standard reduction of $\min_\varphi(f)$. Hence

$$\lim_{r \to \infty} H(<f[r]>)=\min_\psi(f).$$ Q.E.D. 

6. STRUCTURE OF THE PARTIAL ORDERING

The usual reducibility of computable numberings $\psi \leq \varphi$ means the existence of an algorithm which, for arbitrary function $\psi_i$ finds its $\varphi$-index $j$ ($\varphi_j=\psi_i$). There are equivalent numberings (e.g. all Goedel numberings are equivalent). The partial ordering $P_1$ of the equivalence classes of the computable numberings of partial recursive functions is studied intensively (see [Ersh 77]). It is known that $P_1$ is upper semilattice but it is not lower semilattice.

We considered another reducibility of the same computable numberings in [Fre 74]. We use $\psi \leq EX_\psi$ to denote that for arbitrary class $U$ of total recursive functions $U \in EX_\psi$ implies $U \in EX_\varphi$. It is easy to see that if $\psi \leq \varphi$ then $\psi \leq EX \varphi$.

Let $P_2$ be the partial ordering of the equivalence classes with respect to $\leq EX$. It was proved in [Fre 74] that $P_1$ and $P_2$ are very much different.

Now we introduce another reducibility $\psi \leq min \varphi$. We use $\psi \leq min \varphi$ to denote that if for arbitrary class $U$ of total recursive functions $U \in EX_\psi$ implies $U \in EX_\varphi$. Note that $\psi \leq \varphi$ does not imply $\psi \leq min \varphi$. Theorems in Section 1 show that even Goedel numberings which are recursively isomorphic [Rog 58] are far from being min-equivalent.
The numbering constructed in Theorem 1.1 is in the minimum element of \( P_3 \) (the partial ordering of computable numberings with respect to \( \leq_{\text{min}} \)) but it is known that there is no minimum element in \( P_1 \).

Our Theorem 2.2 shows that there is no maximum element (and even no maximal elements) in \( P_3 \). On the contrary, there is the maximum element in \( P_1 \). This maximum element consist of Goedel numberings.

Now we will prove that \( P_3 \) is lower semilattice. To this, we will prove that for arbitrary \( \phi, \chi \in \mathcal{G} \) there is the "best" numbering among the Goedel numberings \( \psi \) such that \( \exists \psi_{\text{min}} \subseteq \exists \phi_{\text{min}} \) and \( \exists \psi_{\text{min}} \subseteq \exists \chi_{\text{min}} \).

**THEOREM 6.1** Let \( \phi, \chi \in \mathcal{G} \). There is a Goedel numbering \( \psi \) such that:

(a) \( \psi \leq \phi, \psi \leq \chi \),

(b) \( (\forall x \in \mathcal{G})( (x \leq \phi) \& (x \leq \chi) \Rightarrow (x \leq \psi)) \).

**PROOF.** Let \( c(i,j) \) be a Cantor numbering of pairs:

\[
c(0,0) = 0, c(0,1) = 1, c(1,0) = 2, c(0,2) = 3, c(2,0) = 5, c(0,3) = 6, ...
\]

For all \( i,j,x \) we define

\[
q^i(x) \text{ if } c^i(x) = x_j(x),
\]

\[
q^i(x) \text{ undefined, if otherwise.}
\]

\( \psi(c(i,j)) \) can be total only if \( \psi_i \) is total, \( \chi_j \) is total and \( \psi_i = \chi_j \). For our \( c(i,j) \) and for arbitrary total recursive function \( f \), \( \min_{\psi}(f) = c(\min_{\phi}(f), \min_{\chi}(f)) \).

This makes the properties (a) and (b) obvious. \( \square \)

Unfortunately, we do not know whether \( P_3 \) is upper semilattice. This problem has resisted all attempts since 1977 when it was posed in [FK 77].

Another open problem in [FK 77] questions whether or not for two arbitrary classes \( U_1 \in \exists \min \) and \( U_2 \in \exists \min \) there is a Goedel numbering \( \phi \) such that \( U_1 \in \exists \phi_{\text{min}} \) and \( U_1 \in \exists \phi_{\text{min}} \).

I would not be much surprised to find out that the answers to these two problems may be opposite.

**THEOREM 6.2** There are Goedel numberings \( \omega, \psi \in \mathcal{G} \) and classes \( U_1, U_2 \) of total recursive functions such that \( U_1 \in \exists \omega_{\text{min}} \setminus \exists \psi_{\text{min}} \) and \( U_2 \in \exists \psi_{\text{min}} \setminus \exists \omega_{\text{min}} \).

