Prediction of color change after tooth bleaching using fuzzy logic for Vita Classical shades identification

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Tooth bleaching is becoming increasingly popular among patients and dentists since it is a relatively noninvasive approach for whitening and lightening teeth. Instruments and visual assessment with respect to commercial shade guides are currently used to evaluate tooth color. However, the association between these procedures is imprecise and the degree of color change after tooth bleaching is known to vary substantially between studies; there are currently no objective guidelines to predict the effectiveness of a tooth-bleaching treatment. We propose a new methodology based on fuzzy logic as a natural means of representing the imprecision present when modeling the color change produced by a tooth-bleaching treatment on the basis of a tooth’s initial chromatic values. This system has the advantage of producing a set of interpretable fuzzy rules that can subsequently be used by scientists and dental practitioners. The fuzzy system obtained has the special characteristic whereby the rule antecedents correspond to prebleaching shades of the well-known Vita commercial shade guide. Additionally, the rule consequents directly correspond with the expected CIELAB postbleaching values for each Vita shade, thanks to a modification of the system’s inference structure. Finally, the values of these postbleaching CIELAB coordinates have been associated with Vita shades by evaluating their respective membership functions, thereby approximating which posttreatment Vita shades are to be expected for each prebleaching shade. © 2010 Optical Society of America

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1. Introduction

Tooth bleaching, a nonrestorative treatment for whitening discolored teeth, has become an essential component of conservative esthetic dentistry. Current bleaching methods are based on the application of an agent that releases hydrogen peroxide to external tooth surfaces. This then penetrates the tooth and produces free radicals, which oxidize organic stains. Hydrogen peroxide or carbamide peroxide, at various concentrations, are the primary bleaching agents currently used by dentists. It has been reported [1] that bleaching with 10%–20% carbamide peroxide is a simple, user-friendly, effective, and inexpensive technique, and numerous clinical studies have documented the effectiveness of bleaching in changing tooth color through whitening and lightening [2]. Tooth color is a complex phenomenon that results from a combination of the tooth’s intrinsic color and the presence of any extrinsic stains that may form on its surface (due to smoking, tannin-rich foods, chlorhexidine, etc.). Intrinsic tooth color is associated with the light-scattering and absorption properties of the enamel and dentine, with the properties of dentine playing a major role.

Tooth color can be measured in a number of ways, including instruments specifically designed for use
with teeth or any object color measurement (colorimeter, spectrophotometer, spectroradiometer) and visual assessment with respect to commercial shade guides [3]. These shade guides are the most commonly used method in clinical practice due to their inexpensiveness and ease of use. The Vita Classical shade guide (Lumin Vacuum Shade Guide, Vita Zahnfabrik, H. Rauter GmbH & Co. KG. Bad Säckingen, Germany) is the most widespread due to its relationship with dental restoration systems and its frequent use in bleaching studies. However, visual assessment with respect to a shade guide is a subjective method, and it is often difficult to match a natural tooth with a shade.

The high frequency of errors associated with the use of commercial shade guides has been well documented [4,5]; therefore, instrumental techniques are needed to obtain an adequate color specification and determine the color change resulting from a tooth-bleaching procedure [6]. In this context, a good clinical practice should, therefore, be able to rely on a direct association between instrumental color measurements and commercial guides. Such an association would allow objective guidelines that inform the practitioner and the patient about the expected tooth shade resulting from a bleaching procedure to be established.

To achieve this objective, a set of instrumental in vivo tooth-color measurements were taken for a set of patients before and after a bleaching treatment using the CIELAB color space [7]. A color measurement using the Vita Classical shade guide was also carried out. A fuzzy inference system in which the component rules’ antecedents correspond to the CIELAB values of the Vita Classical shade guide was designed on the basis of these data with the aim of extracting useful information that can subsequently be used by experts dental practitioners.

