Online Learning with Pairwise Loss Functions

Yuyang Wang
Roni Khardon
Department of Computer Science, Tufts University, Medford, MA 02155, USA

Dmitry Pechyony
Rosie Jones
Akamai Technologies, 8 Cambridge Center, Cambridge, MA 02142, USA

Abstract
Efficient online learning with pairwise loss functions is a crucial component in building large-scale learning system that maximizes the area under the Receiver Operator Characteristic (ROC) curve. In this paper we investigate the generalization performance of online learning algorithms with pairwise loss functions. We show that the existing proof techniques for generalization bounds of online algorithms with a univariate loss can not be directly applied to pairwise losses. In this paper, we derive the first result providing data-dependent bounds for the average risk of the sequence of hypotheses generated by an arbitrary online learner in terms of an easily computable statistic, and show how to extract a low risk hypothesis from the sequence. We demonstrate the generality of our results by applying it to two important problems in machine learning. First, we analyze two online algorithms for bipartite ranking; one being a natural extension of the perceptron algorithm and the other using online convex optimization. Secondly, we provide an analysis for the risk bound for an online algorithm for supervised metric learning.

Keywords: Generalization bounds, Pairwise loss functions, Online learning, Loss bounds, Bipartite Ranking, Metric Learning

1. Introduction
The standard framework in learning theory considers learning from examples \( Z^n = \{(x_t, y_t) \in \mathcal{X} \times \mathcal{Y}\}, t = 1, 2, \cdots, n \), (independently) drawn at random from an unknown probability distribution \( \mathcal{D} \) on \( \mathcal{Z} := \mathcal{X} \times \mathcal{Y} \) (e.g. \( \mathcal{X} = \mathbb{R}^d \) and \( \mathcal{Y} = \mathbb{R} \)). Typically a univariate loss function \( \ell(h, (x, y)) \) is adopted to measure the performance of the hypothesis \( h : \mathcal{X} \rightarrow \mathcal{Y} \), for example, \( \ell(h, (x, y)) = (h(x) - y)^2 \) for regression or \( \ell(h, x, y) = \mathbb{I}[h(x) \neq y] \) for classification. The aim of learning is to find a hypothesis that generalizes well, i.e. has small expected risk \( \mathbb{E}_{(x,y)} \ell(h, (x, y)) \).

In this paper we study learning in the context of pairwise loss functions, that depend on pairs of examples and can be expressed as \( \ell(h, (x, y), (u, v)) \) where the hypothesis is applied to pairs of examples, i.e. \( h : \mathcal{X} \times \mathcal{X} \rightarrow \mathbb{R} \). Pairwise loss functions capture ranking problems that are important for a wide range of applications. For example, in the supervised ranking problem one wishes to learn a ranking function that predicts the correct ordering of objects. The hypothesis \( h \) is called a ranking rule such that \( h(x, u) > 0 \) if \( x \) is ranked higher than \( u \) and vice versa. The misranking loss (Clemençon et al., 2008; Peel et al., 2010) is a pairwise
loss such that
\[ \ell_{\text{rank}}(h, (x, y), (u, v)) = \mathbb{I}_{(y-v)(h(x, u)) < 0}, \]
where \( \mathbb{I} \) is the indicator function and the loss is 1 when the examples are ranked in the wrong order. The goal of learning is to find a hypothesis \( h \) that minimizes the expected misranking risk \( R(h) \),

\[ R(h) := \mathbb{E}_{(x, y)} \mathbb{E}_{(u, v)} [\ell(h, (x, y), (u, v))]. \tag{1} \]

In many interesting cases, finding a ranking rule amounts to learning a good scoring function \( s : \mathcal{X} \rightarrow \mathbb{R} \) such that \( h(x, u) = s(x) - s(u) \). Therefore, higher ranked examples will have higher scores. Another application comes from distance metric learning, where the learner wishes to learn a distance metric such that examples that share the same label should be close while ones from different labels are far away from each others.

This problem, especially the bipartite ranking problem where \( Y = \{+1, -1\} \), has been extensively studied over the past decade in the batch setting, i.e., where the entire sequence \( Z^n \) is presented to the learner in advance of learning. Freund et al. (2003) gave generalization bounds for the RankBoost algorithm, based on the uniform convergence results for classification. Agarwal et al. (2005) derived uniform convergence bounds for the bipartite ranking loss, using a quantity called rank-shatter coefficient, which generalizes ideas from the classification setting. Agarwal et al. provided bounds for the bipartite ranking problem (Agarwal and Niyogi, 2005) and the general ranking problem (Agarwal and Niyogi, 2009) using ideas from algorithmic stability. Rudin et al. (2005) approached a closely related problem where the goal is to correctly rank only the top of the ranked list, and derived generalization bounds based on \( L_{\infty} \) covering number. Recently, several authors investigated oracle inequalities for pairwise-based quantities via the formalization of \( U \)-statistics (Clemençon et al., 2008; Rejchel, 2012) using empirical processes. Peel et al. (2010) gave an empirical Bernstein inequality for higher order \( U \)-statistics. Another thread comes from the perspective of reducing ranking problems to the more familiar classification problems (Kotlowski et al., 2011; Ertekin and Rudin, 2011; Agarwal, 2012).

In this paper we investigate the generalization performance of online learning algorithms, where examples are presented in sequence, in the context of pairwise loss functions. Specifically, on each round \( t \), an online learner receives an instance \( x_t \) and predicts a label \( \hat{y}_t \) according to the current hypothesis \( h_{t-1} \). The true label \( y_t \) is revealed and \( h_{t-1} \) is updated. The goal of the online learner is to minimize the expected risk w.r.t. a pairwise loss function \( \ell \).

Over the past two decades, online learning algorithms have been studied extensively, and theoretical results provide relative loss bounds, where the online learner competes against the best hypothesis (with hindsight) on the same sequence. Conversions of online learning algorithms and their performance guarantees to provide generalization performance in the batch setting have also been investigated (e.g., (Kearns et al., 1987; Littlestone, 1990; Freund and Schapire, 1999; Zhang, 2005)). Cesa-Bianchi et al. (2004) provided a general online-to-batch conversion result that holds under some mild assumptions on the loss function. Given a univariate loss function \( \ell \), a sample \( Z^n \) and an ensemble of hypotheses \( \{h_1, h_2, \cdots, h_n\} \) generated by an online learner \( \mathcal{A} \), the following cumulative loss of \( \mathcal{A} \) is
defined as
\[ M_n = M_n(Z^n) = \frac{1}{n} \sum_{t=1}^{n} \ell(h_{t-1}, Z_t). \]

Cesa-Bianchi and Gentile (2008) showed (as a refined version of the bound in (Cesa-Bianchi et al., 2004)) that one can extract a hypothesis \( \hat{h} \) from the ensemble such that
\[ P\left( \mathcal{R}(\hat{h}) \geq M_n + O\left( \frac{\ln^2 n}{n} + \sqrt{M_n \frac{\ln n}{n}} \right) \right) \leq \delta. \]

Therefore, if one can develop an online learning algorithm with bounded cumulative loss for every possible realization of \( Z^n \), then its generalization performance is guaranteed. A sharper bound exists when the loss function is strongly convex (Kakade and Tewari, 2009). The key step of these derivations is to realize that \( V_{t-1} = R(h_{t-1}) - \ell(h_{t-1}, Z_t) \) is a martingale difference sequence. Thus one can use martingale concentration inequalities (Azuma’s inequality or Friedman Inequality) to bound \( \sum V_t \). Unfortunately, this property no longer holds for pairwise loss functions.

Of course, as mentioned for example in the work of Peel et al. (2010, Sec. 4.2), one can slightly adapt an existing online learning classification algorithm (e.g., perceptron), feeding it with data sequence \( \tilde{z}_t := (z_{2t-1}, z_{2t}) \) and modifying the update function accordingly. In this case, previous analysis (Cesa-Bianchi and Gentile, 2008) does apply. However, this does not make full use of the examples in the training sequence. In addition, empirical results show that this naive algorithm, which corresponds to the algorithm for online maximization of the area under the ROC curve (AUC) with a buffer size of one in (Zhao et al., 2011), is inferior to algorithms that retain some form of the history of the sequence. Alternatively, it is tempting to consider feeding the online algorithm with pairs \( \tilde{z}_t^i = (z_i, z_t) \), \( i < t \) on each round. However, in this case, existing results would again fail because \( \tilde{z}_t^i \) are not i.i.d. Hence, a natural question is whether we can prove data dependent generalization bounds based on the online pairwise loss.

This paper provides a positive answer to this question for a large family of pairwise loss functions. On each round \( t \), we measure \( M_t \), the average loss of \( h_{t-1} \) on examples \( (z_i, z_t), i < t \). Let \( M^n \) denote the average loss, averaging \( M_t \) over \( t \geq (1-c)n \) on a training sequence of length \( n \) where \( c \) is a small constant. The main result of this paper, provides a model selection mechanism to select one of the hypotheses of an arbitrary online learner, and states that the probability that the risk of the chosen hypothesis \( \hat{h} \) satisfies,
\[ \mathcal{R}(\hat{h}) \geq M^n + \epsilon \]
is at most
\[ 2 \left[ \mathcal{N}\left( \mathcal{H}, \frac{\epsilon}{32\text{Lip}(\phi)} \right) + 1 \right] \exp \left\{ -\frac{(cn - 1)\epsilon^2}{256} + 2 \ln n \right\}. \]

Here \( \mathcal{N}(\mathcal{H}, \eta) \) is the \( L_\infty \) covering number for the hypothesis class \( \mathcal{H} \) and \( \text{Lip}(\phi) \) is determined by the Lipschitz constant of the loss function (definitions and details are provided in the following sections). Thus, our results provide an online-to-batch conversion for pairwise loss functions. We demonstrate our results with the following two applications:
1. We analyze two online learning algorithms for the bipartite ranking problem. We first provide an analysis of a natural generalization of the perceptron algorithm to work with pairwise loss functions, that provides loss bounds in both the separable case and the inseparable case. As a byproduct, we also derive a new simple proof of the best $L_1$ based mistake bound for the perceptron algorithm in the inseparable case. Combining with our main results we provide the first online algorithm with corresponding risk bound for bipartite ranking. Secondly, we analyze another algorithm using the online convex optimization techniques, with similar risk bounds.

2. Several online metric learning algorithms have been proposed with corresponding regret analyzes, but the generalization performance of these algorithms has been left open, possibly because no tools existed to provide online-to-batch conversion with pairwise loss functions. We provide risk bounds for an online algorithm for distance metric learning combining with the results for online convex optimization with matrix argument.

The rest of this paper is organized as follows. Section 2 defines the problem and states our main technical theorem and Section 3 provides a sketch of the proof. We provide model selection results and risk analysis for convex and general loss functions in Section 4. In Section 5, we describe our online algorithm for bipartite ranking and analyze it. The results in sections 2-5 are given for a model and algorithms with an ”infinite buffer”, that is, where the update of the online learner at step $t$ depends on the entire history of the sequence, $z_1, \cdots, z_{t-1}$. Section 6 shows that the results and algorithms can be adopted to a buffer of limited size. Interestingly, to guarantee convergence our results require that the buffer size grows logarithmically with the sequence size. Section 7 is devoted to the analysis of online metric learning. Finally, we conclude the paper and discuss possible future directions in Section 8.

2. Main Technical Result

Given a sample $Z^n = \{z_1, \cdots, z_n\}$ where $z_i = (x_i, y_i)$ and a sequence of hypotheses $h_0, h_1, \cdots, h_n$ generated by an online learning algorithm, we define the sample statistic $M^n$ as

$$M^n(Z^n) = \frac{1}{n-c_n} \sum_{t=c_n}^{n-1} M_t(Z^t), \quad M_t(Z^t) = \frac{1}{t-1} \sum_{i=1}^{t-1} \ell(h_{t-1}, z_i, z_i), \quad (2)$$

where $c_n = \lceil c \cdot n \rceil$ and $c \in (0, 1)$ is a small positive constant. $M_t(Z^t)$ measures the performance of the hypothesis $h_{t-1}$ on the next example $z_t$ when paired with all previous examples. Note that instead of considering all the $n$ generated hypotheses, we only consider the average of the hypotheses $h_{c_n-1}, \cdots, h_{n-2}$ where the statistic $M_t$ is reliable and the last two hypotheses $h_{n-1}, h_n$ are discarded for technical reasons. In the following, to simplify the notation, $M^n$ denotes $M^n(Z^n)$ and $M_t$ denotes $M_t(Z^t)$. We define $f \wedge g \equiv \min(f, g)$ and $f \vee g \equiv \max(f, g)$.

As in (Cesa-Bianchi et al., 2004), our goal is to bound the average risk of the sequence of hypotheses in terms of $M^n$, which can be obtained using the following theorem.
Theorem 1 Assume the hypothesis space \((\mathcal{H}, \| \cdot \|_\infty)\) is compact. Let \(h_0, h_1, \ldots, h_n \in \mathcal{H}\) be the ensemble of hypotheses generated by an arbitrary online algorithm working with a pairwise loss function \(\ell\) such that,

\[
\ell(h, z_1, z_2) = \phi(y_1 - y_2, h(x_1, x_2)),
\]

where \(\phi : \mathcal{Y} \times \mathcal{Y} \to [0, 1]\) is a Lipschitz function w.r.t. the second variable with a finite Lipschitz constant \(\text{Lip}(\phi)\). Then, \(\forall \epsilon > 0, \forall \epsilon > 0\), we have for sufficiently large \(n\)

\[
\mathbb{P} \left\{ \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} \mathcal{R}(h_{t-1}) \geq \mathcal{M}^n + \epsilon \right\} \leq \left[ 2N \left( \mathcal{H}, \frac{\epsilon}{16\text{Lip}(\phi)} \right) + 1 \right] \exp \left\{ - \frac{(cn - 1)\epsilon^2}{64} + \ln n \right\}.
\]

(3)

Here the \(L_\infty\) covering number \(\mathcal{N}(\mathcal{H}, \eta)\) is defined to be the minimal \(\ell\) in \(\mathbb{N}\) such that there exist \(\ell\) disks in \(\mathcal{H}\) with radius \(\eta\) that cover \(\mathcal{H}\). We make the following remarks.