**PROOF.** Let \( \phi \) be arbitrary Goedel numbering. We define

\[
\omega_{3i+2}(x) = \psi_{3i+2}(x) = \psi_i(x),
\]

\[
\omega_{3i}(x) = \begin{cases} 
2a, & \text{if } (\exists n)(F'(a)^{<(2a)^{x+n}>}) \neq 3i, \\
2a+1, & \text{if } (\exists n)(F'(a)^{<(2a)^{x+n}>}) = 3i,
\end{cases}
\]

undefined, otherwise,
\( \psi_{3i+1}(x) = \begin{cases} 
2a, & \text{if } (\psi_{1}(0)=2a) \land (\exists n)(F'_{a}(<(2a)^{x+n}>) \neq 3i+1), \\
2a+1, & \text{if } \psi_{1}(0)=2a+1, \\
\text{undefined}, & \text{otherwise}, 
\end{cases} \)

\( \psi_{3i}(x) = \begin{cases} 
2a+1, & \text{if } (\psi_{1}(0)=2a+1) \land (\exists n)(F'_{a}(<(2a)^{x+n}>) \neq 3i), \\
2a, & \text{if } \psi_{1}(0)=2a, \\
\text{undefined}, & \text{otherwise}, 
\end{cases} \)

\( \psi_{3i+1}(x) = \begin{cases} 
2a+1, & \text{if } (\psi_{1}(0)=2a+1) \land (\exists n)(F'_{a}(<(2a+1)^{x+n}>) \neq 3i+1), \\
2a, & \text{if } \psi_{1}(0)=2a, \\
\text{undefined}, & \text{otherwise}, 
\end{cases} \)

\( U_{1} = \{2a+1 \mid a \in \mathbb{N}\} \)

\( U_{2} = \{2a \mid a \in \mathbb{N}\} \)

\( U_{1} \subseteq \text{EX}_{\psi_{3}}^{\text{min}} \) via strategy searching for the minimal \( 3i \) such that \( \psi_{1}(0)=\phi(0) \). The same strategy serves to prove that \( U_{2} \not\subseteq \text{EX}_{\psi_{3}}^{\text{min}} \).

Assume \( U_{1} \subseteq \text{EX}_{\psi_{3}}^{\text{min}(F'_{a})} \) and consider the constant function \( (2a+1)^{\infty} \in U_{1} \). Either \( (\exists n)(F'_{a}(<(2a+1)^{n}>) \neq 3i) \) or \( (\exists n)(F'_{a}(<(2a+1)^{n}>) \neq 3i+1) \), or both. Any case, the minimal \( \psi_{3} \)-index is either \( 3i \) or \( 3i+1 \) but \( F'_{a} \) does not identify it in the limit.

Similarly, \( U_{2} \not\subseteq \text{EX}_{\psi_{3}}^{\text{min}} \).

7. NEARLY MINIMAL INDICES

We allow only "special" (e.g. minimal) indices to be produced as the results of the identification in this paper. The restrictions are softened in this section and we prove that at least some properties that differ \( \text{EX}_{\psi_{3}}^{\text{min}} \) from \( \text{EX} \) are caused not by the minimality of the required results but merely by the fact that they are taken from a set of small cardinality.

Let \( \psi \) be an arbitrary computable numbering of 1-argument partial recursive functions and \( c \) be an arbitrary positive integer. We consider a set \( S \) of the "special" \( \psi_{3} \)-indices and demand that every total recursive function has at least one index in \( S \). We call the set \( S \) \( c \)-bounded if for arbitrary total recursive function the set of the \( \psi_{3} \)-indices for \( f \) intersects with \( S \) in a set of cardinality not exceeding \( c \). We say that a strategy \( F \) identifies in the limit the \( S \)-\( \psi_{3} \)-indices for a class \( U(U \in \text{EX}_{\psi_{3}}^{S}) \) if for arbitrary \( f \in U \) the limit \( \lim_{x \rightarrow \infty} F(<f^{[x]}>) \) exists and equals to a such that \( a \in S \) and \( \psi_{a}=f \).