The aim of the designed fuzzy system was to model the postbleaching tooth color for a given prebleaching color. An interesting property with which the designed system is endowed is that the rule consequents define which CIELAB coordinate value is expected for each Vita prebleaching shade without any previous inference. This property is possible thanks to a modification of the inference process inspired by previous works that obtained similar results for other types of fuzzy model. Finally, the proposed system uses the values of the rule consequents to estimate which Vita shades would be expected after the procedure for each prebleaching shade.

2. Tooth-Color Measurements

A total of 40 subjects (22 males and 18 females), with ages ranging from 22 to 65 years (mean age was 42.8), were enrolled in a home-bleaching study at the Department of Dentistry of the University of Granada, with corresponding approval by the Ethical Committee of Human Investigation at the University of Granada. Subjects were required to have healthy maxillary anterior dentitions without restorations, caries, or prior bleaching history.

All subjects used a 20% carbamide peroxide tooth-bleaching agent (Opalescence 20% PF, Ultradent Products Inc., South Jordan, Utah, USA) in the custom-made trays with reservoirs for 2 h once daily for two weeks. Two alginate impressions of each subject’s maxillary arch were taken to fabricate the bleaching trays. The labial surfaces of anterior teeth on the dental casts were covered with a block-out material (LC Block-out Resin, Ultradent Products Inc.) to create tray reservoirs on these teeth. A vacuum-formed custom tray was then fabricated from a 0.060-in. (1.5 mm) soft tray sheet (Ultradent Products Inc.).

The color of the patients’ teeth was subsequently obtained using a spectroradiometer (SpectraScan PR-704, Photo Research Inc., Chatsworth, California, USA) with a 4% measurement accuracy. These measurements were repeated three times for each tooth at baseline on the day before and 14 days following initiation of the bleaching procedure. In order to ensure that all the measurements were realized under standard lighting conditions, we used a Demetron Shade Light (KerrHawe SA, Bioggio, Switzerland), with a correlated color temperature similar to CIE standard illuminant D65 as the light source. The illumination and measurement configurations were CIE d/0°, and the CIE 1964 10° Supplementary Standard Colorimetric Observer was used.

To ensure that the different color measurements were performed for the same region of the tooth, a custom acrylic positioner with labial openings (Essix type A, Orthospain) was prepared for each patient [8,9]. This custom template was prepared with labial openings (3 mm in diameter) at the center of the tooth under study. Furthermore, the jaw and forehead of each subject were fixed with a chinrest.

Prebleaching (\(L'_i, a'_i, b'_i\)) and postbleaching (\(L'_j, a'_j, b'_j\)) CIELAB coordinate values were obtained. The \(L'_i, a'_i,\) and \(b'_i\) values were averaged to establish a single set of values for each tooth. In each case, the resulting standard deviations were lower than the instrumental accuracy (4%). Table 1 shows the ranges of prebleaching and postbleaching values for each coordinate. The \(L'_i, a'_i,\) and \(b'_i\) values for the Vita Classical shade guide were also obtained under the same measurement conditions (Table 2).

Figure 1 shows the data distribution in two CIELAB diagrams, together with the Vita Classical shade distribution within the same diagrams. It can be seen from this figure that the bleaching procedure results, in principle, in an approach to (0,0) in the coordinates \(a'\) and \(b'\), and an increase in the \(L'\) coordinate. This proved to be the case as the color differences \(\Delta L', \Delta a',\) and \(\Delta b'\) followed similar-to-normal distributions with mean and deviations equal to (2.9, 7.2), (−1.8, 1.7) and (−3.5, 2.5), respectively. The standard deviation in coordinate \(L'\) was strong and showed a larger dispersion of the mea-
measurements. As reported by other authors [6], the overall bleaching effectiveness appeared to be higher for the $b^*$ coordinate.