Remark 2 Let \(\mathbb{E}_t\) denote \(\mathbb{E}_{z_t} [\cdot | z_1, \ldots, z_{t-1}]\). It can be seen that \(\mathbb{E}_t [M_t] - \mathcal{R}(h_{t-1})\) is no longer a martingale difference sequence. Therefore, martingale concentration inequalities that are usually used in online-to-batch conversion do not directly yield the desired bound.

Remark 3 We need the assumption that the hypothesis space \(\mathcal{H}\) is compact so that its covering number \(\mathcal{N}(\mathcal{H}, \eta)\) is finite. As an example, suppose \(\mathcal{X} \subset \mathbb{R}^d\) and the hypothesis space is the class of linear functions that lie within a ball \(B_R(\mathbb{R}^d) = \{ w \in \mathbb{R}^d : \sup_{x \in \mathcal{X}} \langle w, x \rangle \leq R \}\).

It can be shown (see Cucker and Zhou, 2007, chap. 5) that the covering number is one if \(\eta > R\) and otherwise

\[
\mathcal{N}(B_R, \eta) \leq \left( \frac{2R}{\eta} + 1 \right)^d.
\]

(4)

Remark 4 We say that \(f(s, t)\) is Lipschitz w.r.t the second argument if \(\forall s, |f(s, t_1) - f(s, t_2)| \leq \text{Lip}(f)||t_1 - t_2||\). This form of the pairwise loss function is not restrictive and is widely used. For example, in the supervised ranking problem, we can take the hinge loss as

\[
\ell_{\text{hinge}}(h, z_1, z_2) = \phi(y_1 - y_2, h(x_1) - h(x_2)) = [1 - (h(x_1) - h(x_2))(y_1 - y_2)]_+,
\]

which can be thought as a surrogate function for \(\ell_{\text{rank}}\). Since \(\phi\) is not bounded, we define the bounded hinge loss using \(\tilde{\phi}(s, t) = \min((1 - |st|)_+, 1) \in [0, 1]\) if \(s \neq 0\) and \(0\) otherwise. We next show that \(\tilde{\phi}\) is Lipschitz. This is trivial for \(y = 0\). For \(y \neq 0\), when the first argument is bounded by a constant \(C\), \(\tilde{\phi}(y, \cdot)\) satisfies

\[
|\tilde{\phi}(y, x_1) - \tilde{\phi}(y, x_2)| \leq |1 - yx_1|_+ - |1 - yx_2|_+ \leq \|yx_1 - yx_2\| \leq C\|x_1 - x_2\|.
\]

Alternatively, one can take the square loss, i.e. \(\ell(h, z_1, z_2) = [1 - (h(x_1) - h(x_2))(y_1 - y_2)]^2\).

If its support is bounded then \(\ell\) is Lipschitz.
3. Proof of the Main Technical Result

The proof is inspired by the work of (Cucker and Smale, 2002; Agarwal et al., 2005; Rudin, 2009). The proof makes use of the Hoeffding-Azuma inequality, McDiarmid’s inequality, symmetrization techniques and covering numbers of compact spaces.

**Proof** [Proof of Theorem 1] By the definition of $\mathcal{M}^n$ (see (2)), we wish to bound

$$P_{Z^n \sim D^n} \left( \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} R(h_{t-1}) - \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} M_t \geq \epsilon \right),$$

which can be rewritten as

$$P \left( \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} \left[ R(h_{t-1}) - \mathbb{E}_t[M_t] \right] + \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} \left[ \mathbb{E}_t[M_t] - M_t \right] \geq \epsilon \right)$$

$$\leq P \left( \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} \left[ R(h_{t-1}) - \mathbb{E}_t[M_t] \right] \geq \frac{\epsilon}{2} \right) + P \left( \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} \left[ \mathbb{E}_t[M_t] - M_t \right] \geq \frac{\epsilon}{2} \right).$$

(6)

Thus, we can bound the two terms separately. The proof consists of four parts, as follows.

**Step 1: Bounding the Martingale difference**

First consider the second term in (6). We have that $V_t = (\mathbb{E}_t[M_t] - M_t)/(n - c_n)$ is a martingale difference sequence, i.e. $\mathbb{E}_t[V_t] = 0$. Since the loss function is bounded in $[0, 1]$, we have $|V_t| \leq 1/(n - c_n), \ t = 1, \cdots , n$. Therefore by the Hoeffding-Azuma inequality, $\sum_t V_t$ can be bounded such that

$$P_{Z^n \sim D^n} \left( \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} \left[ \mathbb{E}_t[M_t] - M_t \right] \geq \frac{\epsilon}{2} \right) \leq \exp \left\{ - \frac{(1 - c) n \epsilon^2}{2} \right\}.$$  

(7)

**Step 2: Symmetrization by a ghost sample $\Xi^n$**

In this step we bound the first term in (6). Let us start by introducing a ghost sample $\Xi^n = \{\xi_j\} = \{(\tilde{z}_j, \tilde{y}_j)\}, j = 1, \cdots , n$ where each $\xi_j$ follows the same distribution as $z_j$.

Recall the definition of $M_t$ and define $\tilde{M}_t$ as

$$M_t = \frac{1}{t - 1} \sum_{j=1}^{t-1} \ell(h_{t-1}, z_t, z_j), \quad \tilde{M}_t = \frac{1}{t - 1} \sum_{j=1}^{t-1} \ell(h_{t-1}, z_t, \xi_j).$$

(8)

The difference between $\tilde{M}_t$ and $M_t$ is that $M_t$ is the sum of the loss incurred by $h_{t-1}$ on the current instance $z_t$ and all the previous examples $z_j, j = 1, \cdots , t - 1$ on which $h_{t-1}$ is trained, while $\tilde{M}_t$ is the loss incurred by the same hypothesis $h_{t-1}$ on the current instance $z_t$ and an independent set of examples $\xi_j, j = 1, \cdots , t - 1$. 


Claim 1: The following equation holds
\[
\mathbb{P}_{Z^n \sim D^n} \left( \frac{1}{n - c_n} \sum_{t = c_n}^{n-1} \left[ \mathcal{R}(h_{t-1}) - \mathbb{E}_t[M_t] \right] \geq \epsilon \right) \leq 2 \mathbb{P}_{Z^n \sim D^n} \left( \frac{1}{n - c_n} \sum_{t = c_n}^{n-1} \left[ \mathbb{E}_t[\tilde{M}_t] - \mathbb{E}_t[M_t] \right] \geq \frac{\epsilon}{2} \right),
\]
whenever \( n > 2/(\epsilon^2 c^2) \).

Notice that the probability measure on the right hand side of (9) is on \( Z^n \times \Xi^n \).

Proof [Sketch of the proof of Claim 1] It can be seen that the RHS (without the factor of 2) of (9) is at least
\[
\mathbb{P}_{Z^n \sim D^n} \left( \frac{1}{n - c_n} \sum_{t = c_n}^{n-1} \left[ \mathcal{R}(h_{t-1}) - \mathbb{E}_t[M_t] \right] \geq \epsilon \right) \geq \mathbb{P}_{Z^n \sim D^n} \left( \left\{ \mathcal{R}(h_{t-1}) - \mathbb{E}_t[M_t] \leq \frac{\epsilon}{2} \right\} \cap \left\{ \frac{1}{n - c_n} \sum_{t = c_n}^{n-1} \left[ \mathbb{E}_t[\tilde{M}_t] - \mathcal{R}(h_{t-1}) \right] \leq \frac{\epsilon}{2} \right\} \right) \]
\[
= \mathbb{P}_{Z^n \sim D^n} \left( \left\{ \frac{1}{n - c_n} \sum_{t = c_n}^{n-1} \left[ \mathcal{R}(h_{t-1}) - \mathbb{E}_t[M_t] \right] \leq \frac{\epsilon}{2} \right\} \right) \cdot \mathbb{P}_{Z^n \sim D^n} \left( \left\{ \frac{1}{n - c_n} \sum_{t = c_n}^{n-1} \left[ \mathbb{E}_t[\tilde{M}_t] - \mathcal{R}(h_{t-1}) \right] \leq \frac{\epsilon}{2} \right\} \right) \cdot Z^n \right).
\]

Since \( \mathbb{E}_{Z^n \sim D^n} \mathbb{E}_t[\tilde{M}_t] = \mathcal{R}(h_{t-1}) \), by Chebyshev’s inequality
\[
\mathbb{P}_{Z^n \sim D^n} \left( \left\{ \frac{1}{n - c_n} \sum_{t = c_n}^{n-1} \left[ \mathbb{E}_t[\tilde{M}_t] - \mathcal{R}(h_{t-1}) \right] \leq \frac{\epsilon}{2} \right\} \right) \geq 1 - \frac{\mathbb{V}ar \left\{ \frac{1}{n - c_n} \sum_{t = c_n}^{n-1} \mathbb{E}_t[\tilde{M}_t] \right\}}{\epsilon^2/4}.
\]

To bound the variance, we first investigate the largest variation when changing one random variable \( \xi_j \) with others fixed. From (8), it can be easily seen that changing any of the \( \xi_j \) varies each \( \mathbb{E}_t[\tilde{M}_t] \), where \( t \geq j \) by at most by 1/(\( t - 1 \)). Recall that we are only concerned with \( \mathbb{E}_t[\tilde{M}_t] \) when \( t \geq c_n \). Therefore, we can see that the variation of \( \frac{1}{n - c_n} \sum_{t = c_n}^{n-1} \mathbb{E}_t[\tilde{M}_t] \) regarding the \( j \)th example \( \xi_j \) is bounded by
\[
c_j = \frac{1}{n - c_n} \left[ \sum_{t = (j \cap c_n + 1)}^{n-1} \frac{1}{t - 1} \right] \leq \frac{1}{n - c_n} \left[ \sum_{t = c_n + 1}^{n-1} \frac{1}{t - 1} \right] \leq \frac{1}{cn}.
\]

Thus, by Theorem 9.3 in (Devroye et al., 1996), we have
\[
\mathbb{V}ar \left( \frac{1}{n - c_n} \sum_{t = c_n}^{n-1} \mathbb{E}_t[\tilde{M}_t] \right) \leq \frac{1}{4} \sum_{i = 1}^{n} c_i^2 \leq \frac{1}{4c^2 n}.
\]

Thus, whenever \( \epsilon^2 c^2 n > 2 \), the LHS of (10) is greater or equal than 1/2. This completes the proof of Claim 1.

Step 3: Uniform Convergence

In this step, we show how one can bound the RHS of (9) using uniform convergence techniques, McDiarmid’s inequality and \( L_\infty \) covering number. Our task reduces to bound the
following quantity

\[
P_{Z^n \sim \mathcal{D}^n, \Xi^n \sim \mathcal{D}^n} \left( \frac{1}{n - c_n} \sum_{t = c_n}^{n-1} \left[ \mathbb{E}_t [\hat{M}_t] - \mathbb{E}_t [M_t] \right] \geq \epsilon \right) . \tag{13}
\]

Here we want to bound the probability of the large deviation between the empirical performance of the ensemble of hypotheses on the sequence \(Z^n\) on which they are learned and on an independent sequence \(\Xi^n\). Since \(h_t\) relies on \(z_1, \ldots, z_t\) and is independent of \(\{\xi_t\}\), we resort to uniform convergence techniques to bound this probability. Define \(L_t(h_{t-1}) = \mathbb{E}_t [\hat{M}_t] - \mathbb{E}_t [M_t]\). Thus we have

\[
P_{Z^n \sim \mathcal{D}^n, \Xi^n \sim \mathcal{D}^n} \left( \frac{1}{n - c_n} \sum_{t = c_n}^{n-1} L_t(h_{t-1}) \geq \epsilon \right) \leq \mathbb{P} \left( \sup_{\hat{h}_{c_n}, \ldots, \hat{h}_{n-1}} \left[ \frac{1}{n - c_n} \sum_{t = c_n}^{n-1} L_t(\hat{h}_{t-1}) \right] \geq \epsilon \right)
\]

\[
\leq \sum_{t = c_n}^{n-1} P_{Z^t \sim \mathcal{D}^t, \Xi^t \sim \mathcal{D}^t} \left( \sup_{\hat{h} \in \mathcal{H}} \left[ L_t(\hat{h}) \right] \geq \epsilon \right) . \tag{14}
\]

To bound the RHS of (14), we start with the following lemma.

**Lemma 5** Given any function \(f \in \mathcal{H}\) and any \(t \geq 2\)

\[
\mathbb{P}_{Z^t \sim \mathcal{D}^t, \Xi^t \sim \mathcal{D}^t} (L_t(f) \geq \epsilon) \leq \exp \left\{ - \left( t - 1 \right) \epsilon^2 \right\} . \tag{15}
\]

The proof which is given in the appendix shows that \(L_t(f)\) has a bounded variation of \(1/(t - 1)\) when changing each of its \(2(t - 1)\) variables and applies McDiarmid’s inequality. Finally, our task is to bound \(\mathbb{P}(\sup_{f \in \mathcal{H}} [L_t(f)] \geq \epsilon)\). Consider the simple case where the hypothesis space \(\mathcal{H}\) is finite, then using the union bound, we immediately get the desired bound. Although \(\mathcal{H}\) is not finite, a similar analysis goes through based on the assumption that \(\mathcal{H}\) is compact. We will follow Cucker and Smale (2002) and show how this can be bounded. The next two lemmas (see proof of Lemma 6 in the appendix) are used to derive Lemma 8.