THEOREM 7.1. Given an arbitrary computable numbering \( \psi \), \( c \geq 1 \) and \( c \)-bounded set \( S \) of \( \psi \)-indices, there exists an effectively enumerable class \( U \) such that \( U \subseteq \text{EX}_{\psi_{3}}^{S} \).
\textbf{Proof.} Assume that there is $\psi$, $c \geq 1$ and a $c$-bounded $S$ such that all effectively enumerable classes are in $\text{EX}_\psi^S$. Since the strategy $F'_i$ (see Section 1) produces the same result as the corresponding strategy $F_i$, we can assume that the abovementioned effectively enumerable classes can be identified by strategies in $\{F'_i\}$.

We will construct classes $U_1, U_2, U_3, \ldots$ in this proof such that simultaneous $\text{EX}_\psi^S$-identifiability of $U_1, U_2, U_3, \ldots, U_{i+1}$ contradicts the $c$-boundedness of $S$ since the strategies produce pairwise distinct results.

Induction basis. $U_1$ is defined as the class of all total functions $f$ such that $\lim_{x \to \infty} f(x) = 0$. The class $U_1$ is effectively enumerable. From the assumption, there is a strategy $G_1$ which identifies in the limit $S$-indices for $U_1$.

Induction step. Let $j \in \{1, 2, 3, \ldots\}$. Assume by induction that effectively enumerable classes $U_1, \ldots, U_j$ have already been constructed and they are $\text{EX}_\psi^S$-identifiable in the limit by strategies $G_1, \ldots, G_j$, respectively, and $G_1, \ldots, G_{j-1}$ are the strategies used to define the classes $U_2, \ldots, U_j$. Assume that there is an algorithm $M_j$ uniform in $n$ and $p$ where $U_j \in \text{EX}_\psi^S(F_n)$ and $p$ is the Cantor index of an arbitrary string of integers $\{y_0, y_1, \ldots, y_m\}$ ($m$ is arbitrary as well) and $M_j$ produces a $\psi$-index of a total function $f$ such that:

1) $f(0) = y_0, f(1) = y_1, \ldots, f(m) = y_m,$
2) $f \in U_1 \cap U_2 \cap \ldots \cap U_j,$
3) the strategies $G_1, \ldots, G_j$ produce pairwise distinct $\psi$-indices of $f$.

Returning to the induction basis, note that $M_1$ produces the program for the function $f(0) = y_0, \ldots, f(m) = y_m, f(m+1) = f(m+2) = \ldots = 0$.

We construct $U_{j+1} = \{h_z(x)\}$ by describing an algorithm uniform in $z$ and $x$ to compute $h_z(x)$. For every $z$ the construction will be performed in stages. The function $h_z$ is defined following a certain example $f$. We call it the current master-function.

\textbf{Stage 0.} Let $z = c(i,p)$ and $p = <y_0, y_1, \ldots, y_r>$. We define $h_z(0) = y_0$, $h_z(1) = y_1$, ..., $h_z(r) = y_r$. If $p = 0$, i.e. $p$ is the index of the empty string then $h_z(0) = 0$. We take for the master-function the function which is produced by $M_j$ from $G_j$ and $p$.

\textbf{Stage $s+1$.} Let $z = c(i,p)$, $p = <y_0, y_1, \ldots, y_r>$. Assume that $h_z(0), h_z(1), \ldots, h_z(k)$ have been already defined and $k \geq r$, $h_z(0) = y_0, h_z(1) = y_1, \ldots, h_z(r) = y_r$. Assume that the master-function $f$ is such that:

1) $f \in U_1 \cup U_2 \cup \ldots \cup U_j,$
2) $h_z(0) = f(0), h_z(1) = f(1), \ldots, h_z(k) = f(k)$
3) no two strategies among $G_1, \ldots, G_j$ identify in the limit the same $\psi$-index of $f$.

To define $h_z(k+1)$ we compute in parallel: 1) the sequences
Now we describe a chain of tests connected with these computations. This chain may be cut because of several obstacles described below.