3. Fuzzy Logic Approach to Shade Identification of Tooth Color and the Use of Fuzzy Inference to Predict Color Change

Since its invention [10], fuzzy logic has found increasing use in optics and the color literature. Optoelectronic inference fuzzy logic systems have been used for control applications [11], although the optimization and implementation of fuzzy systems from a set of samples is traditionally performed by numerical methods, a field of intensive research [12–14]. Color modeling and the color naming problems have also been tackled using fuzzy logic [15,16]. Specifically, several studies in the field of dentistry have used fuzzy control to characterize a tooth’s surface in periodontal practice and the soft removal of dental calculus [17], or to detect and quantify dental plaque [18]. Here we propose a methodology for the study of tooth color change during a bleaching procedure based on the use of fuzzy logic, an approach which, to our knowledge, has not been used to date in the tooth-color and color-change literature. The advantages of fuzzy logic for this purpose include its ability to incorporate a priori knowledge into the models, to deal with imprecise data, and to explain relationships between the data using natural language.

A fuzzy set is defined by a membership function, which assigns a membership value to any possible data in its domain. The singularity of fuzzy logic with respect to traditional logic is that the membership of a sample $x$ to a fuzzy set $A$ takes values within the range $\mu_A(x) \in [0, 1]$, instead of crisp values $\{0, 1\}$ (belonging or not belonging to the set; see [19,20] for an introduction to fuzzy logic and fuzzy inference systems). This work will define fuzzy sets in CIELAB space corresponding to Vita shades on the basis of their respective measured CIELAB values. A given tooth-color sample will have a certain membership value with respect to a Vita shade, which will depend on its designated membership function. Tooth CIELAB values closer to the Vita CIELAB values will have higher membership values, whereas more distant CIELAB values will have lower or null membership values.

These fuzzy sets were used as antecedents in a fuzzy rule set whose consequents define the expected postbleaching CIELAB color for each prebleaching shade. The consequent values were obtained in a training process using a set of samples for which the expected output values obtained for each respective input value were known. The input/output data set for the problem tackled in this work corresponds to the set of available samples of postbleaching (output) tooth-color measurements and their respective prebleaching values (input). The fuzzy inference process designed models the behavior of the physical whitening process using the set of rules described.
Obtaining significant rules from the modeled process is not, however, something totally inherent to any designed fuzzy model. Interrelation of the model’s component rules is relevant during the training procedure for a fuzzy system, and the consequent values obtained depend greatly on the existing overlap between the antecedent membership functions. To avoid this dependency, a modified inference process that corrects these interrule interactions to avoid their overlap, inspired by previous studies on grid-based fuzzy systems [14,21], has been designed (Subsection 3B).

A. Problem Statement and Fuzzy Rules Design
The model input variables are the prebleaching CIELAB values $\vec{x} = \{L^*, a^*, b^*\}$. Given a prebleaching input value vector, the designed inference system provides an estimate of the postbleaching CIELAB values $\vec{y}(\vec{x}) = \{L^*_f(\vec{x}), a^*_f(\vec{x}), b^*_f(\vec{x})\}$. The designed CIELAB-based fuzzy sets $\mu(\vec{x})$, obtained through the product $T$-norm operator of the respective sets in the three dimensions $L^*$, $a^*$, and $b^*$, $\mu(\vec{x}) = \mu_{L^*}(\vec{x}) \cdot \mu_{a^*}(\vec{x}) \cdot \mu_{b^*}(\vec{x})$, were used as antecedents for the model set of rules. According to the Takagi–Sugeno–Kang (TSK) model [22], the rules (for each prebleaching Vita shade $S$) have the following shape:

\[
\text{IF } L^*_f \text{ is } \mu_{L^*}^{Vita-S} \text{ AND } a^*_f \text{ is } \mu_{a^*}^{Vita-S} \text{ AND } b^*_f \text{ is } \mu_{b^*}^{Vita-S} \text{ THEN } L^*_f = \hat{L}^*_f, a^*_f = \hat{a}^*_f, \text{ and } b^*_f = \hat{b}^*_f,
\]

where $\mu_{L^*}^{Vita-S}$, $\mu_{a^*}^{Vita-S}$, and $\mu_{b^*}^{Vita-S}$ are fuzzy membership functions centered on the corresponding shade values $S$ for coordinates $L^*$, $a^*$, and $b^*$ given in Table 2. The consequent parameters $\hat{L}^*_f$, $\hat{a}^*_f$, and $\hat{b}^*_f$ correspond to the expected CIELAB value for Vita shade $S$.