**Lemma 6** For any two functions \(h_1, h_2 \in \mathcal{H}\), the following equation holds

\[
L_t(h_1) - L_t(h_2) \leq 2 \text{Lip}(\phi) \| h_1 - h_2 \|_{\infty} .
\]

**Lemma 7** Let \(\mathcal{H} = S_1 \cup \cdots \cup S_\ell\) and \(\epsilon > 0\). Then

\[
\mathbb{P} \left( \sup_{h \in \mathcal{H}} L_t(h) \geq \epsilon \right) \leq \sum_{j = 1}^{\ell} \mathbb{P} \left( \sup_{h \in S_j} L_t(h) \geq \epsilon \right)
\]

**Lemma 8** For every \(2 \leq t \leq n\), we have

\[
\mathbb{P} \left( \sup_{h \in \mathcal{H}} [L_t(h)] \geq \epsilon \right) \leq \mathcal{N} \left( \mathcal{H}, \frac{\epsilon}{4 \text{Lip}(\phi)} \right) \exp \left\{ - \frac{(t - 1)\epsilon^2}{4} \right\} . \tag{16}
\]
Proof [Proof of Lemma 8] Let \( \ell = N \left( \mathcal{H}, \frac{\epsilon}{2\Lip(\phi)} \right) \) and consider \( h_1, \ldots, h_\ell \) such that the disks \( D_j \) centered at \( h_j \) and with radius \( \frac{\epsilon}{2\Lip(\phi)} \) cover \( \mathcal{H} \). By Lemma 6, we have
\[
|L_t(h) - L_t(h_j)| \leq 2\Lip(\phi)\|h - h_j\|_\infty \leq \epsilon.
\]
Thus, we get
\[
P \left( \sup_{h \in D_j} L_t(h) \geq 2\epsilon \right) \leq P (L_t(h_j) \geq \epsilon)
\]
Combining this with (15), and Lemma 7 and replacing \( \epsilon \) by \( \epsilon/2 \), we have (16).

Combining (16) and (14), we have
\[
P \left( \frac{1}{n - cn} \sum_{t=cn}^{n-1} L_t(h_{t-1}) \geq \epsilon \right) \leq \mathcal{N} \left( \mathcal{H}, \frac{\epsilon}{4\Lip(\phi)} \right) n \exp \left\{ -\frac{(cn - 1)\epsilon^2}{4} \right\}.
\]
(17)
This shows why we need to discard the first \( cn \) hypotheses in the ensemble. If we include \( h_2 \) for example, according to (16), we have
\[
P (L_2(f) \geq \epsilon) \leq e^{-\epsilon^2/2}.
\]
As \( n \) grows, this heavy term remains in the sum, and the desired bound cannot be obtained.

Step 4: Putting it all together
From (9) and (14) and substituting \( \epsilon \) with \( \epsilon/4 \) in (17), we have
\[
P_{Z^n \sim \mathcal{D}^n} \left( \frac{1}{n - cn} \sum_{t=cn}^{n-1} (\mathcal{R}(h_{t-1}) - \mathbb{E}_t[M_t]) \geq \frac{\epsilon}{2} \right) \leq \mathcal{N} \left( \mathcal{H}, \frac{\epsilon}{16\Lip(\phi)} \right) n \exp \left\{ -\frac{(cn - 1)\epsilon^2}{64} \right\}.
\]
(18)
From (18) and (7), accompanied with the fact that (18) decays faster than (7), we complete the proof for Theorem 1.

4. Model Selection
Following Cesa-Bianchi et al. (2004) our main tool for finding a good hypothesis from the ensemble of hypotheses generated by the online learner is to choose the one that has a small empirical risk. We measure the risk for \( h_t \) on the remaining \( n-t \) examples, and penalize each \( h_t \) based on the number of examples on which it is evaluated, so that the resulting upper bound on the risk is reliable. Our construction and proofs (in the appendix) closely follow the ones in (Cesa-Bianchi et al., 2004), using large deviation results for \( U \)-statistics (see Clemençon et al., 2008, Appendix) instead of the Chernoff bound.

4.1. Risk Analysis for Convex losses
If the loss function \( \phi \) is convex in its second argument and \( \mathcal{Y} \) is convex, then we can use the average hypothesis \( \bar{h} = \frac{1}{n-cn} \sum_{t=cn}^{n-1} h_{t-1} \). It is easy to show that \( \bar{h} \) achieves the desired bound (the proof is in the Appendix, i.e.
\[
P \left( \mathcal{R}(\bar{h}) \geq M^n(Z^n) + \epsilon \right) \leq \left[ 2\mathcal{N} \left( \mathcal{H}, \frac{\epsilon}{16\Lip(\phi)} \right) + 1 \right] \exp \left\{ -\frac{(cn - 1)\epsilon^2}{64} + \ln n \right\}.
\]
(19)
4.2. Risk Analysis for General Losses

Define the empirical risk of hypothesis \( h_t \) on \( \{z_{t+1}, \cdots, z_n\} \) as \( \hat{R}(h_t, t+1) \)

\[
\hat{R}(h_t, t+1) = \left( \frac{n - t}{2} \right)^{-1} \sum_{k > i, i \geq t+1} \ell(h_t, z_i, z_k).
\]

The hypothesis \( \hat{h} \) is chosen to minimize the following penalized empirical risk,

\[
\hat{h} = \arg\min_{c_{n-1} \leq t < n-1} (\hat{R}(h_t, t+1) + c_\delta(n-t)), \tag{20}
\]

where

\[
c_\delta(x) = \sqrt{\frac{1}{x-1} \ln \frac{2(n-c_n)(n-c_n+1)}{\delta}}.
\]

Notice that we discard the last two hypotheses so that any \( \hat{R}(h_t, t+1), c_{n-1} \leq t \leq n-2 \) is well defined. The following theorem, which is the main result of this paper, shows that the risk of \( \hat{h} \) is bounded relative to \( \mathcal{M}^n \). The proof of Theorem 9 is in Appendix E.

**Theorem 9** Let \( h_0, \cdots, h_n \) be the ensemble of hypotheses generated by an arbitrary online algorithm \( A \) working with a pairwise loss \( \ell \) which satisfies the conditions given in Theorem 1. \( \forall \epsilon > 0 \), if the hypothesis is chosen via (20) with the confidence \( \delta \) chosen as

\[
\delta = 2(n-c_n+1) \exp \left\{ -\frac{(n-c_n)^2}{64} \right\},
\]

then, when \( n \) is sufficiently large, we have

\[
P\left( R(\hat{h}) \geq M^n + \epsilon \right) \leq 2 \left[ \mathcal{N}\left( \frac{\epsilon}{32\text{Lip}(\phi)} \right) + 1 \right] \exp \left\{ -\frac{(cn-1)^2}{256} + 2 \ln n \right\}.
\]

5. Application: Online Algorithms for Bipartite Ranking

In the bipartite ranking problem we are given a sequence of labeled examples \( z_t = (x_t, y_t) \in \mathbb{R}^d \times \{-1, +1\}, t = 1, \cdots, n \). Minimizing the misranking loss \( \ell_{\text{rank}} \) under this setting is equivalent to maximizing the AUC, which measures the probability that \( f \) ranks a randomly drawn positive example higher than a randomly drawn negative example. This problem has been studied extensively in the batch setting, but the corresponding online problem has not been investigated until recently. In this section, we investigate two online algorithms, analyze their relative loss bounds and combine them with the main result to derive risk bounds for them.

5.1. Online AUC Max with Infinite Buffer (OAM-I)

Recently, Zhao et al. (2011) proposed an online algorithm using linear hypotheses for this problem based on reservoir sampling, and derived bounds on the expectation of the regret of this algorithm. Zhao et al. (2011) use the hinge loss (that bounds the 0-1 loss) to derive the regret bound. The hinge loss is Lipschitz, but it is not bounded and therefore not suitable.
Online Learning with Pairwise Loss Functions

for our risk bounds. Therefore, in the following we use a modified loss function where we bound the Hinge loss in $[0, 1]$ such that

$$\ell(f, z_t, z_j) = \tilde{\phi}(\frac{y_t - y_j}{2}, f(x_t) - f(x_j))$$

where $\tilde{\phi}$ is defined in Remark 4. Using this loss function together with Theorem 9 all we need is an online algorithm that minimizes $M^n$ (or an upper bound of $M^n$) and this guarantees generalization ability of the corresponding online learning algorithm. To this end, we propose the following perceptron-like algorithm, shown in Algorithm 1, and provide loss bounds for this algorithm. Notice that the algorithm does not treat each pair of examples separately, and instead for each $z_t$ it makes a large combined update using its loss relative to all previous examples. Our algorithm corresponds to the algorithm of Zhao et al. (2011) with an infinite buffer, but it uses a different learning rate and different loss function which are important in our proofs.

---

**Algorithm 1:** Online AUC Maximization (OAM) with Infinite Buffer.

**Theorem 10** Suppose we are given an arbitrary sequence of examples $z_t = (x_t, y_t), t = 1, \cdots, n$, and let $u$ be any unit vector. Assume $\max_t \|x_t\| \leq R$ and define

$$M = \sum_{t=2}^{n} \frac{1}{t-1} \left[ \sum_{j=1}^{t-1} \ell_j^t \right], \quad M^* = \sum_{t=2}^{n} \frac{1}{t-1} \left[ \sum_{j=1}^{t-1} \hat{\ell}_j^t(u) \right],$$

where

$$\hat{\ell}_j^t(u) = \mathbb{I}_{y_t \neq y_j} \cdot \left[ \gamma - \langle u, \frac{1}{2}(y_t - y_j)(x_t - x_j) \rangle \right]_+.$$

That is, $M^*$ is the cumulative average hinge loss $u$ suffers on the sequence with margin $\gamma$. Then, after running Algorithm 1 on the sequence, we have

$$M \leq \left( \frac{\sqrt{4R^2 + 2} + \sqrt{\gamma M^*}}{\gamma} \right)^2.$$
When the data is linearly separable by margin $\gamma$, i.e., there exists an unit vector $u$ such that $\hat{\ell}_j^t = 0, \forall t \leq n, j < t$, we have $M^* = 0$ and the bound is constant.

**Proof [Proof of Theorem 10]** First notice that $w_0 = w_1 = 0$ and we also have the following fact

$$\gamma - \langle u, \frac{1}{2}(y_t - y_j)(x_t - x_j) \rangle \geq \gamma - \langle u, \frac{1}{2}(y_t - y_j)(x_t - x_j) \rangle,$$

which implies that when $y_t \neq y_j$,

$$\langle u, y_t(x_t - x_j) \rangle = \langle u, \frac{1}{2}(y_t - y_j)(x_t - x_j) \rangle \geq \gamma - \hat{\ell}_j^t(u). \quad (21)$$

On the other hand, when $y_t = y_j$, then $\hat{\ell}_j^t = 0$. Thus we can write

$$\langle w_t, u \rangle = \langle w_{t-1}, u \rangle + \frac{1}{t-1} \sum_{j=1}^{t-1} \hat{\ell}_j^t \langle u, y_t(x_t - x_j) \rangle$$

$$\geq \langle w_{t-1}, u \rangle + \frac{1}{t-1} \sum_{j=1}^{t-1} \hat{\ell}_j^t (\gamma - \hat{\ell}_j^t(u)) = \langle w_{t-1}, u \rangle + \frac{\gamma}{t-1} \sum_{j=1}^{t-1} \hat{\ell}_j^t - \frac{1}{t-1} \sum_{j=1}^{t-1} \hat{\ell}_j^t \cdot \hat{\ell}_j^t(u)$$

$$\geq \langle w_{t-1}, u \rangle + \frac{\gamma}{t-1} \sum_{j=1}^{t-1} \hat{\ell}_j^t - \frac{1}{t-1} \sum_{j=1}^{t-1} \hat{\ell}_j^t (u) \quad (\therefore \hat{\ell}_j^t \in [0, 1])$$

$$\Rightarrow \langle w_t, u \rangle \geq \sum_{t=2}^{n} \left[ \frac{\gamma}{t-1} \sum_{j=1}^{t-1} \hat{\ell}_j^t - \frac{1}{t-1} \sum_{j=1}^{t-1} \hat{\ell}_j^t (u) \right] = \gamma M - M^*. \quad (22)$$

On the other hand, we have,

$$\|w_t\|^2 = \|w_{t-1}\|^2 + \frac{2}{t-1} \sum_{j=1}^{t-1} \hat{\ell}_j^t (w_{t-1}, y_t(x_t - x_j)) + \frac{1}{t-1} \sum_{j=1}^{t-1} \hat{\ell}_j^t y_t(x_t - x_j)$$

$$\leq \|w_{t-1}\|^2 + \frac{2}{t-1} \sum_{j=1}^{t-1} \hat{\ell}_j^t + 4R^2 \left( \frac{1}{t-1} \right)^2 \left( \sum_{j=1}^{t-1} \hat{\ell}_j^t \right) \cdot \left( \sum_{j=1}^{t-1} \hat{\ell}_j^t \right) \quad (\therefore \hat{\ell}_j^t > 0 \Rightarrow (w_{t-1}, y_t(x_t - x_j)) \leq 1)$$

$$\leq \|w_{t-1}\|^2 + \frac{2}{t-1} \sum_{j=1}^{t-1} \hat{\ell}_j^t + 4R^2 \left( \frac{1}{t-1} \right)^2 \left( \sum_{j=1}^{t-1} \hat{\ell}_j^t \right) \cdot (t-1) \quad (\therefore \hat{\ell}_j^t \in [0, 1])$$

$$= \|w_{t-1}\|^2 + (4R^2 + 2) \left[ \frac{1}{t-1} \sum_{j=1}^{t-1} \hat{\ell}_j^t \right]$$

$$\Rightarrow \|w_n\|^2 \leq (4R^2 + 2) \sum_{t=2}^{n} \left[ \frac{1}{t-1} \sum_{j=1}^{t-1} \hat{\ell}_j^t \right] = (4R^2 + 2)M \quad (23)$$
Combining (22) and (23), we have \((\gamma M - M^*)^2 \leq (4R^2 + 2)M\), which yields
\[
M \leq \frac{\gamma M^* + (2R^2 + 1) + \sqrt{(2R^2 + 1)(\gamma M^* + 2R^2 + 1)}}{\gamma^2} \\
\leq \frac{\gamma M^* + (4R^2 + 2) + \sqrt{(2R^2 + 1)\gamma M^*}}{\gamma^2} \leq \left(\frac{\sqrt{4R^2 + 2 + \gamma M^*}}{\gamma}\right)^2
\]

We therefore get the risk bound for the proposed algorithm as follows.