The chain of tests starts with testing whether or not $\psi_{a(k)}(k+1) = f(k+1)$. If yes then we use $\psi_{a(k)}(k+1) = f(k+1)$ and to test whether $\psi_{a(k)}(k+1) = f(k+1)$. If it does not converge or the result differs then we test whether there are $r \in \{1,2,\ldots,j\}$ such that $G_r(f[k]) = G_r(f[k+1])$. If there are then we test whether $F_{i'}(f[k+1])$ differs from all such $G_r(f[k+1])$. If it differs not from all such $G_r(f[k+1])$ then we test whether $F_{i'}(f[k+2])$. If yes then we use $F_{i'}(f[k+2]) = a(k)$ to compute 2 steps of $\psi_{a(k)}(k+1)$ and to test whether $\psi_{a(k)}(k+1) = f(k+1)$. If it does not converge or the result differs then we test whether there are $r \in \{1,2,\ldots,j\}$ such that $G_r(f[k]) = G_r(f[k+1]) = G_r(f[k+2])$. If there are then we test whether $F_{i'}(f[k+2])$ differs from all such $G_r(f[k+2])$. If it differs not from all such $G_r(f[k+2])$ then we test whether $F_{i'}(f[k+3])$, etc.

This chain of tests can be cut because of several obstacles:

a) $\psi_{a(k)}(k+1)$ may converge and equal $f(k+1)$;

b) an $m$ can be found such that $F_{i'}(f[m]) 
eq F_{i'}(f[m])$;

c) an $m > k$ can be found such that there are no $r \in \{1,2,\ldots,j\}$ such that $G_r(f[k]) = G_r(f[k+1]) = \ldots = G_r(f[m])$;

d) an $m > k$ can be found such that $F_{i'}(f[m])$ differs from all $G_r(f[m])$ ($r \in \{1,2,\ldots,j\}$) such that $G_r(f[k]) = G_r(f[k+1]) = \ldots = G_r(f[m])$.

In the case a) we define $h_2(k+1) = f(k+1) + 1$ and take another master - function being the result of $G_j$ from $G_j$ and $h_2(0), h_2(1), \ldots, h_2(k+1)$.

In the cases b), c) and d) we define $h_2(k+1) = f(k+1)$, $h_2(k+2) = f(k+2), \ldots, h_2(m) = f(m)$ and the master - function is not changed.

This concludes the description of Stage s+1. Go to the Stage s+2.

We prove now that every stage definitely ends. To this, we prove that every chain of tests end. Indeed, the master - function $f$ is in $U_1, U_2, \ldots, U_j$. The strategies EX - identify these classes. Hence the limits $\lim_{n \to \infty} G_r(f[n])$ exist and equal correct $\psi$ - indices of $f$.

Assume, the chain never ends. Then $F_{i'}(f[k]) = \lim_{n \to \infty} F_{i'}(f[n])$ and there is an $r \in \{1,2,\ldots,j\}$ such that $\lim_{n \to \infty} G_r(f[n]) = \lim_{n \to \infty} F_{i'}(f[n])$. But then $F_{i'}(f[k]) = a(k)$ is
a correct $\psi$-index of $f$ and $\psi_{a(k)}(k+1)=f(k+1)$. Assumption on the infiniteness of the chain fails.

Now we prove that for arbitrary $i$ and $p$ the function $h_{c(i,p)}$ has the following properties:

A) Let $p=<y_0, y_1, \ldots, y_r>$. Then $h_{c(i,p)}(0)=y_0, h_{c(i,p)}(1)=y_1, \ldots, h_{c(i,p)}(r)=y_r$.

B) If the strategy $F'_{i}$ identifies in the limit a $\psi$-index for $h_{c(i,p)}$ then $h_{c(i,p)}$ is in all classes $U_1, U_2, \ldots, U_j, U_{j+1}$.

C) If the strategy $F'_{i}$ identifies in the limit a $\psi$-index for $h_{c(i,p)}$ then the strategies $G_1, G_2, \ldots, G_j, F'_{i}$ identify pairwise distinct $\psi$-indices for this function.

The property A) is obvious. We prove B), C).

Every stage in the construction of $h_{c(i,p)}$ ends in a), b), c), d). At least one of them occurs infinitely often. In our proof we distinguish between:

1) the case a) occurs infinitely often. Then $h_{c(i,p)} \not\in \text{EX}_\psi(F'_{i})$ since $F'_{i}$ makes infinitely many errors;

2) case b) occurs infinitely often. Then $F'_{i}$ has infinitely many mindchanges;

3) the cases a) and b) occur no more than finite number of times. Then the master-function is changed only finite number of times. Hence $h_{c(i,p)}$ is in $U_1 \cap \ldots \cap U_j$. It is in $U_{j+1}$ by the definition of $U_{j+1}$. We have proved B).

The strategies $G_1, G_2, \ldots, G_j$ have only finite number of mindchanges on the master-function, and their limits are pairwise distinct. If $F_{i}$ and $F'_{i}$ as well have only finite number of mindchanges on $h_{c(i,p)}$ then almost all stages end in d). We have proved C).

