Unfriendly Partitions of a Graph

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Communicated by the Managing Editors
Received March 8, 1988

It has been conjectured by Cowan and Emerson [3] that every graph has an unfriendly partition; i.e., there is a partition of the vertex set \( V = V_1 \cup V_2 \) such that every vertex of \( V_1 \) is joined to at least as many vertices in \( V_1 \) as to vertices in \( V_2 \). It is easily seen that every finite graph has such a partition, and hence by compactness so does any locally finite graph. We show that the conjecture is also true for graphs which satisfy one of the following two conditions: (i) there are only finitely many vertices having infinite degrees; (ii) there are a finite number of infinite cardinals \( m_0 < m_1 < \cdots < m_k \) such that \( m_i \) is regular for \( 1 \leq i \leq k \), there are fewer than \( m_0 \) vertices having finite degrees, and every vertex having infinite degree has degree \( m_i \) for some \( i \leq k \).}

1. INTRODUCTION

By a partition of a set \( X \) we always mean a 2-partition, i.e., a map \( \pi : X \to 2 \). In this paper a graph is a pair \( G = (V, E) \), where \( V \) is the set of vertices of \( G \) and \( E \), the set of edges of \( G \), is a subset of \( [V]^2 = \{ X \subseteq V : |X| = 2 \} \). If \( G = (V, E) \) is a graph, a partition of \( G \) is a partition of

* This paper was written when the first and third authors visited the University of Calgary in 1986. Research supported by NSERC Grant A5198.
the vertex set \( V \), and a partial partition of \( G \) is a partition of some subset \( V' \subseteq V \).

If \( \pi \) is a partial partition of \( G \) and \( X, Y \subseteq V \), then we define

\[
A_\pi(X, Y) = \{ e \in E : e = \{ x, y \} \subseteq \text{dom}(\pi), x \in X, y \in Y, \pi(x) \neq \pi(y) \},
\]

\[
B_\pi(X, Y) = \{ e \in E : e = \{ x, y \} \subseteq \text{dom}(\pi), x \in X, y \in Y, \pi(x) = \pi(y) \}.
\]

Thus \( A_\pi(X, Y)(B_\pi(X, Y)) \) is the set of edges joining a point of \( X \) to a point of \( Y \) on the opposite side (the same side) of the partition \( \pi \). For brevity, we write \( A_\pi(X) = A_\pi(X, V) \), \( B_\pi(X) = B_\pi(X, V) \). Also, by an abuse of notation we write \( A_\pi(x) \) instead of \( A_\pi(\{x\}) \), etc. We define \( a_\pi(X, Y) = |A_\pi(X, Y)| \), \( b_\pi(X, Y) = |B_\pi(X, Y)| \), etc. The degree of a vertex \( x \) is denoted by \( d(x) \).

An unfriendly partition of the graph \( G = (V, E) \) is a partition \( \pi \) of \( G \) such that

\[
a_\pi(x) \geq b_\pi(x)
\]

holds for every \( x \in V \). It is easily seen (Corollary 1.1) that any finite graph has an unfriendly partition and so by a standard compactness argument (Corollary 2.1) any locally finite graph has such a partition. For finite graphs a more general result is stated without proof in [1] and the short proof is given in [2]. Cowan and Emerson [3] asked if every graph has an unfriendly partition; in particular they asked if this is true for a graph having a single vertex with infinite degree. We could not answer the general question, but the following partial results suggest a positive answer.

**Theorem 1.** If \( G = (V, E) \) has only finitely many vertices of infinite degree, then there is an unfriendly partition of \( G \).

**Theorem 2.** Let \( k < \omega \) and let \( m_0 < m_1 < \cdots < m_k \) be infinite cardinals, with \( m_i \) regular for \( 1 \leq i \leq k \). If \( G = (V, E) \) is a graph such that \( |\{ x \in V : d(x) \text{ is finite} \}| < m_0 \) and such that \( d(x) \in \{m_0, \ldots, m_k\} \) for every vertex \( x \) of infinite degree, then \( G \) has an unfriendly partition.