**Theorem 11** Let \(w_0, \ldots, w_{n-1}\) be the ensemble of hypotheses generated by Algorithm 1. \(\forall \epsilon > 0\), if the hypothesis \(\hat{w}\) is chosen via (20) with the confidence \(\delta\) chosen to be
\[
\delta = 2(n - c_n + 1) \exp \left\{ -\frac{(n - c_n)\epsilon^2}{64} \right\},
\]
then the probability that
\[
R(\hat{w}) \geq \frac{1}{n - c_n} \left[ \left(\frac{\sqrt{4R^2 + 2 + \gamma M^*}}{\gamma}\right)^2 \right] + \epsilon
\]
is at most
\[
2 \left[ \left(\frac{128R^2\sqrt{5n}}{\epsilon} + 1\right)^d + 1 \right] \exp \left\{ -\frac{(cn - 1)\epsilon^2}{256} + 2 \ln n \right\}.
\]

**Proof** [Proof of Theorem 11] By (23), we can easily see that \(\|w_t\| \leq \sqrt{n(4R^2 + 2)}, t = 1, \ldots, n\), therefore we have \(\|w_t\| \cdot \|x\| \leq R^2\sqrt{5n}, \forall t \leq n\). Therefore, we can take the hypothesis space to be
\[
\mathcal{H} = \{ w \in \mathbb{R}^d : \max_{\|x\| \leq R} |\langle w, x \rangle| \leq R^2\sqrt{5n} \}.
\]
By (4), the covering number can be calculated. On the other hand, from the definition in (2), it is easy to see that \(\mathcal{M}^n \leq M/(n - c_n)\). Finally, combining Theorem 9 and Theorem 10 concludes the proof.

5.2. Mistake Bound for Perceptron

Interestingly, we can apply our proof strategy in Theorem 10 to analyze the Perceptron algorithm in the inseparable case. This recovers the best known bound in terms of the one-norm of the hinge losses (given by (Gentile, 2003, Theorem 8) and (Shalev-Shwartz and Singer, 2005, Theorem 2)), but using a simple direct proof.
Theorem 12 (Gentile, 2003; Shalev-Shwartz and Singer, 2005) Let \((x_1, y_1), \cdots, (x_n, y_n)\) be a sequence of examples with \(\|x_i\| \leq R\). Let \(u\) be any unit vector and let \(\gamma > 0\). Define the one-norm of the hinge losses as

\[
D_1 = \sum_{t=1}^{n} \ell_t, \quad \ell_t = [\gamma - y_t(u, x_t)]_+.
\]

Then the number of mistakes the perceptron algorithm makes on this sequence is bounded by

\[
\left( \frac{R + \sqrt{\gamma D_1}}{\gamma} \right)^2.
\]

Proof Let \(m_t = \mathbb{I}_{\text{sgn}(w_{t-1} \cdot x_t) \neq y_t}\) so that the total number of mistakes is \(M = \sum_t m_t\). Then, as usual, the upper bound is \(\|w_n\|^2 \leq R^2 M\). On the other hand, using the fact that \(\ell_t = [\gamma - y_t(u, x_t)]_+ \geq \gamma - y_t(u, x_t)\), which implies \(y_t(u, x_t) \geq \gamma - \ell_t\), we have the lower bound

\[
\langle w_{t+1}, u \rangle = \langle w_t, u \rangle + y_t(u, x_t) m_t \geq \langle w_t, u \rangle + (\gamma - \ell_t) m_t
\]

\[
= \langle w_t, u \rangle + \gamma m_t - \ell_t m_t \geq \langle w_t, u \rangle + \gamma m_t - \ell_t \quad (\because m_t \leq 1)
\]

\[
\Rightarrow \langle w_n, u \rangle \geq \sum_{t=1}^{n} \gamma m_t - \sum_{t=1}^{n} \ell_t = \gamma M - D_1.
\]

Combining the upper bound \(R^2 M\) with (24), we get \((\gamma M - D_1)^2 \leq R^2 M\). Solving the quadratic equation, we have

\[
M \leq \frac{1}{2\gamma^2} \left[ 2\gamma D_1 + R^2 + \sqrt{4\gamma R^2 D_1 + R^4} \right] \leq \frac{1}{2\gamma^2} \left[ 2\gamma D_1 + 2R^2 + \sqrt{4\gamma R^2 D_1} \right]
\]

\[
= \frac{1}{\gamma^2} \left[ R^2 + \gamma D_1 + R\sqrt{D_1} \right] \leq \left( \frac{R + \sqrt{\gamma D_1}}{\gamma} \right)^2.
\]

5.3. Online Projected Gradient Descent for bipartite ranking

We start by reviewing the online learning problem with univariate loss functions under the framework of online convex optimization (OCO). We are given a convex set \(K\) and at each step \(t\), the algorithm selects a hypothesis \(w_{t-1} \in K\). Nature then reveals a convex loss function \(f_t\) and the algorithm suffers a loss \(f_t(w_{t-1})\). The goal of the online learner is to perform well comparing to the best \(w^*\) which is obtained as if the whole data sequence is observed beforehand. More formally, we wish to develop algorithms that achieve a low value of regret \(R(T)\) after round \(T\), which is defined as follows:

\[
R(T) = \sum_{t=1}^{T} f_t(w_{t-1}) - \inf_{w \in K} \sum_{t=1}^{T} f_t(w).
\]

One algorithm that has performance guarantees is the Online Projected Gradient Descent algorithm, which consists of the following three steps:
1. Choose a learning rate $\eta$.

2. Choose $w_0$ to be an arbitrary point in the convex set $K$.

3. For all $t = 1, 2, \cdots, T$,
   \[ w_{t+1} = P_K(x_t - \eta \nabla f_t(w_{t-1})) , \]
   where $P_K$ is the projection operator.

The following theorem (Zinkevich, 2003) shows that this algorithm achieves $O(\sqrt{T})$ regret.

**Theorem 13** Assume that $K$ is bounded, closed and non-empty. Let

\[ D = \max_{w \in K} \| w_0 - w \|. \]

Assume $f_t$ is convex and $\nabla f_t$ exists for all $t$, and define

\[ G = \max_{t \in [T], w \in K} \| \nabla f_t(w) \| \]

to be the maximum $l_2$ norm of the gradient of any $f_t$ in the set $K$. Choose $\eta = D/G\sqrt{T}$, then the regret of the Online Projected Gradient Descent algorithm after time $T$ is at most:

\[ R(T) \leq GD\sqrt{T} . \]

Next, consider applying this algorithm to the bipartite ranking problem, where the hypothesis $w$ is restricted to reside in a convex set $K$. Suppose $\| x_t \| \leq R$ and $\max_{w \in K} \| w \| \leq U$, we define the normalized hinge loss as follows

\[ \ell_{hinge}(w, z_1, z_2) = \frac{1}{1 + 4RU} \left[ 1 - w^T(x_1 - x_2)(y_1 - y_2) \right]_+, \]

which is convex in $w$ and bounded in $[0, 1]$. In each step, nature selects the following convex loss function

\[ f_t(w_{t-1}) = \frac{1}{t-1} \sum_{j=1}^{t-1} \ell_{hinge}(w_{t-1}, z_t, z_j) , \]

which is also a convex function because it is a convex combination of convex functions.

Thus, we can bound the sub-gradient as

\[ \| \nabla f_t(w) \| = \frac{1}{t-1} \frac{1}{1 + 4RU} \left\| - \sum_{j=1}^{t-1} (x_t - x_j)(y_t - y_j) \right\| \leq \frac{4R}{1 + 4RU} \leq \frac{1}{U} . \]

Matching the terminology in Theorem 13, we have $D = U$ and $G = 1/U$. It is easy to see that the Lipschitz constant of (25) is also upper bounded by $1/U$. Therefore, we have the algorithm for online bipartite ranking given in Algorithm 2.

From Theorem 13, we see that the regret of Algorithm 2 is bounded by $\sqrt{n}$. Let $M^*$ denote $\inf_w \sum f_t(w)$, i.e., the online loss of the optimal $w$. Using (19), we have the following theorem
Initialize: \( w_0 = 0 \) and \( \eta = \frac{U^2}{\sqrt{T}} \).

repeat

At the \( t \)-th iteration, receive a training instance \( z_t = (x_t, y_t) \in \mathbb{R}^d \times \{-1, +1\} \).

Update the weight vector such that

\[

w_t = P_K \left( w_{t-1} - \frac{1}{1 + 4RU} \cdot \frac{1}{t-1} \sum_{j=1}^{t-1} \eta (y_t - y_j) (x_t - x_j) \cdot I_{\ell_{\text{hinge}}(w_{t-1}, z_t, z_j) > 0} \right).

\]

until the last instance;

Algorithm 2: Online Projected Gradient Descent for Bipartite Ranking.

Theorem 14 Let \( w_0, \ldots, w_{n-1} \) be the ensemble of hypotheses generated by Algorithm 2 and let

\[

\bar{w} = \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} w_t

\]

then the probability that

\[

\mathcal{R}(\bar{w}) \geq \frac{1}{n - c_n} \left( M^* + \sqrt{n} \right) + \epsilon

\]

is at most

\[

\left\lfloor 2 \left( \frac{32R}{\epsilon} + 1 \right)^d + 1 \right\rfloor \exp \left\{ -\frac{(c n - 1) \epsilon^2}{64} + \ln n \right\}.

\]

Notice that the above bound is worse than the one in Theorem 11 when the data is linearly separable. In the inseparable case, when the optimal cumulative loss \( M^* = \mathcal{O}(n) \), it yields a better bound.

6. Risk Bounds for Algorithms with Finite Buffers

A natural criticism is that Algorithm 1 and 2 are not real online algorithms due to the fact that the entire sample is stored and at each iteration \( t \), the update requires \( \mathcal{O}(t) \) time while online algorithms should have \( \mathcal{O}(1) \) time per step.

To make it a real online algorithm, one can constrain the number of updates at each iteration. The idea is that at the \( t \)-th iteration, instead of keeping all previous \( t-1 \) examples, we keep buffer \( B_t \), whose cardinality can not exceed a predefined size \( |B| \), that has a sample of the history. We call this type of online bipartite ranking algorithm OAM with finite buffer.

One realization of this idea is using the “reservoir sampling” techniques from (Zhao et al., 2011) (Random OAM) where the buffer \( B_t \) is maintained via reservoir sampling. Zhao et al. (2011) gave a bound on the expectation of the cumulative loss \( \mathcal{L} = \sum_t \sum_j \ell_j \). Translating their bound to our notation we get \( \mathbb{E}[M] = M^* + \mathcal{O}(\sqrt{n}) \) where the expectation is over randomly sampled instances in the buffer. However, when the data are linearly separable, the cumulative loss given by this bound grows as \( \mathcal{O}(\sqrt{n}) \) which is worse than the bound we...
provided. In principle, one could turn the results of Zhao et al. (2011) into a high probability bound on \( M \) using the Chebyshev’s inequality and then use Theorem 9 to analyze its risk. However, this does not yield exponential convergence as above. Therefore, a natural question is whether we can provide similar analysis for OAM with finite buffer. The answer is positive.

In the following, we provide the complete analysis. We give the generalization bounds for online learners with the finite buffer. Let us redefine the sample statistic \( M^n_B \) as

\[
M^n_B(Z^n) = \frac{1}{n-c_n} \sum_{t=c_n}^{n-1} M^B_t(Z^t), \quad M^B_t(Z^t) = \frac{1}{|B_{t-1}|} \sum_{j \in B_{t-1}} \ell(h_{t-1}, z_t, z_j). \tag{26}
\]

The difference is that for each hypothesis \( h_{t-1} \), the performance is evaluated on \( x_t \) and the examples in the buffer \( B_t \), which is a subset of the previous examples. Throughout this section, we assume a sequential buffer update strategy, or First In First Out (FIFO), as this simplifies the analysis. We believe that the same results hold for other random buffer strategies, e.g., the reservoir sampling, but leave this for future work. At iteration \( t \), if \( B_{t-1} \) already hits the maximum size \( |B| \), we substitute the oldest example in \( B_{t-1} \) with \( z_t \); otherwise \( z_t \) is added to \( B_{t-1} \). We can extend Theorem 1 as follows. The proof is similar and is provided in Appendix G for completeness.

**Theorem 15** Assume the preconditions in Theorem 1 hold and \( |B_t| = (t-1) \wedge |B| \) where \( |B| \) is a predefined upper bound for the buffer size. Then, \( \forall c > 0, \forall \epsilon > 0 \), we have for sufficiently large \( n \)

\[
\mathbb{P}\left\{ \frac{1}{n-c_n} \sum_{t=c_n}^{n-1} \mathcal{R}(h_{t-1}) \geq M^n_B + \epsilon \right\} \leq \left[ 2N\left( \mathcal{H}, \frac{\epsilon}{16 \text{Lip}(\phi)} \right) + 1 \right] \exp \left\{ -\frac{(\lfloor cn \rfloor - 1) \epsilon^2}{64} + \ln n \right\}. \tag{27}
\]

Similarly, using the same technique of extracting a single hypothesis from an ensemble as in Section 4.2, we have the following theorem

**Theorem 16** Assume the preconditions in Theorem 1 hold. \( \forall \epsilon > 0 \), if the hypothesis is chosen via (20) with the confidence \( \delta \) chosen as

\[
\delta = 2(n-c_n + 1) \exp \left\{ -\frac{(n-c_n) \epsilon^2}{64} \right\},
\]

then, when \( n \) is sufficiently large, we have

\[
\mathbb{P}\left\{ \mathcal{R}(\hat{h}) \geq M^n_B + \epsilon \right\} \leq \left[ N\left( \mathcal{H}, \frac{\epsilon}{16 \text{Lip}(\phi)} \right) + 1 \right] \exp \left\{ -\frac{(\lfloor cn \rfloor - 1) \epsilon^2}{256} + 2 \ln n \right\}.
\]

Thus, we can see that the buffer size must grow faster than \( \ln(n) \). We then introduce Algorithm 3, the finite buffer analog of Algorithm 1.

Algorithm 3 is very similar to the algorithm in (Zhao et al., 2011) where the random buffer strategy is substituted with FIFO. We obtain the following mistake bound.
Initialize: $w_0 = 0, B_0 = \emptyset$; 

repeat
At the $t$-th iteration, receive a training instance $z_t = (x_t, y_t) \in \mathbb{R}^d \times \{-1, +1\}$. Update the weight vector such that

$$w_t = w_{t-1} + \frac{1}{|B_{t-1}|} \sum_{j \in B_{t-1}} \ell^t_j y_t (x_t - x_j).$$

Update the buffer using FIFO such that $B_t = B_{t-1} \cup z_t$; if $|B_t| > |B|$, remove $z_{t-|B|+1}$ out of the buffer.

until the last instance;

Algorithm 3: Online AUC Maximization (OAM) with Finite Buffer.