To complete the induction step it remains to note that the algorithm $M_j$ for the number $i$ of the strategy $F_{i}$ and the Cantor index $p$ of $\{y_0, y_1, \ldots, y_n\}$ produces a $\phi$-index of the function $h_{c(i,p)}$.

We conclude the proof of the Theorem by pointing at the contradiction between the assumed $\text{EX}_\phi^S$ - identifiability of $U_{c+1}$ for the $c$-bounded $S$ and the property c) for $U_{c+1}$ which says that $(c+1)$ strategies produce $c+1$ pairwise distinct results when $\text{EX}_\phi^S$ - identifying a function.

8. KOLMOGOROV NUMBERINGS

Proofs of the most of theorems above are based on the following construction: an arbitrary Goedel numbering $\varphi$ is taken and a new computable numbering $\psi$ is constructed such that for a total recursive function $g(i)$ it holds $\psi_{g(i)}=\varphi_i$. We conclude that computability of the numbering $\psi$ implies its being Goedel, and we are free to use the intervals between $\psi_{g(i-1)}$ and $\psi_{g(i)}$ to construct sets of functions either helping to identify the $\psi$-minimal indices or vice versa: to fail attempts to identify them.

Only in Theorem 2.1 we have been able to keep these intervals of constant length. In most cases the method of the proof demand these intervals being of growing length.
Hence the function $g(i)$ reducing $\varphi$ to $\psi$ cannot be linearly bounded and $\psi$ cannot be a Kolmogorov numbering.

Theorems in this Section show that rather many of the above proved theorems fail to have counterparts for Kolmogorov numberings.

$\text{CONSTANTS} = \{ f \mid (\exists c)(\forall x)(f(x) = c) \}$.

**Theorem 8.1.** There is a Kolmogorov numbering $\psi$ such that $\text{CONSTANTS} \not\in \text{EX}_{\psi}^{\text{min}}$.

**Proof.** Let $\varphi$ be an arbitrary Kolmogorov numbering. Let $\psi_{2i+1} = \varphi_i$ and

$$
\psi_{2i}(x) = \begin{cases} 
\varphi_i(0), & \text{if } \varphi_i(0) \text{ defined,} \\
\text{undefined, if } \varphi_i(0) \text{ not defined.}
\end{cases}
$$

The strategy always output the minimal $2i$ such that $\varphi_i(0) = f(0)$ for the given $f$. \qed

**Theorem 8.2.** There is a Kolmogorov numbering $\psi$ such that $\text{CONSTANTS} \not\in \text{EX}_{\psi}^{\text{min}}$.

**Proof.** Let $\psi$ be an arbitrary Kolmogorov numbering and $\{F'_1\}$ be the family of total recursive strategies described in Section 1. Let $\psi_{3i+2} = \varphi_i$,

$$
\psi_{3i}(x) = \begin{cases} 
\varphi_i(0), & \text{if } F'_f(<f[y]>) \neq 3i \text{ for no less than } x \text{ distinct values of } y, \\
\text{undefined, otherwise,}
\end{cases}
$$

$$
\psi_{3i+1}(x) = \begin{cases} 
\varphi_i(0), & \text{if } F'_f(<f[y]>) \neq 3i+1 \text{ for no less than } x \text{ distinct values of } y, \\
\text{undefined, otherwise,}
\end{cases}
$$

Note that if $\varphi_i(0)$ is defined then at least one of the functions $\psi_{3i}$, $\psi_{3i+1}$ is the constant $\varphi_i(0)$. Hence the minimal $\psi$-index for every constant function always equals 0 or 1 (modulo 3).

On the other hand, assume that $F'_j \not\in \text{EX}_{\psi}^{\text{min}}$ -identifies $\text{CONSTANTS}$. Denote by $i$ the minimal $\varphi$-index for functions such that $\varphi_i(0) = j$. Then the minimal $\psi$-index of the constant $j$ equals either $3i$ or $3i+1$. However it follows from the definitions of $\psi_{3i}$ and $\psi_{3i+1}$ that $\lim_{y \to \infty} F'_f(<f[y]>) \neq 3i$ and $\lim_{y \to \infty} F'_j(<f[y]>) \neq 3i-1$.