## 2. Preliminary Lemmas

For a partial partition \( \pi \) of a graph \( G \) and a subset \( A \subseteq \text{dom}(\pi) \), we define a partial partition \( \pi' = \pi^*A \) by

\[
\pi'(x) = \begin{cases} 
\pi(x) & \text{if } x \in \text{dom}(\pi) \setminus A \\
1 - \pi(x) & \text{if } x \in A.
\end{cases}
\]
If \( \pi \) is a partition and \( F \) is a finite set of vertices of \( G \), we say that \( \pi \) is \( F \)-good if

\[
a_{\pi}(F) \geq a_{\pi}(F)
\]

holds for every partition \( \pi' \) such that \( \pi' \uparrow V \setminus F = \pi \uparrow V \setminus F \), where as usual \( \pi \uparrow V \setminus F \) denotes the restriction of \( \pi \) to \( V \setminus F \). In other words, \( \pi \) is \( F \)-good if the value \( a_{\pi}(F) \) cannot be increased simply by reshuffling the vertices in \( F \) from one side of the partition to the other. For partial partitions \( \pi, \pi' \), we say that \( \pi' \) extends \( \pi \) if \( \pi = \pi' \uparrow \text{dom}(\pi) \).

**Lemma 1.** Let \( \pi \) be a partition of \( G = (V, E) \), and let \( x \in X \subseteq V, F \subseteq V, F \cap X = \emptyset \). Then

(i) \( a_{\pi \uparrow \{x\}}(X) + a_{\pi}(x) = a_{\pi}(X) + b_{\pi}(x) \),

(ii) \( a_{\pi \uparrow F}(X) + a_{\pi}(F, X) = a_{\pi}(X) + b_{\pi}(F, X) \).

**Proof.**

(i) \( A_{\pi \uparrow \{x\}}(X) \cup A_{\pi}(x) = A_{\pi}(X) \cup B_{\pi}(x) \) and \( A_{\pi \uparrow \{x\}}(X) \cap A_{\pi}(x) = \emptyset = A_{\pi}(X) \cap B_{\pi}(x) \).

(ii) \( A_{\pi \uparrow F}(X) \cup A_{\pi}(F, X) = A_{\pi}(X) \cup B_{\pi}(F, X) \) and \( A_{\pi \uparrow F}(X) \cap A_{\pi}(F, X) = \emptyset = A_{\pi}(X) \cap B_{\pi}(F, X) \).

**Corollary 1.1.** Any finite graph has an unfriendly partition.

**Proof.** Consider the partition \( \pi \) for which \( a_{\pi}(V) \) is maximum. Then \( a_{\pi}(x) \geq b_{\pi}(x) \) holds for every \( x \in V \) by Lemma 1(i) (with \( X = V \)).

**Lemma 2.** Let \( \rho \) be a partial partition of \( G, \text{dom}(\rho) = D \). If each vertex \( x \in V \setminus D \) has finite degree, then there is a partition \( \pi \) which extends \( \rho \) and is \( F \)-good for every finite set \( F \subseteq V \setminus D \).

**Proof.** For each finite set \( K \subseteq V \) choose a partition \( \pi_{K} \) of \( K \) which extends \( \rho \uparrow K \cap D \) such that \( a_{\pi_{K}}(K) \) is maximum. By Rado's selection lemma [4], there is a partition \( \pi \) of \( V \) such that

\[
\forall L \in [V]^{<\omega} \exists K \in [V]^{<\omega} (L \subseteq K \text{ and } \pi \uparrow L = \pi_{K} \uparrow L)
\]

(where \( [V]^{<\omega} \) is the set of finite subsets of \( V \)). Since \( \pi_{K}(x) = \rho(x) \) whenever \( K \in [V]^{<\omega} \) and \( x \in K \cap D \), it follows that \( \pi \) extends \( \rho \). We have to show that \( \pi \) is \( F \)-good for every finite set \( F \subseteq V \setminus D \).