**Theorem 17** Suppose we are given a sequence of examples $z_t, t = 1, \cdots, n$, and let $u$ be any unit vector. Assume $\max_t \|x_t\| \leq R$ and define

$$M_B = \sum_{t=2}^{n} \frac{1}{|B_{t-1}|} \left[ \sum_{j \in B_{t-1}} \ell_j^t (u) \right], M^*_B = \sum_{t=2}^{n} \frac{1}{|B_{t-1}|} \left[ \sum_{j \in B_{t-1}} \hat{\ell}_j^t \right].$$

Then, after running Algorithm 3 on the sequence, we have

$$M_B \leq \left( \frac{\sqrt{4R^2 + 2} + \sqrt{\gamma M^*_B}}{\gamma} \right)^2.$$

The proof is almost identical to that of Theorem 10 and we include it in Appendix F for completeness.

Similarly, using the model selection approach describe before, we have

**Theorem 18** Let $w_0, \cdots, w_{n-1}$ be the ensemble of hypotheses generated by Algorithm 3. $\forall \epsilon > 0$, if the hypothesis $\hat{w}_B$ is chosen via (20) with the confidence $\delta$ chosen to be

$$\delta = 2(n - c_n + 1) \exp \left\{ -\frac{(n - c_n) \epsilon^2}{64} \right\},$$

then the probability that

$$\mathcal{R}(\hat{w}_B) \geq 1 \left[ \frac{\sqrt{4R^2 + 2} + \sqrt{\gamma M^*_B}}{\gamma} \right]^2 + \epsilon$$

is at most

$$2 \left[ \frac{32R^2 \sqrt{5n}}{\epsilon} + 1 \right]^d \exp \left\{ -\frac{(|B_{cn}| - 1) \epsilon^2}{256} + 2 \ln n \right\}.$$
Suppose the pairwise loss $\ell$ satisfies the conditions given in Theorem 1. Assuming the sequential updating rule for the buffer, then $\forall \epsilon > 0$, we have

$$\mathbb{P}\left\{ \sup_{\mathbf{w}} \left[ \frac{1}{n-2} \hat{M}_B^*(\mathbf{w}) - \frac{1}{n - c_n} M^*_c(\mathbf{w}) \right] \geq \epsilon \right\} \leq \mathcal{N}\left( \mathcal{H}, \frac{\epsilon}{8 \text{Lip}(\phi)} \right) e^{-c^2(1-c)^2 n \epsilon^2}$$

**Proof** To prove this, define $\Omega(\mathbf{w})$ as

$$\Omega(\mathbf{w}) = \frac{1}{n - 2} \hat{M}_B^*(\mathbf{w}) - \frac{1}{n - c_n} M^*_c(\mathbf{w})$$

$$= \frac{1}{n - 2} \sum_{t=2}^{n-1} \frac{1}{|B_t|} \sum_{j \in B_t} \ell(\mathbf{w}, z_t, z_j) - \frac{1}{n - c_n} \sum_{t=cn}^{n-1} \frac{1}{t - 1} \sum_{j=1}^{t-1} \ell(\mathbf{w}, z_t, z_j).$$

Suppose the sequential buffer strategy is used where $B_t = \{x_{t-1}, x_{t-2}, \ldots, x_{t-|B|+1}\}$. We next compute the variation of $\Omega(\mathbf{w})$ when changing any of its $n$ random variables. For both terms, there are two situations when one substitutes $x_i$ with $x_i'$, i.e., when $t = i$ and $t > i$.

- For the first term, when $t = i$, its variation is bounded by $1/(n - 2)$. When $t > i$, as we use FIFO, $x_i$ can only be kept in the buffer for $B$ rounds, thus the variation is bounded by $1/(n - 2)$.
- For the second term, when $t = i$, the variation bounded by $1/(n - c_n)$. We have previously shown in (11) that changing $x_i$ when $i > t$ alters the second term by $1/cn$. 

**Algorithm 4:** Online Projected Gradient Descent for Bipartite Ranking with Finite Buffer.

| Initialize: $\mathbf{w}_0 = 0$ and $\eta = \frac{U^2}{\sqrt{T}}$. |
| repeat |
| At the $t$-th iteration, receive a training instance $z_t = (x_t, y_t) \in \mathbb{R}^d \times \{-1, +1\}$. |
| Update the weight vector such that |
| $$\mathbf{w}_t = P_K \left[ \mathbf{w}_{t-1} - \frac{1}{4RU + 1} \cdot \frac{1}{t - 1} \sum_{j \in B_{t-1}} \eta(y_t - y_j)(x_t - x_j) \cdot \mathbb{I}_{\text{hinge}(\mathbf{w}_{t-1}, z_t, z_j) > 0} \right].$$ |
| until the last instance; |

We have similar extension for Algorithm 2, as described in Algorithm 4. It has the same risk bound as in Theorem 13 where $M^*$ is substituted by $M^*_B$.

Finally we relate $M^*_B(\mathbf{w})$ to $M^*(\mathbf{w})$, which is defined as

$$M^*(\mathbf{w}) = \frac{1}{n - c_n} \sum_{t=cn}^{n-1} \frac{1}{t - 1} \sum_{j=1}^{t-1} \ell(\mathbf{w}, z_t, z_j).$$

It is easy to see that for any fixed $\mathbf{w}$, $M^*(\mathbf{w})$ is an unbiased estimator of $\mathcal{R}(\mathbf{w})$. Therefore, if $M^*_B(\mathbf{w})$ is close to $M^*(\mathbf{w})$ with high probability, we can say that $\mathcal{R}(\hat{w}_B)$ is close to $\mathcal{R}(\mathbf{w})$ (Notice that $\mathbf{w}$ can be arbitrary) with high probability. We have the following lemma,

**Lemma 19** Suppose the pairwise loss $\ell$ satisfies the conditions given in Theorem 1. Assuming the sequential updating rule for the buffer, then $\forall \epsilon > 0$, we have

$$\mathbb{P}\left\{ \sup_{\mathbf{w}} \left[ \frac{1}{n-2} \hat{M}_B^*(\mathbf{w}) - \frac{1}{n - c_n} M^*_c(\mathbf{w}) \right] \geq \epsilon \right\} \leq \mathcal{N}\left( \mathcal{H}, \frac{\epsilon}{8 \text{Lip}(\phi)} \right) e^{-c^2(1-c)^2 n \epsilon^2}$$
Consequently, $\Omega(\mathbf{w})$ is bounded by $\frac{2}{(c(1 - c)n)}$ when one variable is changed. It is easy to see $E[\Omega(\mathbf{w})] = 0$. By McDiarmid’s inequality, we have

$$\mathbb{P}(\Omega(\mathbf{w}) \geq \epsilon) \leq \exp^{-c^2(1-c)^2n\epsilon^2}.$$  

Using the covering number technique again, we have

$$\mathbb{P}(\sup_{\mathbf{w}} \Omega(\mathbf{w}) \geq \epsilon) \leq \mathcal{N}\left(\mathcal{H}, \frac{\epsilon}{\delta \text{Lip}(\phi)}\right) e^{-c^2(1-c)^2n\epsilon^2}.$$  

7. Application: Online Metric Learning

In the past decade, metric learning has become an active field in machine learning with numerous applications in information retrieval, classification, etc. In this section, we consider online learning algorithms for supervised metric learning. Generally speaking, supervised metric learning seeks to find a Mahalanobis distance metric that makes instances “agree with” their labels. The intuition is that under the desired metric, examples that share the same label should be close while ones from different labels should be far away from each other. The metric is parameterized by a positive semi-definite matrix $\mathbf{A}$ such that for any two example $\mathbf{z}_i, \mathbf{z}_j$, we have

$$d_{\mathbf{A}}(\mathbf{z}_i, \mathbf{z}_j) = \sqrt{(\mathbf{x}_i - \mathbf{x}_j)^T \mathbf{A} (\mathbf{x}_i - \mathbf{x}_j)}.$$  

Under the batch setting, recent work derived the generalization bounds for supervised metric learning (Jin et al., 2009; Cao et al., 2012; Bellet and Habrard, 2012). Several online metric learning algorithms have been proposed (Davis et al., 2007; Jain et al., 2008; Jin et al., 2009; Kunapuli and Shavlik, 2012); these analyzed to provide regret bounds but to date the generalization performance of these algorithms has not been analyzed, possibly because no tools existed to provide online-to-batch conversion with pairwise loss functions. In this section, we provide such an analysis.

We consider the hypothesis space to be the vector space of symmetric semi-definite matrices $S^+_d$ of size $d \times d$, equipped with the inner product, $\langle \mathbf{X}, \mathbf{Y} \rangle := \text{Tr}(\mathbf{X}^T \mathbf{Y})$. $\|\mathbf{X}\|_F^2 = (\mathbf{X}, \mathbf{X})$ denotes the Frobenius norm of matrix $\mathbf{X}$. Following Jin et al. (2009), we will work with the following pairwise loss function

$$\ell(\mathbf{A}, \mathbf{z}_i, \mathbf{z}_j) = g\left(y_{ij} \left[1 - (\mathbf{x}_i - \mathbf{x}_j)^T \mathbf{A} (\mathbf{x}_i - \mathbf{x}_j)\right]\right),$$

where $g$ is a normalized version of the hinge loss and $y_{ij} = 1$ if $y_i = y_j$ and $-1$ otherwise. With this setting, examples in the same class must have distance 0 to obtain zero loss and examples in different classes must have distance larger than 2 to yield zero loss.

Before further development, we first state and prove the following theorem for online gradient descent over matrices. Its proof is similar to the case where the hypothesis space is $\mathbb{R}^d$ (Zinkevich, 2003), but we include a proof for completeness. More sophisticated analysis for learning with matrices is provided by Kakade et al. (2012).
Theorem 20 Assume that $K \subset S^+_d$ is convex, closed, non-empty and bounded such that
\[
\sup_{A,B} \|A - B\|_F \leq U.
\]
Assume that at round $t$ we are working with a convex loss function $\ell_t : S^+_d \to \mathbb{R}^+$ such that,
\[
\|\nabla \ell_t(A_t)\|_F \leq D.
\]
Consider an online learner with update rule
\[
A_{t+1} = \mathcal{P}_K[A_t - \eta \nabla \ell_t(A_t)],
\]
where $\mathcal{P}_K$ is the projection operator. If we set the learning rate $\eta = \frac{U}{D} \sqrt{\frac{T}{T}}$, we have
\[
\sum_{t=1}^{T} \ell_t(A_t) - \inf_{B \in K} \sum_{t=1}^{T} \ell_t(B) \leq UD\sqrt{\frac{T}{T}}.
\]

Proof Since $K \subset S^+_d$ is convex, closed, non-empty and bounded subspace of a Hilbert space, we have (Rudin, 2006), $\forall A, B \in S^+_d$,
\[
\|\mathcal{P}_K(A) - \mathcal{P}_K(B)\|_F^2 \leq \|A - B\|_F^2.
\]
For an arbitrary $B \in K$, we have
\[
\|A_{t+1} - B\|_F^2 - \|A_t - B\|_F^2
\]
\[
= \|\mathcal{P}_K[A_t - \eta \nabla \ell_t(A_t)] - B\|_F^2 - \|A_t - B\|_F^2
\]
\[
\leq \|A_t - B - \eta \nabla \ell_t(A_t)\|_F^2 - \|A_t - B\|_F^2
\]
\[
= \|\eta \nabla \ell_t(A_t)\|_F^2 - 2\eta \langle \nabla \ell_t(A_t), A_t - B \rangle
\]
which gives
\[
\langle \nabla \ell_t(A_t), A_t - B \rangle \leq \frac{1}{2\eta} \left(\|A_{t+1} - B\|_F^2 - \|A_t - B\|_F^2 + \eta^2 \|\nabla \ell_t(A_t)\|_F^2\right).
\]
Therefore,
\[
\sum_{t=1}^{T} \ell_t(A_t) - \sum_{t=1}^{T} \ell_t(B) \leq \sum_{t=1}^{T} \langle \nabla \ell_t(A_t), A_t - B \rangle
\]
\[
\leq \frac{1}{2\eta} \sum_{t=1}^{T} \left(\|A_{t+1} - B\|_F^2 - \|A_t - B\|_F^2 + \frac{\eta^2}{2} D^2 T\right)
\]
\[
\leq \frac{1}{2\eta} U^2 + \frac{\eta}{2} D^2 T.
\]
Setting the learning rate $\eta = \frac{U}{D} \sqrt{\frac{T}{T}}$ yields the result. ■
For the metric learning, at each round the loss function is
\[
\ell_t(A) = \frac{1}{t-1} \sum_{j=1}^{t-1} \ell^t_j(A),
\]
where
\[
\ell^t_j(A) = \left[ 1 - y_{tj}(1 - (x_t - x_j)^T A (x_t - x_j)) \right]_+,
\]
where \( y_{tj} \in \{+1, -1\} \) and \( X_{tj} = (x_t - x_j)(x_t - x_j)^T \). It is easy to see that \( \ell^t_j \) is a convex function of \( A \).

Next we choose the convex set \( K \) to be
\[
K = \{ A : A \in S^+_d, \|A\|_F \leq U \}.
\]
We assume \( \sup_t \|x\|_2 \leq R \). To bound \( \ell_t \) in \([0, 1]\), we redefine \( \ell^t_j \) scaling it by a factor of \( \frac{1}{2+UR^2} \). To utilize Theorem 20, we need to bound the subgradient of \( \ell^t_j \) as follows,
\[
\|\nabla \ell^t_j(A)\| \leq \frac{1}{2+UR^2} \|y_{tj}X_{ij}\|_F \leq \frac{R^2}{2+UR^2} \leq \frac{1}{U}.
\]

**Algorithm 5:** Online Projected Gradient Descent for Metric Learning.

Initialize: \( \eta = \frac{U^2}{\sqrt{T}} \) and \( A_0 \) to be any PSD matrix with \( \|A_0\|_F \leq U \).

repeat

At the \( t \)-th iteration, receive a training instance \( z_t = (x_t, y_t) \in \mathbb{R}^d \times \{-1, +1\} \).