Contradiction. \qed

**Theorem 8.3.** For arbitrary Kolmogorov numbering $\psi$ there is an infinite class $U$ of total recursive functions such that $U \not\in \text{EX}_{\psi}^{\text{min}}$. 
**Proof.** Let \( \{ \alpha_i \} \) be arbitrary Kolmogorov numbering of 1-argument partial recursive functions. Consider auxiliary Kolmogorov numbering \( \varphi = \{ \varphi_i \} : \)

\[
\varphi_i(x) =
\begin{cases}
\alpha_j(x), & \text{if } i = 2j+1, \\
\ j, & \text{if } i = 2j.
\end{cases}
\]

Since \( \psi \) is Kolmogorov numbering, \( \varphi \) is reducible to \( \psi \) by linearly bounded total recursive function. Hence for \( \psi \), there is a positive \( c \) such that for arbitrary \( n \) the minimal \( \psi \)-indices of all the constants 0, 1, 2, ..., \( n \) do not exceed \( c \cdot n \).

Consider the set \( A_n \) of all \( d \in \{ 0, 1, 2, ..., n \} \) such that the interval \([0, c \cdot n]\) contain no more than \( 2c \) \( \psi \)-indices of functions \( \psi_j \) such that \( \psi_j(0) = d \). It is easy to see that the cardinality of \( A_n \) is less than \( n/2 \). Note that for these \( n/2 \) values \( d \) the interval \([0, c \cdot n]\) contains at least one \( \psi \)-index of the constant function \( d \).

Now consider \( 2c \) strategies \( G_1, G_2, ..., G_{2c} \). The strategy \( G_k \) having learned \( f(0) \) identify in the limit k-th minimal \( \psi \)-index of a function \( \psi_j \) such that \( \psi_j(0) = f(0) \). At least one of these strategies identify the minimal \( \psi \)-index for infinitely many constants. \( \Box \)

**9. Abstract Theory of Identification Types**

In this Section we prove a theorem saying that for "natural" identification types there is a "typical" class of total recursive function such that it is the most difficult for the identification.

When we say "identification types" we have in mind specific types studied many times by many researchers, such types as \( \text{EX}, \text{EX}_n^m, \text{EX}^\text{cons} \text{BC}, \text{FIN} \) ([Gold 67], [Bar 72], [Pod 74], [Pod 75], [Wie 77], [Smi 82]).

**Definition 9.1.** We define identification type \( M \) as a predicate \( M(f, \{ n_i \}) \) over functions and admissible recursive sequences of integers.

The predicate is understood as: "the sequence \( \{ n_i \} \) might be the sequence of hypotheses in \( M \)-identification of \( f \)."

We denote the set of all \( M \)-admissible sequences for \( f \) by \( D_M(f) \). We denote the class of all identification types by \( \Delta \) (\( M \in \Delta \)).

For instance, \( \text{EX}_\varphi \) is characterized by a predicate which is true for every pair (\( f, \{ n_i \} \)) where \( f \) is a total recursive function and \( \{ n_i \} \) is a sequence such that

\[
(\exists i_0)(\forall n)(\forall i > i_0)((n_i = n) \& (\forall x)(\varphi_{n_i}(x) = f(x))).
\]

\( \text{BC} \) is characterized by a predicate which is true for every pair (\( f, \{ n_i \} \)) where \( f \) is a total recursive function and \( \{ n_i \} \) is a sequence such that

\[
(\exists i_0)(\forall x)(\forall i > i_0)(\varphi_{n_i}(x) = f(x)).
\]

\( \text{EX}^\text{cons} \) is characterized by a predicate which is true for every pair (\( f, \{ n_i \} \)) where \( f \) is a total recursive function and \( \{ n_i \} \) is a sequence such that

\[
(\forall i)(\forall x)(\forall i_0)(\exists n)(\forall i > i_0)((n_i = n)).
\]
The family of all $\mathcal{M}$-identifiable classes of total recursive functions is denoted by $\text{EX}_\mathcal{M}$.

We prove going to prove that, provided certain "naturality" properties of the types, every identification type $\mathcal{M}$ has complete class $U$ of total recursive functions such that:

1) $U \in \mathcal{M}$,

2) for arbitrary "natural" type $\mathcal{L}$ such that $\mathcal{L} \subseteq \mathcal{M}$ the class $U \subseteq \mathcal{L}$.