Suppose for a contradiction that \( F \in [V \setminus D]^{<\omega} \) and that \( \pi \) is not \( F \)-good. Then there is a partition \( \pi' \) of \( V \) such that \( \pi' \uparrow V \setminus F = \pi \uparrow V \setminus F \) and \( a_{\pi'}(F) > a_{\pi}(F) \). Let \( L = \{x \mid \{x, y\} \in E \text{ for some } y \in F \} \cup F \). Then \( L \) is finite.
and there is $K \subseteq V$ such that $L \subseteq K$ and $\pi \upharpoonright L = \pi_K \upharpoonright L$. Now consider the partition $\pi'_K$ of $K$ defined by

$$\pi'_K(x) = \begin{cases} \pi_K(x) & \text{if } x \in K \setminus F, \\ \pi'(x) & \text{if } x \in F. \end{cases}$$

Since $A_{\pi'_K}(K) \setminus A_{\pi_K}(F) = A_{\pi_K}(K) \setminus A_{\pi_K}(F)$ and since $A_{\pi_K}(F) = A_\pi(F)$, $A_{\pi'_K}(F) = A_\pi(F)$, it follows that $a_{\pi'_K}(K) > a_{\pi_K}(K)$ and this contradicts the choice of $\pi_K$ since $\pi'_K$ also extends $\rho$.

**COROLLARY 2.1.** Every locally finite graph has an unfriendly partition.

**Proof.** Apply Lemma 2 with $\rho = \emptyset$. If $\pi$ is $\{x\}$-good for the vertex $x$, then $a_\pi(x) \geq b_\pi(x)$.

The main idea required for the proof of Theorem 1 is the following result.

**LEMMA 3.** Let $G = (V, E)$ be a countable graph and let $\rho$ be a partition of a subset $D \subseteq V$. If there are only finitely many vertices $x \in V \setminus D$ with infinite degree, then there is a partition $\pi$ of $G$ which extends $\rho$ and satisfies $a_\pi(x) > b_\pi(x)$ for every $x \in V \setminus D$.

**Proof.** Let $I$ be the set of vertices $x \in V \setminus D$ with infinite degree. Then $I$ is finite. Let $\rho_0 = \rho \cup \{(i, 0): i \in I\}$. By Lemma 2 there is a partition $\pi_0$ of $G$ which extends $\rho_0$ and is $F$-good for every finite set $F \subseteq V \setminus (D \cup I)$. Thus $a_{\pi_0}(x) \geq b_{\pi_0}(x)$ for $x \in V \setminus (D \cup I)$.

Let $I_0 = \{i \in I: a_{\pi_0}(i) \text{ is finite}\}$, and let $\pi_1 = \pi_0 \ast I_0$. Then $a_{\pi_1}(i)$ is infinite for every $i \in I$ and hence $a_{\pi_1}(i) \geq b_{\pi_1}(i)$ since $G$ is countable. Unfortunately, it need not be true that $a_{\pi_1}(x) \geq b_{\pi_1}(x)$ for $x \in V \setminus (D \cup I)$. However, for any finite set $F \subseteq V \setminus (D \cup I)$, we have that $a_{\pi_1 \ast F}(i)$ is infinite for every $i \in I$ and so to prove the lemma it is enough to show that there is some finite $F \subseteq V \setminus (D \cup I)$ such that $a_{\pi_1 \ast F}(x) \geq b_{\pi_1 \ast F}(x)$ holds for every $x \in V \setminus (D \cup I)$.

Suppose for a contradiction that this is false, so that, for every finite set $F \subseteq V \setminus (D \cup I)$, there is some $x \in V \setminus D$ such that $a_{\pi_1 \ast F}(x) < b_{\pi_1 \ast F}(x)$. In this case, we can successively choose vertices $x_1, x_2, \ldots \in V \setminus (D \cup I)$ (not necessarily distinct) so that

$$a_{\pi_1}(x_i) < b_{\pi_1}(x_i),$$

where $\pi_{i+1} = \pi_1 \ast \{x_i\}$ $(i = 1, 2, 3, \ldots)$.

Since $a_{\pi_0}(i)$ is finite for each $i \in I_0$, it follows that $k = a_{\pi_0}(I_0)$ is finite. Consider the finite set $F = \{x_1, x_2, \ldots, x_{2k+1}\}$. By Lemma 1(i), $a_{\pi_{i+1}}(F) = a_{\pi_i}(F) + b_{\pi_i}(x_i) - a_{\pi_i}(x_i) > a_{\pi_i}(F)$ $(1 \leq i \leq 2k+1)$. Therefore,