Update the weight vector such that
\[
A_t = P_K \left[ A_{t-1} - \frac{1}{2+UR^2} \cdot \frac{1}{t-1} \sum_{j=1}^{t-1} \eta y_{tj}X_{ij} \cdot \mathbb{I} \ell^t_j(A_{t-1}) > 0 \right].
\]

until the last instance;

Finally, as in previous results, define \( M^* = \inf_B \sum_t \ell_t(B) \). Using (19) we bound the risk of Algorithm 5 with the following theorem.

**Theorem 21** Let \( A_0, \ldots, A_{n-1} \) be the ensemble of hypotheses generated by Algorithm 5 and let
\[
\tilde{A} = \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} A_t,
\]
then the probability that
\[
R(\tilde{A}) \geq \frac{1}{n - c_n} \left( M^* + \sqrt{n} \right) + \epsilon
\]
is at most
\[
\left[ 2 \left( \frac{32R}{\epsilon} + 1 \right)^d + 1 \right] \exp \left\{ -\frac{(cn - 1)\epsilon^2}{64} + \ln n \right\}.
\]
The key computational step is to project the matrix to $S_d^+$. Jin et al. (2009) showed an efficient way to perform the projection. Notice that Theorem 21 also applies to the online learning algorithm proposed in Jin et al. (2009) with proper normalization. In general, we believe our results are applicable for all online metric learning algorithms with proved regret guarantees.

8. Conclusion and Future work

In this paper, we provide generalization bounds for online learners using pairwise loss functions and apply these to bipartite ranking and supervised metric learning. These are the first results to provide online-to-batch conversion for pairwise loss functions and as we demonstrate, they are applicable to multiple problems.

There are several directions for possible future work. From an empirical perspective, although the random Online AUC Maximization (OAM) is simple and easy to implement, it seems that it does not maintain buffers in an optimal way. Intuitively, one might want to store important examples that help build the correct ranker instead of using a random buffer. We are currently exploring ideas on building a smart buffer to improve its performance.

From the theoretical point of view, one direction is to improve the current bounds to achieve faster convergence rates. Another direction is deriving tighter mistake bounds for random OAM. Finally, our buffered based results require a buffer size of $O(\log n)$ to guarantee convergence. It would be interesting to investigate whether this is necessary, or alternatively improve the results to show convergence for constant size buffer. It is also interesting to extend our results to dependent data, for example, assuming the data to be stationary mixing sequences (Agarwal and Duchi, 2013) or utilizing new tools such as the sequential Rademacher complexity (Rakhlin et al., 2012).

Acknowledgments

YW and DP thank Nicolò Cesa-Bianchi for early discussions. YW and RK were partly supported by NSF grant IIS-0803409. Part of this research was done when YW was an intern at Akamai Technologies in 2011.

References


Appendix A. Complete Proof of Claim 1

**Proof** [Proof of Claim 1] The required probability can be bounded as follows.

\[
\mathbb{P}_{Z^n \sim D^n, \Xi^n \sim D^n} \left( \frac{1}{n - cn} \sum_{t = c_n}^{n-1} \left[ \mathbb{E}_t [\widetilde{M}_t] - \mathbb{E}_t [M_t] \right] \leq \frac{\epsilon}{2} \right) \\
\geq \mathbb{P}_{Z^n \sim D^n, \Xi^n \sim D^n} \left( \left\{ \frac{1}{n - cn} \sum_{t = c_n}^{n-1} (\mathcal{R}(h_{t-1}) - \mathbb{E}_t [M_t]) \geq \epsilon \right\} \cap \left\{ \left| \frac{1}{n - cn} \sum_{t = c_n}^{n-1} [\mathbb{E}_t [\widetilde{M}_t] - \mathcal{R}(h_{t-1})] \right| \leq \frac{\epsilon}{2} \right\} \right) \\
= \mathbb{E}_{Z^n \sim D^n, \Xi^n \sim D^n} \left[ \left\{ \frac{1}{n - cn} \sum_{t = c_n}^{n-1} (\mathcal{R}(h_{t-1}) - \mathbb{E}_t [M_t]) \geq \epsilon \right\} \times \left\{ \left| \frac{1}{n - cn} \sum_{t = c_n}^{n-1} [\mathbb{E}_t [\widetilde{M}_t] - \mathcal{R}(h_{t-1})] \right| \leq \frac{\epsilon}{2} \right\} \right] \\
= \mathbb{E}_{Z^n \sim D^n} \left[ \mathbb{E}_{\Xi^n \sim D^n} \left[ \left\{ \frac{1}{n - cn} \sum_{t = c_n}^{n-1} (\mathcal{R}(h_{t-1}) - \mathbb{E}_t [M_t]) \geq \epsilon \right\} \times \left\{ \left| \frac{1}{n - cn} \sum_{t = c_n}^{n-1} [\mathbb{E}_t [\widetilde{M}_t] - \mathcal{R}(h_{t-1})] \right| \leq \frac{\epsilon}{2} \right\} \right] Z^n \right] \\
= \mathbb{E}_{Z^n \sim D^n} \left( \frac{1}{n - cn} \sum_{t = c_n}^{n-1} \left[ \mathbb{E}_t [\widetilde{M}_t] - \mathcal{R}(h_{t-1}) \right] \right) Z^n \\
= \mathbb{E}_{\Xi^n \sim D^n} \left( \frac{1}{n - cn} \sum_{t = c_n}^{n-1} \mathbb{E}_t \left[ \frac{1}{t - 1} \sum_{j = 1}^{t-1} \ell(h_{t-1}, z_t, \xi_j) \right] - \mathcal{R}(h_{t-1}) \right) Z^n \\
= \frac{1}{n - cn} \sum_{t = c_n}^{n-1} \left[ \frac{1}{t - 1} \sum_{j = 1}^{t-1} \mathbb{E}_t \left[ \ell(h_{t-1}, z_t, \xi_j) \right] - Z^n \right] \\
= \frac{1}{n - cn} \sum_{t = c_n}^{n-1} \left[ \frac{1}{t - 1} \sum_{j = 1}^{t-1} \mathcal{R}(h_{t-1}) \right] - \mathcal{R}(h_{t-1}) = 0.
\]

We next show that for sufficiently large \( n \),

\[
\mathbb{P}_{Z^n \sim D^n} \left( \frac{1}{n - cn} \sum_{t = c_n}^{n-1} \left[ \mathbb{E}_t [\widetilde{M}_t] - \mathcal{R}(h_{t-1}) \right] \leq \frac{\epsilon}{2} \right) \geq \frac{1}{2},
\]

which combined with (28) implies (9). To begin with, we first show that the corresponding random variable has mean zero

\[
\mathbb{E}_{\Xi^n \sim D^n} \left( \frac{1}{n - cn} \sum_{t = c_n}^{n-1} \left[ \mathbb{E}_t [\widetilde{M}_t] - \mathcal{R}(h_{t-1}) \right] \right) Z^n \\
= \mathbb{E}_{\Xi^n \sim D^n} \left( \frac{1}{n - cn} \sum_{t = c_n}^{n-1} \mathbb{E}_t \left[ \frac{1}{t - 1} \sum_{j = 1}^{t-1} \ell(h_{t-1}, z_t, \xi_j) \right] - \mathcal{R}(h_{t-1}) \right) Z^n \\
= \frac{1}{n - cn} \sum_{t = c_n}^{n-1} \left[ \frac{1}{t - 1} \sum_{j = 1}^{t-1} \mathbb{E}_t \left[ \ell(h_{t-1}, z_t, \xi_j) \right] - Z^n \right] \\
= \frac{1}{n - cn} \sum_{t = c_n}^{n-1} \left[ \frac{1}{t - 1} \sum_{j = 1}^{t-1} \mathcal{R}(h_{t-1}) \right] - \mathcal{R}(h_{t-1}) = 0.
\]

Thus, we can use Chebyshev’s inequality to bound the conditional probability as follows

\[
\mathbb{P} \left( \left\{ \frac{1}{n - cn} \sum_{t = c_n}^{n-1} \left[ \mathbb{E}_t [\widetilde{M}_t] - \mathcal{R}(h_{t-1}) \right] \leq \frac{\epsilon}{2} \right\} \right) Z^n \geq 1 - \frac{\text{Var} \left\{ \frac{1}{n - cn} \sum_{t = c_n}^{n-1} \mathbb{E}_t [\widetilde{M}_t] \right\}}{\epsilon^2/4}.
\]
To bound the variance, we resort to the following Theorem (see Devroye et al., 1996, Theorem 9.3)

**Theorem 22** Let $X_1, \cdots, X_n$ be independent random variables taking values in a set $A$, and assume that $f : A^n \to \mathbb{R}$ satisfies

$$\sup_{x_1, x_2, \cdots, x_n, x'_n} |f(x_1, \cdots, x_i, \cdots, x_n) - f(x_1, \cdots, x'_i, \cdots, x_n)| \leq c_i \quad \forall 1 \leq i \leq n.$$

Then

$$\text{Var}(f(X_1, \cdots, X_n)) \leq \frac{1}{4} \sum_{i=1}^{n} c_i^2.$$

To bound the variance, we first investigate the largest variation when changing one random variable $\xi_j$ with others fixed. From (8), it can be easily seen that changing any of the $\xi_j$ varies each $\mathbb{E}_t[\tilde{M}_t]$, where $t > j$ by at most by $1/(t - 1)$. Remember we are only concerned with $\mathbb{E}_t[\tilde{M}_t]$ when $t \geq c_n$. Therefore, we can see that the variation of $\frac{1}{n-c_n} \sum_{t=c_n}^{n-1} \mathbb{E}_t[\tilde{M}_t]$ regarding the $j$th example $\xi_j$ is bounded by

$$c_j = \frac{1}{n - c_n} \left[ \sum_{t=(j\lor c_n)+1}^{n-1} \frac{1}{t-1} \right] \leq \frac{1}{n - c_n} \left[ \sum_{t=c_n+1}^{n-1} \frac{1}{c_n} \right] \leq \frac{1}{cn}.$$

Thus, by Theorem 9.3 in (Devroye et al., 1996), we have

$$\text{Var}\left( \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} \mathbb{E}_t[\tilde{M}_t] \right) \leq \frac{1}{4} \sum_{i=1}^{n} c_i^2 \leq \frac{1}{4cn^2}.$$

Thus, whenever $\epsilon^2cn^2 > 2$, the LHS of (10) is greater or equal than $1/2$. This completes the proof of Claim 1.

**Appendix B. Proof of Lemma 5**

**Proof [Proof of Lemma 5]** To bound the probability, we use the McDiarmid’s inequality.

**Theorem 23 (McDiarmid’s Inequality)** Let $X_1, \cdots, X_N$ be independent random variables with $X_k$ taking values in a set $A_k$ for each $k$. Let $\phi : (A_1 \times \cdots \times A_N) \to \mathbb{R}$ be such that

$$\sup_{x_k \in A_k, x'_k \in A_k} |\phi(x_1, \cdots, x_N) - \phi(x_1, \cdots, x_{k-1}, x'_k, x_{k+1}, \cdots, x_N)| \leq c_k.$$

Then for any $\epsilon > 0$,

$$\mathbb{P}\{\phi(x_1, \cdots, x_N) - \mathbb{E}\phi(x_1, \cdots, x_N) \geq \epsilon\} \leq e^{-2\epsilon^2/\sum_{k=1}^{N} c_k^2},$$

and

$$\mathbb{P}\{|\phi(x_1, \cdots, x_N) - \mathbb{E}\phi(x_1, \cdots, x_N)| \geq \epsilon\} \leq 2e^{-2\epsilon^2/\sum_{k=1}^{N} c_k^2}.$$
For any fixed \( f \in \mathcal{H} \), we have

\[
\mathbb{E}_{\mathbf{z}_{1:t-1}, \xi_{1:t-1}} [L_t(f)] = \frac{1}{t-1} \sum_{j=1}^{t-1} \mathbb{E}_{\mathbf{z}_{1:t-1}, \xi_{1:n-1}} \mathbb{E}_z [\ell(f, z, \xi_j) - \ell(f, z, z_j)]
\]

\[
= \frac{1}{t-1} \sum_{j=1}^{t-1} (\mathbb{E}_{\xi_j} \mathbb{E}_z [\ell(f, z, \xi_j)] - \mathbb{E}_{\xi_j} \mathbb{E}_z [\ell(f, z, z_j)]) = 0
\]

Now, \( L_t(f) \) is a function of \( 2(t-1) \) variables with each affecting its value at most by \( c_i = 1/(t-1), i = 1, 2, \cdots, 2(t-1) \). Thus, we have \( \sum_{i=1}^{2(t-1)} c_i^2 = \frac{2}{t-1} \). Finally, using the McDiarmid’s inequality, we get

\[
\mathbb{P}_{Z^t \sim \mathcal{D}^t, \xi^t \sim \mathcal{D}^t} (L_t(f) \geq \epsilon) \leq \exp \{-(t-1)\epsilon^2\}.
\]