**DEFINITION 9.2.** We say that the identification type $\mathcal{M}$ is reducible to the identification type $\mathcal{L}$ ($\mathcal{M} \leq \mathcal{L}$) if there is an algorithm which transforms pairs (graph of a function $f$, $\mathcal{M}$-admissible sequence for $f$) into a $\mathcal{L}$-admissible sequence for the same $f$.

**DEFINITION 9.3.** We say that identification type $\mathcal{M}$ is closed ($\mathcal{M} \in \Delta_0$) if there are functions $u(x,y)$ and $v(z)$ such that:

1) for arbitrary $f \in R$ and $\{n_i\} \in D_{\mathcal{M}}(f)$, the sequence $\{k_i\} = \{u(f(i),n_i)\}$ is in $D_{\mathcal{M}}(g)$ for $g(x) = c(f(x),n_x)$;

2) for arbitrary $g \in R$, the sequence $\{n_i\} = \{v(g(i))\}$ is in $D_{\mathcal{M}}(f)$ for $f(x) = l(g(x))$.

Informally, an identification type is closed when identifiability of a function implies identifiability of similar functions as well. The types $\text{EX}$, $\text{EX}_{n,m}$, $\text{EX}_{\text{cons}}$, $\text{BC}$ are closed but identification of the minimal numbers in a Gödel numbering is not.

**THEOREM 9.1.** For arbitrary identification type $\mathcal{M} \in \Delta_0$ there is a class $V_\mathcal{M}$ of total recursive functions such that:

1) $V_\mathcal{M} \subseteq \text{EX}_\mathcal{M}$,

2) if $\mathcal{L} \in \Delta_0$ and $V_\mathcal{M} \subseteq \text{EX}_\mathcal{L}$ then $\text{EX}_\mathcal{M} \subseteq \text{EX}_\mathcal{L}$,

3) if $V_\mathcal{M} \subseteq \text{EX}_\mathcal{L}$ then $\text{EX}_\mathcal{M} \not\subseteq \text{EX}_\mathcal{L}$.

**PROOF.** We define $V_\mathcal{M}$ to consist of all the functions $g(x) = c(f(x),n_x)$ where $f$ is arbitrary total recursive function and $\{n_x\}$ is arbitrary recursive sequence which is $\mathcal{M}$-admissible for $f$. Now we use the function $u(x,y)$ from the property 1) in the definition of closedness for $\mathcal{M}$ and define a strategy $F(<g^{[k]}>) = u(l(g(k)),r(g(k)))$. We see that the strategy $F$ $\mathcal{M}$-identifies the class $V_\mathcal{M}$. Hence $V_\mathcal{M} \in \text{EX}_\mathcal{M}$.

If $V_\mathcal{M} \in \text{EX}_\mathcal{L}$ then, of course, $\text{EX}_\mathcal{M} \not\subseteq \text{EX}_\mathcal{L}$. If $V_\mathcal{M} \in \text{EX}_\mathcal{L}$ then we denote the strategy $\mathcal{L}$-identifying $V_\mathcal{M}$ by $G$. Let $U$ be an arbitrary class in $\text{EX}_\mathcal{L}$. We will prove that $U \in \text{EX}_\mathcal{L}$ as well. There is a strategy $H$ $\mathcal{M}$-identifying $U$. If a total recursive function $f$ is in $U$ then the function $g(x) = c(f(x),H(<f^{[x]}>))$ is in $V_\mathcal{M}$, and the sequence $\{G(<g^{[x]}>)\}$ is a $\mathcal{L}$-admissible one for the function $g$. It follows from the property 2) in the definition of closedness for $\mathcal{L}$ that the sequence $\{V(G(<g^{[x]}>))\}$ is a $\mathcal{L}$-admissible sequence for $f$. Hence $U \in \text{EX}_\mathcal{L}$.
THEOREM 9.2. For arbitrary identification types $\mathcal{M}, \mathcal{L} \in \Delta_0$, $\mathcal{M} \leq \mathcal{L} \iff \text{EX}(\mathcal{M}) \subseteq \text{EX}(\mathcal{L})$.

PROOF. $\Rightarrow$ Let $U$ be a class of total recursive functions and $F$ be a strategy $\mathcal{M}$-identifying $U$. Then for arbitrary $f \in U$ the sequence $\{F(f[x])\}$ is an $\mathcal{M}$-admissible sequence for $\mathcal{M}$. It follows from $\mathcal{M} \leq \mathcal{L}$ that there is an algorithm which transforms pairs (graph of a function $f$, $\mathcal{M}$-admissible sequence for $f$) into a $\mathcal{L}$-admissible sequence for the same $f$. It is easy to combine this algorithm and the strategy $F$ to get the $\mathcal{L}$-identifying strategy for $U$. Hence $\text{EX}(\mathcal{M}) \subseteq \text{EX}(\mathcal{L})$.