The membership functions in the antecedent sets were considered to be Gaussian; therefore, the membership function shape for each fuzzy set is given by the equations (one for each dimension)

\[
\begin{align*}
\mu_{L^*}^{Vita-S}(L^*_f) &= e^{-\frac{(L^*_f - L^*(S))^2}{2\sigma^S_L^2}}, \\
\mu_{a^*}^{Vita-S}(a^*_f) &= e^{-\frac{(a^*_f - a^*(S))^2}{2\sigma^S_a^2}}, \\
\mu_{b^*}^{Vita-S}(b^*_f) &= e^{-\frac{(b^*_f - b^*(S))^2}{2\sigma^S_b^2}},
\end{align*}
\]

where $\{L^*(S), a^*(S), b^*(S)\}$ is the CIELAB coordinate for shade $S$ and $\sigma^S$ is the radius of the respective Gaussian function. A fair partitioning of the input space was then performed by adjusting the radius for each center according to the distance to the nearest center in three-dimensional CIELAB space [23].

B. Optimization of Fuzzy Systems: Local–Global Modeling Approach
The output function of a TSK type fuzzy model is calculated according to the following formula:

\[
\hat{y}(\vec{x}) = \sum_{k=1}^{K} \mu^k(\vec{x}) \hat{y}^k, \quad k = 1, \ldots, K
\]

where $K$ is the number of rules in the system and $\hat{y}^k$ is the rule consequents for output $y$ [there are three outputs ($L^*_f$, $a^*_f$, and $b^*_f$) in the proposed model]. These consequents have to be estimated from the available set of samples in order to obtain the desired modeling behavior. This is normally performed by minimizing the mean square error

\[
J = \sum_{m \in D} \frac{(\hat{y}(\vec{x}^m) - y^m)^2}{\#D},
\]

where $D$ is the input/output training data set composed by $\#D$ samples indexed by index $m$, which leads to the creation of a linear equation system that can be solved using any mathematical method suitable for that purpose. Singular value decomposition (SVD) [24] was used in this work to solve the three linear equation systems constructed, one for each postbleaching CIELAB coordinate, and obtain the optimal consequent coefficients for every rule $L^*_f$, $a^*_f$, and $b^*_f$.

However, during the optimization process, function $\hat{y}$ in Eq. (3) for a given point corresponding to the center of a rule $\hat{c}^k$ will, for all rules activated at that point, be the sum of

\[
\hat{y}(\hat{c}^k) = \sum_{k \mid \mu^k(\hat{c}^k) > 0} \frac{\mu^k(\hat{c}^k) \hat{y}^k}{\sum_{k \mid \mu^k(\hat{c}^k) > 0} \mu^k(\hat{c}^k)}.
\]

The optimal consequent $\hat{y}^k$ for rule $k$ does not, therefore, immediately correspond to the expected value of the system for input $\hat{c}^k$, $\hat{y}(\hat{c})$; rather, it depends on the remaining rules whose membership value for that point is not null. In general, this can occur in any fuzzy system with sparse-partitioning and Gaussian membership functions in which the rules (or their centers) are spread all over the input space as they are needed, as is the case for the model presented in this paper.

Thus, guaranteeing that the rule consequents are consistent with the input space area they cover requires some degree of control. In principle, this could be performed by using regularization in the model optimization, although this does not ensure local significance. Other works have proposed the use of multiobjective genetic algorithms [25] to optimize the global inference model. However, solutions are available that modify the design structure of the fuzzy system by selecting specific membership functions.
for grid-based fuzzy models [26,14]. These membership functions ensure that only the corresponding rule is activated in its own center, thereby avoiding overlap at those points and maintaining the interpolative and approximative properties of the system. The technique adopted in this work makes use of this latter approach for sparse-partitioning systems by modifying the final activation value for the rules. We note here that the desired properties cannot be obtained in sparse systems using a special type of membership function precisely due to the sparse layout of the rules in the input space [21].