Appendix C. Proof of Lemma 6

Proof [Proof of Lemma 6] From the definition of \( L_t \) and the assumption on \( \phi \) we have

\[
L_t(h_1) - L_t(h_2) = \frac{1}{t-1} \sum_{j=1}^{t-1} \left[ \mathbb{E}_z [\ell(h_1, z, \xi_j) - \ell(h_1, z, z_j)] - \mathbb{E}_z [\ell(h_2, z, \xi_j) - \ell(h_2, z, z_j)] \right]
\]

\[
= \frac{1}{t-1} \sum_{j=1}^{t-1} \mathbb{E}_z \left\{ [\phi(y - \tilde{y}_j, h_1(x, \tilde{x}_j)) - \phi(y - y_j, h_1(x, x_j))] - [\phi(y - y_j, h_2(x, \tilde{x}_j)) - \phi(y - y_j, h_2(x, x_j))] \right\}
\]

\[
= \frac{1}{t-1} \sum_{j=1}^{t-1} \mathbb{E}_z \left\{ [\phi(y - \tilde{y}_j, h_1(x, \tilde{x}_j)) - \phi(y - \tilde{y}_j, h_2(x, \tilde{x}_j))] - [\phi(y - y_j, h_1(x, x_j)) - \phi(y - y_j, h_2(x, x_j))] \right\}
\]

\[
\leq \frac{1}{t-1} \sum_{j=1}^{t-1} \left[ \text{Lip}(\phi) \sup_{x', x''} |h_1(x', \tilde{x}_j) - h_2(x', \tilde{x}_j)| + \text{Lip}(\phi) |h_1(x, x_j) - h_2(x, x_j)| \right]
\]

\[
\leq \frac{1}{t-1} \sum_{j=1}^{t-1} \left[ 2\text{Lip}(\phi) \sup_{x', x''} |h_1(x', x'') - h_2(x', x'')| \right] = 2\text{Lip}(\phi) \|h_1 - h_2\|_{\infty}
\]
Appendix D. Proof for the Risk Bound of Convex Losses in Section 4.1

Proof Using Jensen’s inequality, we have

\[
\mathcal{R}(\tilde{h}) = \mathbb{E}_z \mathbb{E}_{z'} \left[ \phi \left( y - y', \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} h_{t-1}(x) - \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} h_{t-1}(x') \right) \right]
\]

\[
= \mathbb{E}_z \mathbb{E}_{z'} \left[ \phi \left( y - y', \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} [h_{t-1}(x) - h_{t-1}(x')] \right) \right]
\]

\[
\leq \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} \mathbb{E}_z \mathbb{E}_{z'}[\phi(y - y', h_{t-1}(x) - h_{t-1}(x'))]
\]

\[
= \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} \mathbb{E}_z \mathbb{E}_{z'}[\ell(h_{t-1}, z, z')] = \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} \mathcal{R}(h_{t-1}).
\]

Combining with Theorem 1, we have

\[
\mathbb{P}(\mathcal{R}(\tilde{h}) \geq M^n(Z^n) + \epsilon) \leq \left[ 2N \left( \mathcal{H}, \frac{\epsilon}{16\text{Lip}(\phi)} \right) + 1 \right] \exp \left\{ - \frac{(cn - 1)\epsilon^2}{64} + \ln n \right\}.
\]

Appendix E. Proof of Theorem 9

Proof [Proof of Theorem 9] The proof is adapted from the proof for Theorem 4 in (Cesa-Bianchi et al., 2004). The main difference is that instead of using the Chernoff bound we use a large deviation bound for the U-statistic as follows.

Lemma 24 (see Clementon et al., 2008, Appendix) Suppose we have i.i.d. random variables \( X_1, \ldots, X_n \in \mathcal{X} \) and the \( U \)-statistic is defined as

\[
U_n = \frac{1}{n(n-1)} \sum_{i \neq j} q(X_i, X_j) = \frac{2}{n(n-1)} \sum_{i > j} q(X_i, X_j),
\]

where the kernel \( q : \mathcal{X} \times \mathcal{X} \to \mathbb{R} \) is a symmetric real-valued function. Then we have,

\[
\mathbb{P}(|U_n - \mathbb{E}[U_n]| \geq \epsilon) \leq 2 \exp\{- (n-1)\epsilon^2 \}. \tag{30}
\]

Therefore, by (30), we have

\[
\mathbb{P} \left( |\hat{\mathcal{R}}(h_t, t+1) - \mathcal{R}(h_t)| \geq \epsilon \right) \leq 2 \exp\{- (n-t-1)\epsilon^2 \},
\]

or equivalently,

\[
\mathbb{P} \left( |\hat{\mathcal{R}}(h_t, t+1) - \mathcal{R}(h_t)| \geq \sqrt{\frac{1}{n-t-1} \ln \frac{2}{\delta}} \right) \leq \delta. \tag{31}
\]
By the definition of $c_\delta$ and (31), one can see that
\[
\mathbb{P}\left(\left|\hat{R}(h_t, t+1) - R(h_t)\right| > c_\delta(n-t)\right) \leq \frac{\delta}{(n-c_n)(n-c_n+1)}.
\] (32)

Next, we show the following lemma,

**Lemma 25** Let $h_0, \ldots, h_{n-1}$ be the ensemble of hypotheses generated by an arbitrary online algorithm $A$ working with a pairwise loss $\ell$ which satisfies the conditions given in Theorem 1. Then for any $0 < \delta \leq 1$, we have
\[
\mathbb{P}\left(R(\hat{h}) \geq \min_{c_n-1 \leq t < n-1} (R(h_t) + 2c_\delta(n-t))\right) \leq \delta.
\] (33)

**Proof [Proof of Lemma 25]** The proof closely follows the proof of Lemma 3 in (Cesa-Bianchi et al., 2004) and is given for the sake of completeness. Let
\[
T^* = \arg\min_{c_n-1 \leq t < n-1} (R(h_t) + 2c_\delta(n-t)),
\]
and $h^* = h_{T^*}$ is the corresponding hypothesis that minimizes the penalized true risk and let $\hat{R}^*$ to be the penalized empirical risk of $h_{T^*}$, i.e.
\[
\hat{R}^* = \hat{R}(h_{T^*}, T^* + 1).
\]
Set, for brevity
\[
\hat{R}_t = \hat{R}(h_t, t+1),
\]
and let
\[
\hat{T} = \arg\min_{c_n-1 \leq t < n-1} (\hat{R}_t + c_\delta(n-t)),
\]
where $\hat{h}$ defined in (20) coincides with $h_{\hat{T}}$. With this notation, and since
\[
\hat{R}_{\hat{T}} + c_\delta(n - \hat{T}) \leq \hat{R}^* + c_\delta(n - T^*)
\]
holds with certainty, we can write
\[
\mathbb{P}\left(\hat{R}(\hat{h}) > R(h^*) + \mathcal{E}\right)
= \mathbb{P}\left(\hat{R}(\hat{h}) > R(h^*) + \mathcal{E}, \hat{R}_{\hat{T}} + c_\delta(n - \hat{T}) \leq \hat{R}^* + c_\delta(n - T^*)\right)
\leq \sum_{t=c_n-1}^{n-2} \mathbb{P}\left(\hat{R}(h_t) > R(h^*) + \mathcal{E}, \hat{R}_t + c_\delta(n-t) \leq \hat{R}^* + c_\delta(n - T^*)\right)
\]
where $\mathcal{E}$ is a positive-valued random variable to be specified. Now if
\[
\hat{R}_t + c_\delta(n-t) \leq \hat{R}^* + c_\delta(n - T^*)
\]
holds, then at least one of the following three conditions:
\[
\hat{R}_t \leq R(h_t) - c_\delta(n-t)
\hat{R}^* > R(h^*) + c_\delta(n - T^*)
R(h_t) - R(h^*) < 2c_\delta(n - T^*)
\]
must hold. Therefore, for any fixed $t$, we can write
\[
P\left(\mathcal{R}(h_t) > \mathcal{R}(h^*) + \mathcal{E}, \hat{R}_t + c_\delta(n-t) \leq \hat{R}^* + c_\delta(n-T^*)\right)
\leq P\left(\hat{R}_t \leq \mathcal{R}(h_t) - c_\delta(n-t)\right) + P\left(\hat{R}^* > \mathcal{R}(h^*) + c_\delta(n-T^*)\right) + P\left(\mathcal{R}(h_t) - \mathcal{R}(h^*) < 2c_\delta(n-T^*), \mathcal{R}(h_t) > \mathcal{R}(h^*) + \mathcal{E}\right).
\]

The last term is zero if we choose $\mathcal{E} = 2c_\delta(n-T^*)$. Hence, we can write
\[
P\left(\mathcal{R}(\hat{h}) > \mathcal{R}(h^*) + 2c_\delta(n-T^*)\right)
\leq \sum_{t=c_n-1}^{n-2} P\left(\hat{R}_t \leq \mathcal{R}(h_t) - c_\delta(n-t)\right) + (n-c_n)P\left(\hat{R}^* > \mathcal{R}(h^*) + c_\delta(n-T^*)\right)
\leq (n-c_n) \times \left[\frac{\delta}{(n-c_n)(n-c_n+1)} \quad \text{(By (32).)}\right]
+ (n-c_n) \left[\sum_{t=c_n-1}^{n-2} P\left(\hat{R}_t > \mathcal{R}(h_t) + c_\delta(n-t)\right)\right]
\leq \frac{\delta}{n-c_n+1} + \frac{(n-c_n)^2 \times \delta}{(n-c_n)(n-c_n+1)} \quad \text{(By (32).)}
= \frac{\delta}{n-c_n+1} + \frac{(n-c_n)\delta}{n-c_n+1} = \delta.
\]

Therefore, we know that
\[
P\left(\mathcal{R}(\hat{h}) \geq \min_{c_n-1 \leq t < n-1} (\mathcal{R}(h_t) + 2c_\delta(n-t))\right) \leq \delta.
\] (34)

The next step is to show that with high probability $\min_{c_n-1 \leq t < n-1} (\mathcal{R}(h_t) + 2c_\delta(n-t))$ is close to $M^n$. To begin with, notice that
\[
\min_{c_n-1 \leq t < n-1} (\mathcal{R}(h_t) + 2c_\delta(n-t))
= \min_{c_n-1 \leq t < n-1} \min_{t \leq i < n-1} (\mathcal{R}(h_i) + 2c_\delta(n-i))
\leq \min_{c_n-1 \leq t < n-1} \frac{1}{n-1-t} \sum_{i=t}^{n-2} (\mathcal{R}(h_i) + 2c_\delta(n-i))
= \min_{c_n-1 \leq t < n-1} \left(\frac{1}{n-1-t} \sum_{i=t}^{n-2} \mathcal{R}(h_i)\right)
\quad + \frac{2}{n-1-t} \sum_{i=t}^{n-2} \sqrt{\frac{1}{n-i-1} \ln \frac{2(n-c_n)(n-c_n+1)}{\delta}}
\leq \min_{c_n-1 \leq t < n-1} \left(\frac{1}{n-1-t} \sum_{i=t}^{n-2} \mathcal{R}(h_i) + \frac{2}{n-1-t} \sum_{i=t}^{n-2} \sqrt{\frac{2}{n-i-1} \ln \frac{2(n-c_n+1)}{\delta}}\right)
\leq \min_{c_n-1 \leq t < n-1} \left(\frac{1}{n-1-t} \sum_{i=t}^{n-2} \mathcal{R}(h_i) + 4 \sqrt{\frac{2}{n-t-1} \ln \frac{2(n-c_n+1)}{\delta}}\right)
\]
where the last equality holds because \( \sum_{i=1}^{n-t-1} \sqrt{1/i} \leq 2\sqrt{n-t-1} \) (see Cesa-Bianchi et al., 2004, Sec. 2.B). Define
\[
M_{m,n} = \frac{1}{n-m} \sum_{t=m}^{n-1} M_t(Z^t).
\]

From Theorem 1, one can see that for each \( t = c_n - 1, \ldots, n - 2 \),
\[
\mathbb{P}
\left( \frac{1}{n-1-t} \sum_{i=t}^{n-2} R(h_i) \geq M_{t,n} + \epsilon \right)
\leq \left[ 2N\left( \mathcal{H}, \frac{\epsilon}{16\text{Lip}(\phi)} \right) + 1 \right] \exp \left\{ -\frac{(t-1)\epsilon^2}{64} + \ln n \right\}.
\]

Then, set for brevity,
\[
K_t = M_{t,n} + 4\sqrt{\frac{2}{n-1-t} \ln \frac{2(n-c_n+1)}{\delta}} + \epsilon.
\]

Using the fact that if \( \min(a_1, a_2) \leq \min(b_1, b_2) \) then either \( a_1 \leq b_1 \) or \( a_2 \leq b_2 \), we can write
\[
\mathbb{P}
\left( \min_{c_n-1 \leq t < n-1} (R(h_t) + 2c_s(n-t)) \geq \min_{c_n-1 \leq t < n-1} K_t \right)
\leq \mathbb{P}
\left( \min_{c_n-1 \leq t < n-1} \left( \frac{1}{n-1-t} \sum_{i=t}^{n-2} R(h_i) \right.
\right.
\left.+ \left. 4\sqrt{\frac{2}{n-1-t} \ln \frac{2(n-c_n+1)}{\delta}} \right) \geq \min_{c_n-1 \leq t < n-1} K_t \right)
\leq \sum_{t=c_n-1}^{n-2} \mathbb{P}
\left( \frac{1}{n-1-t} \sum_{i=t}^{n-2} R(h_i) \right.
\left.+ \left. 4\sqrt{\frac{2}{n-1-t} \ln \frac{2(n-c_n+1)}{\delta}} \right) \geq K_t \right)
\leq (n-c_n-1) \left[ 2N\left( \mathcal{H}, \frac{\epsilon}{16\text{Lip}(\phi)} \right) \right] \exp \left\{ -\frac{(cn-1)\epsilon^2}{64} + \ln n \right\} \quad \text{(By (35).)}
\leq \left[ 2N\left( \mathcal{H}, \frac{\epsilon}{16\text{Lip}(\phi)} \right) \right] \exp \left\{ -\frac{(cn-1)\epsilon^2}{64} + 2 \ln n \right\}.
\]

Therefore, using (34), we get
\[
\mathbb{P}
\left( R(\hat{h}) \geq \min_{c_n-1 \leq t < n-1} \left( M_{t,n} + 4\sqrt{\frac{2}{n-1-t} \ln \frac{2(n-c_n+1)}{\delta}} \right) + \epsilon \right)
\leq \delta + \left[ 2N\left( \mathcal{H}, \frac{\epsilon}{16\text{Lip}(\phi)} \right) \right] \exp \left\{ -\frac{(cn-1)\epsilon^2}{64} + 2 \ln n \right\},
\]
which, in particular, leads to

\[
\mathbb{P}\left( \mathcal{R}(\hat{h}) \geq M^n + 4\sqrt{\frac{2}{n-c_n}} \ln \frac{2(n-c_n+1)}{\delta} + \epsilon \right) \\
\leq \delta + \left[ 2N\left( \mathcal{H}, \frac{\epsilon}{16 \text{Lip}(\phi)} \right) \right] \exp \left\{ -\frac{(cn-1)\epsilon^2}{64} + 2 \ln n \right\}.
\]