$\Leftarrow$ Assume $\text{EX}(\mathcal{M}) \subseteq \text{EX}(\mathcal{L})$. It follows from Theorem 9.1 that $V, \mathcal{M} \in \text{EX}(\mathcal{L})$. Let $V, \mathcal{M}$ be $\mathcal{L}$-identified by a strategy $G$.

It is possible to reduce $\mathcal{M} \leq \mathcal{L}$ in the following way. We take the graph of arbitrary function $f$, an $\mathcal{M}$-admissible sequence for $f$, and apply the function $U(x, y)$ from the property 1) in the definition of closedness for $\mathcal{L}$. Since the $\mathcal{M}$-admissible sequence for $f$ is recursive, we get a total recursive function $g \in V, \mathcal{M}$. Then the sequence $\{G(<g[x]>\}$ is a $\mathcal{L}$-admissing sequence for $g$. It follows from the property 2) in the definition of closedness for $\mathcal{L}$ that the sequence $\{v(G(<g[x]>\}$ is a $\mathcal{L}$-admissing sequence for $f$. Hence $\mathcal{M} \leq \mathcal{L}$.

Note that Theorem 9.2. establishes equivalence between a uniform and a non-uniform notion. $\mathcal{M} \leq \mathcal{L}$ postulates existence of a uniform reduction algorithm valid for all total recursive function and all recursive $\mathcal{M}$-admissible sequences for them. On the other hand, $\text{EX}(\mathcal{M}) \subseteq \text{EX}(\mathcal{L})$ means only that for arbitrary class $U$ of total recursive functions, if $U$ can be $\mathcal{M}$-identifiable then it can be $\mathcal{L}$-identifiable as well. There is no uniformity allowing to find a $\mathcal{L}$-identifying strategy, given the $\mathcal{M}$-identifying strategy.

It follows from Theorem 9.2. that if $\text{EX}(\mathcal{M}) \subseteq \text{EX}(\mathcal{L})$ then $\mathcal{M} \leq \mathcal{L}$ and hence there is a uniform operator transforming $\mathcal{M}$-identifying strategies into $\mathcal{L}$-identifying strategies for the same classes of total recursive functions.

Theorems 9.1 and 9.2 have a certain defect. They are proved only for identification types in $\Delta_0$ (rather than $\Delta$). Unfortunately, the Theorems do not hold for $\Delta$.

THEOREM 9.3. There are identification types $\mathcal{M}, \mathcal{L} \in \Delta$ such that $\text{EX}(\mathcal{M}) \subseteq \text{EX}(\mathcal{L})$ but $\mathcal{M} \nleq \mathcal{L}$.

PROOF. The identification type $\mathcal{M}$ is chosen such that only finite classes of total recursive functions are $\mathcal{M}$-identifiable (see e.g. Theorem 4.1). The type $\mathcal{L}$ is chosen among the continuum of the types such that for every total recursive function $f$ the set $D_\mathcal{L}(f)$ contains only one recursive sequence. Note that for every such $\mathcal{L}$ all finite classes
of total recursive functions are $\mathcal{L}$-identifiable. On the other hand, if $\mathcal{M}$ is fixed and $\mathcal{L}_1$ and $\mathcal{L}_2$ are two different identification types with the abovementioned property then $\mathcal{M} \leq \mathcal{L}_1$ and $\mathcal{M} \leq \mathcal{L}_2$ cannot be reduced by the same reduction algorithm. There is an enumerable set of reduction algorithms. We remove from the continuum of abovementioned types $\mathcal{L}$ an enumerable family of types $\mathcal{L}_i$ such that the type $\mathcal{M}$ is reducible to $\mathcal{L}_i$ via the corresponding reduction algorithm. Every $\mathcal{L}$ from the remaining family of types can be used to complete the proof of our Theorem.

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\section*{REFERENCES}

[Rog 58] H.Rogers. Goedel numberings of the partial recursive functions. J.Symbolic
[Tra 65] B.A.Trakhtenbrot. Optimal computations and the frequential phenomenon by
[Wie 77] R.Wiehagen. Identification of formal languages. Lecture Notes in Computer