The idea is to modify the activation value of a rule \(k\) in an \(n\)-dimensional problem with input \(\tilde{x}\) as follows:

\[
\mu^k(\tilde{x}) = \mu^k(\tilde{x}) \prod_{j=1}^{K}(1 - \mu^j(\tilde{x})).
\]

This forces the activation value of the every rule to vanish in every other rule center, but in the respective center, in which it takes a maximum value. (See Appendix A for an explanatory example of the effect of this modification in the activations of the rules.) Then, using normalization, the general expression for the TSK fuzzy system output is calculated as

\[
\hat{y}(\tilde{x}) = \frac{\sum_{k=1}^{K} \mu^k(\tilde{x}) \hat{y}^k}{\sum_{k=1}^{K} \mu^k(\tilde{x})} = \frac{\sum_{k=1}^{K} \mu^k(\tilde{x}) \prod_{i=1}^{n} (1 - \mu^i(\tilde{x})))\hat{y}^k}{\sum_{k=1}^{K} \mu^k(\tilde{x}) \prod_{i=1}^{n} (1 - \mu^i(\tilde{x}))}.
\]

This new formulation of the system’s output [Eq. (7)] complies with the desired conditions for simultaneous local–global modeling. The output function of the proposed model is, nonetheless, continuous and differentiable since it is a linear composition of continuous and differentiable functions. It is also easy to deduce that, due to the continuity of the Gaussian function, the following properties hold for every rule \(k\):

\[
\frac{\mu^j(\hat{c}^k)}{\sum_{l=1}^{K} \mu^l(\hat{c}^k)} = 1, \quad \frac{\mu^k(\hat{c}^k)}{\sum_{l=1}^{K} \mu^l(\hat{c}^k)} = 0,
\]

\[
\forall j = 1...K, \quad j \neq k \rightarrow \hat{y}(\hat{c}^k) = \hat{y}^k.
\]

Thus, according to this equation, the consequents \(\hat{y}^k\) of the rules are exactly the values of the fuzzy system output around the respective rule center \(\hat{c}^k\).

### C. Vita-Based Fuzzy System Construction

The tooth colorimetric measurements for 32 patients were randomly selected from the set of 40 patients to design and optimize the rule system; the measurements for the remaining eight patients were used as a test set to evaluate the approximation capability of the model. The measurements in \(L^*, a^*,\) and \(b^*\) were normalized to give the same weight to the three coordinates (remember that the chromatic coordi-

nates \(a^*\) and \(b^*\) present a lower range of values than coordinate \(L^*\), and that \(L^*\) presents a larger dispersion; see Section 2). The Vita guide shades used for the fuzzy rule antecedents correspond to those shades that had sufficient data coverage. No colorimetric measurements directly associated with shades B1, A1, D2, and C1 were found on the basis of their membership function values.

The fuzzy rules obtained automatically, without the intervention of a human expert, from the prebleaching and postbleaching tooth colorimetric measurements data set for the remaining 12 shades are given in Appendix B. The training root mean square errors (RMSE) obtained for the designed system were 7.24, 1.39, and 2.30 for coordinates \(L^*, a^*,\) and \(b^*\), respectively [the RMSE formula is equal to \(\text{sqrt}(J)\) in Eq. (4)], and the test RMSEs were 7.29, 1.39, and 2.33, respectively. When using a traditional TSK fuzzy system and not controlling the local modeling, the training and test errors should be similar to those obtained for the proposed model (a training RMSE of 7.44, 1.41, and 2.28 for \(L^*, a^*,\) and \(b^*\), respectively, and a test RMSE of 7.32, 1.42, and 2.40), although the consequents would not reflect the expected CIELAB values for each Vita shade.