$$a_{\pi_i}(F) > a_{\pi_i}(F) + 2k + 1,$$
where \( \pi' = \pi_{2k+2} = \pi_1 \ast F' \) and \( F' = \{ x \in F : |\{ i : 1 \leq i \leq 2k+1 \text{ and } x_i = x \}| \) is odd. Let \( \pi'' = \pi' \ast I_0 \). Then \( \pi'' \upharpoonright V \setminus F = \pi_0 \upharpoonright V \setminus F \). Also, by Lemma 1(ii), we have that
\[
\mu_\pi(F) = \mu_\pi(I_0, F) - a_{\pi}(I_0, F) \geq a_{\pi'}(F) - a_{\pi'}(I_0, F),
\]
and
\[
a_{\pi}(F) = a_{\pi_0}(F) + b_{\pi_0}(I_0, F) - a_{\pi_0}(I_0, F).
\]
Since
\[
a_{\pi'}(I_0, F) \leq a_{\pi_0}(I_0, F) + b_{\pi_0}(I_0, F)
\]
and since \( a_{\pi_0}(I_0, F) = k \), it follows that
\[
a_{\pi'}(F) > a_{\pi_0}(F).
\]
But this contradicts the fact that \( \pi_0 \) is \( F \)-good.

3. PROOF OF THEOREM 1

We prove the following stronger assertion, \( \mathcal{R}_\kappa \), by induction on the infinite cardinal \( \kappa \).

\( \mathcal{R}_\kappa \): Let \( G = (V, E) \) be a graph of cardinality \( |V| \leq \kappa \) and let \( \rho \) be a partial partition of \( G \), \( D = \text{dom}(\rho) \). If there are only finitely many vertices \( x \in V \setminus D \) having infinite degrees, then there is a partition \( \pi \) of \( G \) which extends \( \rho \) and satisfies
\[
a_\pi(x) \geq b_\pi(x)
\]
for each \( x \in V \setminus D \).

\( \mathcal{R}_\omega \) holds by Lemma 3. Assume that \( \kappa > \omega \) and that \( \mathcal{R}_\mu \) holds for every infinite cardinal \( \mu < \kappa \).

Let \( I \) denote the set of vertices of \( V \setminus D \) with degree \( \kappa \), and let \( I' \) be the set of vertices \( x \in I \) that are joined to \( \kappa \) points of \( D \). Put \( D' = D \cup I' \), \( J = I \setminus I' \). Let \( \rho' \) be any partition of \( D' \) which extends \( \rho \) and satisfies \( a_{\rho'}(x) = \kappa \) for each \( x \in I' \). Clearly there is such a \( \rho' \) and whenever \( \pi \) is a partition of \( G \) which extends \( \rho' \), then \( a_\pi(x) \geq b_\pi(x) \) is satisfied for each \( x \in I' \).

Let \( \mathcal{C} \) be the set of connected components of \( G \setminus (D' \cup J) \) and for \( C \in \mathcal{C} \) let \( D'(C), J(C) \) denote respectively the sets of vertices of \( D' \) and \( J \) that are connected to \( C \) by an edge of \( G \). Since only finitely many \( x \in V \setminus D \) have
infinite degree there is an infinite cardinal \( \mu < \kappa \) such that \( d(x) \leq \mu \) for \( x \in V \setminus (D' \cup J) \) and so \( |C| \leq \mu \) and \( |D'(C)| \leq \mu \) for each \( C \in \mathcal{C} \). We may also assume that each \( x \in J \) is joined to at most \( \mu \) points of \( D \).

We prove the assertion \( \mathcal{R}_\kappa \) by (a second) induction on \( |J| \).

Suppose first that \( J = \emptyset \). Since by assumption \( \mathcal{R}_\mu \) holds, it follows that, for each \( C \in \mathcal{C} \), there is a partition \( \pi_C \) of \( C \cup D'(C) \) which extends \( \rho' \upharpoonright D'(C) \) and satisfies \( a_{\pi_C}(x) > b_{\pi_C}(x) \) for every \( x \in C \). The partition \( \pi = \rho' \cup \bigcup \{ \pi_C : C \in \mathcal{C} \} \) extends \( \rho \) and satisfies \( a_{\pi}(x) \geq b_{\pi}(x) \) for all \( x \in V \setminus D' \) and hence for all \( x \in V \setminus D \).