By substituting \( \epsilon \) with \( \epsilon/2 \) and choosing \( \delta \) as in the statement of Theorem 1, that is, satisfying \( 4\sqrt{\frac{2}{n-c_n}} \ln \frac{2(n-c_n+1)}{\delta} = \frac{\epsilon}{2} \), we have for any \( c > 0 \),

\[
\mathbb{P}\left( \mathcal{R}(\hat{h}) \geq M^n + \epsilon \right) \leq 2(n-c_n+1) \exp \left\{ -\frac{(cn-1)\epsilon^2}{256} + 2 \ln n \right\} + 2 \left[ N\left( \mathcal{H}, \frac{\epsilon}{32 \text{Lip}(\phi)} \right) + 1 \right] \exp \left\{ -\frac{(cn-1)\epsilon^2}{256} + 2 \ln n \right\}
\]

\[
\leq 2 \left[ N\left( \mathcal{H}, \frac{\epsilon}{32 \text{Lip}(\phi)} \right) + 1 \right] \exp \left\{ -\frac{(cn-1)\epsilon^2}{256} + 2 \ln n \right\}
\]

\[\blacksquare\]

Appendix F. Proof of Theorem 17

**Proof [Proof of Theorem 17]** First notice that \( w_0 = w_1 = 0 \) and from (21), we also have \( \langle u, y_t(x_t - x_j) \rangle \geq \gamma - \ell^*_j \) whenever \( y_t \neq y_j \) and \( \ell^*_j = 0 \) otherwise. Thus similarly, we have

\[
\langle w_t, u \rangle = \langle w_{t-1}, u \rangle + \frac{1}{|B_{t-1}|} \sum_{j \in B_{t-1}} \ell^*_j \langle u, y_t(x_t - x_j) \rangle \\
\geq \langle w_{t-1}, u \rangle + \frac{1}{|B_{t-1}|} \sum_{j \in B_{t-1}} \ell^*_j (\gamma - \ell^*_j(u)) \\
= \langle w_{t-1}, u \rangle + \frac{\gamma}{|B_{t-1}|} \sum_{j \in B_{t-1}} \ell^*_j - \frac{1}{|B_{t-1}|} \sum_{j \in B_{t-1}} \ell^*_j \cdot \ell_j(u) \\
\geq \langle w_{t-1}, u \rangle + \frac{\gamma}{|B_{t-1}|} \sum_{j \in B_{t-1}} \ell^*_j - \frac{1}{|B_{t-1}|} \sum_{j \in B_{t-1}} \ell_j(u) \quad (\because \ell_j \in [0,1]) \\
\Rightarrow \langle w_t, u \rangle \geq \sum_{t=2}^{n} \left[ \frac{\gamma}{|B_{t-1}|} \sum_{j \in B_{t-1}} \ell^*_j - \frac{1}{|B_{t-1}|} \sum_{j \in B_{t-1}} \ell_j(u) \right] = \gamma M_B - M_{B^*}.
\]
On the other hand, we have,

\[ \|w_t\|^2 = \|w_{t-1}\|^2 + \frac{2}{|B_{t-1}|} \sum_{j \in B_{t-1}} \ell_j \langle w_{t-1}, y_t(x_t - x_j) \rangle + \left\| \frac{1}{|B_{t-1}|} \sum_{j \in B_{t-1}} \ell_j y_t(x_t - x_j) \right\|^2 \]

\[ \leq \|w_{t-1}\|^2 + \frac{2}{|B_{t-1}|} \sum_{j \in B_{t-1}} \ell_j^2 + 4R^2 \left( \frac{1}{|B_{t-1}|} \right)^2 \left( \sum_{j \in B_{t-1}} \ell_j \right) \cdot \left( \sum_{j \in B_{t-1}} \ell_j \right) \]

\[ \vdash \ell_j^2 > 0 \Rightarrow \langle w_{t-1}, y_t(x_t - x_j) \rangle \leq 1 \]

\[ \leq \|w_{t-1}\|^2 + \frac{2}{|B_{t-1}|} \sum_{j \in B_{t-1}} \ell_j^2 + 4R^2 \left( \frac{1}{|B_{t-1}|} \right)^2 \left( \sum_{j \in B_{t-1}} \ell_j \right) \cdot |B_{t-1}| \quad (\vdash \ell_j \in [0, 1]) \]

\[ = \|w_{t-1}\|^2 + (4R^2 + 2) \left[ \frac{1}{|B_{t-1}|} \sum_{j \in B_{t-1}} \ell_j \right] \]

\[ \Rightarrow \|w_n\|^2 \leq (4R^2 + 2) \sum_{t=2}^n \left[ \frac{1}{|B_{t-1}|} \sum_{j \in B_{t-1}} \ell_j \right] = (4R^2 + 2)M_B \tag{37} \]

Combining (36) and (37), we have \((\gamma M_B - M_B^*)^2 \leq (4R^2 + 2)M_B\), which yields the desired bound.

**Appendix G. Proof of Theorem 15**

**Proof** [Proof of Theorem 15] Again, we rewrite our objective

\[ \mathbb{P}_{\mathcal{Z}^n \sim \mathcal{D}^n} \left( \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} \mathcal{R}(h_{t-1}) - \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} M_t^B \geq \epsilon \right), \tag{38} \]

as

\[ \mathbb{P} \left( \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} [\mathcal{R}(h_{t-1}) - \mathbb{E}_t[M_t^B]] + \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} \left[ \mathbb{E}_t[M_t^B] - M_t^B \right] \geq \epsilon \right) \]

\[ \leq \mathbb{P} \left( \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} \mathcal{R}(h_{t-1}) - \mathbb{E}_t[M_t^B] \geq \frac{\epsilon}{2} \right) + \mathbb{P} \left( \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} \mathbb{E}_t[M_t^B] - M_t^B \geq \frac{\epsilon}{2} \right). \tag{39} \]

Thus, we can bound the two terms separately. The proof consists of four parts, as follows.

**Step 1: Bounding the Martingale difference**

First consider the second term in (39). We have that \( V_t = (\mathbb{E}_t[M_t^B] - M_t^B)/(n - c_n) \) is a martingale difference sequence, i.e. \( \mathbb{E}_t[V_t] = 0 \). Since the loss function is bounded in \([0, 1]\),
we have $|V_t| \leq 1/(n - c_n)$, $t = 1, \ldots, n$. Therefore by the Hoeffding-Azuma inequality, $\sum_t V_t$ can be bounded such that

$$
P_{Z_n^n \sim D^n} \left( \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} \left| \mathbb{E}_t[M_t^B] - M_t^B \right| \geq \frac{\epsilon}{2} \right) \leq \exp \left\{ - \frac{(1 - c)n \epsilon^2}{2} \right\}. \quad (40)$$

**Step 2: Symmetrization by a ghost sample $\Xi^n$**

Recall $M_t^B$ and define $\tilde{M}_t^B$ as

$$M_t^B(Z^t) = \frac{1}{B_{t-1}} \sum_{j \in B_{t-1}} \ell(h_{t-1}, z_t, z_j), \quad \tilde{M}_t^B(Z^t) = \frac{1}{B_{t-1}} \sum_{j \in B_{t-1}} \ell(h_{t-1}, z_t, \xi_j). \quad (41)$$

**Claim 2** The following equation holds

$$P_{Z^n \sim D^n} \left( \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} \left| \mathcal{R}(h_{t-1}) - \mathbb{E}_t[M_t^B] \right| \geq \epsilon \right) \leq 2 P_{Z^n \sim D^n} \left( \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} \left| \mathbb{E}_t[\tilde{M}_t^B] - \mathbb{E}_t[M_t^B] \right| \geq \frac{\epsilon}{2} \right), \quad (42)$$

whenever $(1 - c)^2 n \geq 1/2$.

Notice that the probability measure on the right hand side of (42) is on $Z^n \times \Xi^n$.

**Proof** It can be seen that the RHS (without the factor of 2) of (42) is at least

$$P_{Z^n \sim D^n} \left( \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} \left| \mathcal{R}(h_{t-1}) - \mathbb{E}_t[M_t^B] \right| \geq \epsilon \right) \leq \mathbb{E}_{Z^n \sim D^n} \left[ \mathbb{P}_{Z^n \sim D^n} \left( \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} \left| \mathcal{R}(h_{t-1}) - \mathbb{E}_t[M_t^B] \right| \geq \epsilon \right) \right]. \quad (43)$$

Since $\mathbb{E}_{Z^n \sim D^n} \mathbb{E}_t[\tilde{M}_t^B] = \mathcal{R}(h_{t-1})$, by Chebyshev’s inequality

$$P_{Z^n \sim D^n} \left( \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} \left| \mathbb{E}_t[\tilde{M}_t^B] - \mathcal{R}(h_{t-1}) \right| \leq \frac{\epsilon}{2} \right) \geq 1 - \frac{\text{Var} \left\{ \frac{1}{n - c_n} \sum_{t=c_n}^{n-1} \mathbb{E}_t[\tilde{M}_t^B] \right\}}{\epsilon^2/4}. \quad (43)$$

To bound the variance, we first investigate the largest variation when changing one random variable $\xi_j$ with others fixed. From (41), it can be easily seen that changing any of the $\xi_j$ varies each $\mathbb{E}_t[\tilde{M}_t^B]$, where $t > j$ by at most by $1/|B_t|$.

To estimate the difference the variation of $\frac{1}{n - c_n} \sum_{t=c_n}^{n-1} \mathbb{E}_t[\tilde{M}_t^B]$ regarding the $j$th example $\xi_j$, it is easy to see that since we are using the FIFO strategy, any example can only stay in the buffer for $|B|$ round. Changing each example can only result in $1/|B|$ for $\mathbb{E}_t[\tilde{M}_t^B]$. Thus, the variation is bounded by

$$c_j = \frac{1}{n - c_n} \mathbb{E}_t[\tilde{M}_t^B] \leq \frac{1}{n - c_n} |B| \frac{1}{|B|} = \frac{1}{n - c_n}. \quad (44)$$
Thus, we have
\[
\text{Var} \left( \frac{1}{n - cn} \sum_{t=cn}^{n-1} \mathbb{E}_t[I_{E_t}] \right) \leq \frac{1}{4} \sum_{i=1}^{n} c_i^2 = \frac{1}{4(1-c)^2n}.
\] (44)

Thus, whenever \((1-c)^2n \geq 1/2\), the LHS of (43) is greater or equal than 1/2. This completes the proof of Claim 2.

**Step 3: Uniform Convergence**

In this step, we show how one can bound the RHS of (42) using uniform convergence techniques, McDiarmid’s inequality and \(L_\infty\) covering number. Our task reduces to bound the following quantity
\[
\mathbb{P}_{Z^n \sim D^n, \Xi^n \sim D^n} \left( \frac{1}{n - cn} \sum_{t=cn}^{n-1} \mathbb{E}_t[I_{E_t}] - \mathbb{E}_t[I_{E_t}] \geq \epsilon \right).
\] (45)

Define \(L_t(h_{t-1}) = \mathbb{E}_t[I_{E_t}] - \mathbb{E}_t[I_{E_t}]\). Thus we have
\[
\mathbb{P}_{Z^n \sim D^n, \Xi^n \sim D^n} \left( \frac{1}{n - cn} \sum_{t=cn}^{n-1} L_t(h_{t-1}) \geq \epsilon \right) \leq \mathbb{P} \left( \sup_{h_{cn}, \ldots, h_{n-1}} \left[ \frac{1}{n - cn} \sum_{t=cn}^{n-1} L_t(h_t) \right] \geq \epsilon \right)
\leq \sum_{t=cn}^{n-1} \mathbb{P}_{Z^n \sim D^n, \Xi^n \sim D^n} \left( \sup_{h \in \mathcal{H}} [L_t(h)] \geq \epsilon \right). \quad (46)
\]

To bound the RHS of (46), we start with the following lemma.

**Lemma 26** *Given any function \(f \in \mathcal{H}\) and any \(t \geq 2\)*
\[
\mathbb{P}_{Z^n \sim D^n, \Xi^n \sim D^n} \left( L_t(f) \geq \epsilon \right) \leq \exp \left\{ -\left( |\mathcal{B}_t| - 1 \right) \epsilon^2 / 8 \right\}. \quad (47)
\]

We have already proved the case when \(t \leq |\mathcal{B}|\). When, \(|\mathcal{B}| < t\), \(L_t(f)\) has a bounded variation of \(1/|\mathcal{B}_t|\) when changing its \(2|\mathcal{B}_t|\) variables. Applying McDiarmid’s inequality, we immediately get the desired inequality. Using exactly the same reasoning as in Lemma 6-8, we get

**Lemma 27** *For every \(2 \leq t \leq n\), we have*
\[
\mathbb{P} \left( \sup_{h \in \mathcal{H}} [L_t(h)] \geq \epsilon \right) \leq \mathcal{N} \left( \mathcal{H}, \frac{\epsilon}{4\text{Lip}(\phi)} \right) \exp \left\{ -\left( |\mathcal{B}_t| - 1 \right) \epsilon^2 / 8 \right\}. \quad (48)
\]

Combining (48) and (46), we have
\[
\mathbb{P} \left( \frac{1}{n - cn} \sum_{t=cn}^{n-1} L_t(h_{t-1}) \geq \epsilon \right) \leq \mathcal{N} \left( \mathcal{H}, \frac{\epsilon}{4\text{Lip}(\phi)} \right) \exp \left\{ -\left( |\mathcal{B}_n| - 1 \right) \epsilon^2 / 8 \right\}. \quad (49)
\]
Step 4: Putting it all together

We have

$$\Pr_{Z \sim D^n} \left( \frac{1}{n - c_n} \sum_{t \leq c_n} (R(h_{t-1}) - E_t[M^B_t]) \geq \frac{\epsilon}{2} \right) \leq 2N \left( \mathcal{H}, \frac{\epsilon}{16\text{Lip}(\phi)} \right) n \exp \left\{ -\frac{(|B_{cn}| - 1)\epsilon^2}{64} \right\}. \quad (50)$$