For comparison, a set of multivariate linear models were used to estimate the postbleaching \(L^*, a^*,\) and \(b^*\) values on the basis of the prebleaching \(L^*, a^*,\) and \(b^*\) measurements as regressors. In this case, the training RMSEs were 6.86, 1.34, and 2.19, respectively, and the test RMSEs were 7.01, 1.34, and 2.38; in other words, they were similar to those obtained using the proposed fuzzy model. Note, however, that optimal training of an approximative TSK fuzzy system would have to optimize the rule antecedent parameters, as well. The rule antecedents have been selected here to identify the Vita guide shades in order to provide understandability and usability despite the fact that this approach does not guarantee an optimal performance on a least-squares basis [14,13].

### 4. Data Mining from the Fuzzy System

The immediate postbleaching CIELAB values for a tooth unequivocally associated with a certain Vita shade are immediately obtained from the rule set given in Eqs. (B1) and (B2) (Appendix B). The characteristics of the proposed fuzzy model provide this property. As additional information, the association of postbleaching CIELAB shades with the Vita guide shades were estimated. This association was obtained through fuzzification of the postbleaching CIELAB values using the fuzzy sets designed for the Vita shades. Table 3 shows the three Vita shades with the highest correspondence with each postbleaching consequent; in other words, the postbleaching Vita shades that would be expected for each possible prebleaching Vita shade after the procedure. It also includes the membership function values for each selected Vita shade.
Moreover, as a control measure, a confidence value for each fuzzy rule was calculated on the basis of the RMSE contributed by the rule to the error committed in three-dimensional CIELAB space, according to the following formula:

$$\text{confid. value}_k = \sum_{m \in \mathcal{D}} \| y^m - \tilde{y}^m \| \cdot \mu^{k*}(\tilde{x}^m),$$

where \(\tilde{y}^m = \{\tilde{a}_i^*, \tilde{b}_i^*, \tilde{L}_i^*\}^m\) are the estimated postbleaching CIELAB coordinate values for sample \(m\), \(y^m = \{a_i^*, b_i^*, L_i^*\}^m\) are the corresponding postbleaching coordinate values (of sample \(m\)), and \(\mu^{k*}(\tilde{x}^m)\) is the activation value of rule \(k\) for the prebleaching coordinate values \(\tilde{x}^m = \{a_i^*, b_i^*, L_i^*\}^m\). The last column in Table 3 lists the confidence values for the obtained rules (the mean error for the samples in the three-dimensional space is 6.5).

### 5. Conclusions

This study has shown that it is possible to establish a correspondence between colorimetric measurements and a well-known guide commonly used in dental clinics. This work has proposed a fuzzy-logic-based approach in order to model a set of prebleaching and postbleaching tooth colorimetric values by using a set of rules whose antecedents correspond to the chromatic coordinate values of a Vita shade guide. This novel fuzzy model allows us to undertake a globally and locally efficient modeling that obtains the expected colorimetric values for each possible prebleaching shade. Additionally, the association between the expected postbleaching values and Vita shades has been determined, and a confidence value for the rules extracted has been obtained. Moreover, the results obtained have demonstrated that it is possible to predict the color change obtained using a carbamide peroxide bleaching system. This is an outstanding advance in dental esthetics as dental practitioners can now have access to an objective set of indicators regarding the bleaching operation based on the tooth’s initial chromatic properties. The fuzzy-logic-based technique used in this work could be of great interest in other disciplines that require color changes due to any physical, chemical, or natural phenomenon to be predicted and described. Specifically, within the same study area, this work could be extended to other commercial shade guides, as well as to the development of novel shade guides that are better adapted to the colorimetric evaluation of tooth-bleaching treatments.

### Appendix A: Modified Activation Example in a One-Dimensional Case with Two Rules

Let us assume a one-dimensional input space example with domain \([0, 1]\) with two rules with Gaussian membership functions (MFs) centered, for example, at \(c^1 = 0.2\) and \(c^2 = 0.8\), with \(\sigma = 0.3\) [see Fig. 2(a)], and consequents \(y^1\) and \(y^2\), respectively. In this case, there is a moderate overlap between the two rules’ activations. In order to comply with the aforementioned overlap condition (see Subsection 3.B), we will allow the domain of the first rule activation \(\mu^1(x)\) to be limited by the function \(1 - \mu^2(x)\). In other words, when the activation value of the opposite rule is 1, the first rule activation will be forced to take the value 0. The final activation value for

![Fig. 2. (Color online) (a) Original \(\mu^1\) and \(\mu^2\) MFs for the one-dimensional example. (b) Normalized final MF activations using the modified calculation \(\frac{\mu^1(x)}{\mu^1(x) + \mu^2(x)}\) and \(\frac{\mu^2(x)}{\mu^1(x) + \mu^2(x)}\).](image-url)
any point in the system using normalization [see Fig. 2(b)] would be

\[
\mu^V(x) = \mu^I(x)(1 - \mu^2(x)),
\]

\[
\frac{\mu^V(x)}{\sum_{k=1}^{K} \mu^V(x)} = \frac{\mu^I(x)}{\mu^I(x) + \mu^2(x)}, \tag{A1}
\]

\[
\mu^2(x) = \mu^2(x)(1 - \mu^4(x)),
\]

\[
\frac{\mu^2(x)}{\sum_{k=1}^{K} \mu^2(x)} = \frac{\mu^2(x)}{\mu^I(x) + \mu^2(x)}. \tag{A2}
\]

Then, from the continuity of the Gaussian function and of the compounded functions, the following properties can be directly extracted for the given example:

\[
\mu^1(c^1) = 1, \quad \mu^2(c^1) = 0 \Rightarrow \tilde{y}(c^1) = \tilde{y}^1,
\]

\[
\mu^2(c^2) = 1, \quad \mu^1(c^2) = 0 \Rightarrow \tilde{y}(c^2) = \tilde{y}^2, \tag{A3}
\]

showing that the consequents \( \tilde{y}^1 \) and \( \tilde{y}^2 \) of the rules are exactly the values of the fuzzy system output around the respective rule center. These results can be directly extrapolated to the \( n \)-dimensional case due to the continuity of the compounded functions.

### Appendix B: Fuzzy Rules

The fuzzy rules obtained in the design of the proposed fuzzy system (see Section 3) are the following:

\[
\text{IF } L_i^j \text{ is } \widehat{B}_2, \text{ AND } a_i^j \text{ is } \widehat{B}_2, \text{ THEN }
\]

\[
L_i^j = 68.89 \quad \text{AND} \quad a_i^j = 4.68 \quad \text{AND} \quad b_i^j = 9.49
\]

\[
\text{IF } L_i^j \text{ is } \widehat{A}_2, \text{ AND } a_i^j \text{ is } \widehat{A}_2, \text{ THEN }
\]

\[
L_i^j = 60.39 \quad \text{AND} \quad a_i^j = 4.49 \quad \text{AND} \quad b_i^j = 8.15
\]

\[
\text{IF } L_i^j \text{ is } \widehat{C}_2, \text{ AND } a_i^j \text{ is } \widehat{C}_2, \text{ THEN }
\]

\[
L_i^j = 53.29 \quad \text{AND} \quad a_i^j = 7.12 \quad \text{AND} \quad b_i^j = 11.86
\]

\[
\text{IF } L_i^j \text{ is } \widehat{D}_4, \text{ AND } a_i^j \text{ is } \widehat{D}_4, \text{ THEN }
\]

\[
L_i^j = 57.87 \quad \text{AND} \quad a_i^j = 4.08 \quad \text{AND} \quad b_i^j = 10.90
\]

\[
\text{IF } L_i^j \text{ is } \widehat{D}_3, \text{ AND } a_i^j \text{ is } \widehat{D}_3, \text{ THEN }
\]

\[
L_i^j = 62.82 \quad \text{AND} \quad a_i^j = 2.80 \quad \text{AND} \quad b_i^j = 3.51
\]

\[
\text{IF } L_i^j \text{ is } \widehat{A}_3, \text{ AND } a_i^j \text{ is } \widehat{A}_3, \text{ THEN }
\]

\[
L_i^j = 64.24 \quad \text{AND} \quad a_i^j = 6.14 \quad \text{AND} \quad b_i^j = 11.74,
\tag{B1}
\]

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