Now assume that \( J \neq \emptyset \). Each vertex \( x \in J \) is joined to \( \kappa \) different components \( C \in \mathcal{C} \). Hence there are \( J^* \subseteq J \) and \( \mathcal{C}' \subseteq \mathcal{C} \) such that \( |\mathcal{C}'| = \kappa \) and \( J^*(C) = J^* \neq \emptyset \) for each \( C \in \mathcal{C}' \). Since \( \mathcal{R}_\mu \) holds, it follows that, for each \( C \in \mathcal{C}' \), there is a partition \( \pi_C \) of \( D'(C) \cup J^* \cup C \) which extends \( \rho' \upharpoonright D'(C) \) and satisfies \( a_{\pi_C}(x) \geq b_{\pi_C}(x) \) for all \( x \in J^* \cup C \). There is \( \mathcal{C}'' \subseteq \mathcal{C}' \) such that \( |\mathcal{C}''| = \kappa \) and such that \( \pi_C \upharpoonright J^* \) is the same for all \( C \in \mathcal{C}'' \). Put

\[
\rho_1 = \rho' \cup \bigcup \{ \pi_C : C \in \mathcal{C}'' \}.
\]

Then \( \rho_1 \) is a partition of \( D_1 = D' \cup J^* \cup \bigcup \mathcal{C}'' \) which extends \( \rho \). If \( x \in C \in \mathcal{C}'' \), then \( a_{\rho_1}(x) = a_{\pi_C}(x) \geq b_{\pi_C}(x) = b_{\rho_1}(x) \) since \( a_{\pi_C}(x) \geq 1 \) for each \( C \in \mathcal{C}'' \).

Since \( |J \setminus J^*| < |J| \), it follows from our second induction hypothesis that there is a partition \( \pi \) of \( V \) which extends \( \rho_1 \) and satisfies \( a_\pi(x) \geq b_\pi(x) \) for all \( x \in V \setminus D_1 \). Since \( a_\pi(x) = a_{\rho_1}(x) \geq b_{\rho_1}(x) = b_\pi(x) \) for \( x \in \bigcup \mathcal{C}'' \), and since \( a_\pi(x) \geq a_{\rho_1}(x) = \kappa \) for \( x \in J^* \cup J' \), it follows that

\[
a_\pi(x) \geq b_\pi(x)
\]

holds for all \( x \in V \setminus D \).

4. PROOF OF THEOREM 2

We use the alternative notation \( \pi = [A_0, A_1] \) to indicate that \( \pi : A_0 \cup A_1 \to 2 \) is a partition with sides \( A_i = \pi^{-1}(i) \) \((i = 0, 1)\). If \( x \) is a vertex of the graph \( G \) having infinite degree, then we say that the partial partition \( \pi = [A_0, A_1] \) is **satisfactory** for \( x \) if, for \( i = 0 \) or \( 1 \),

\[
x \in A_i \quad \text{and} \quad d_{A_{1-i}}(x) = d(x),
\]

where \( d_S(x) = |\{ y \in S : \{x, y\} \in E \}| \). An unfriendly partition of \( G \) is satisfactory for every vertex of infinite degree.

Let \( \pi = [A_0, A_1] \) be a partial partition of \( G \), \( D = V \setminus (A_0 \cup A_1) \). An element \( x \in D \) of infinite degree such that \( d_\rho(x) < d(x) \) and \( d_{A_0}(x) \neq d_{A_1}(x) \)
is immediately forced by $\pi$ in the sense that the side to which $x$ belongs in any extension $[A_0', A_1']$ of $[A_0, A_1]$ that is satisfactory for $x$ is determined ($x \in A'$ if $d_{A'}(x) \neq d(x)$). Let $\pi^*$ be the partition obtained from $\pi$ by adjoining all immediately forced elements so that $\pi^*$ is satisfactory for these. Then we may define an increasing sequence of partial partitions $\pi_x$ ($\pi$ an ordinal) by setting $\pi_0 = \pi$, $\pi_{x+1} = \pi^*_x$, $\pi_x = \bigcup \{\pi^*_\beta : \beta < \alpha\}$ ($\alpha$ a limit). The $\pi_x$ are eventually constant, $\pi_x = \bar{\pi}$ for $x$ sufficiently large, and we say that $\bar{\pi}$ is the partial partition forced by $\pi$. $\bar{\pi} = [A_0', A_1']$ is satisfactory for every vertex $x \in (A_0 \cup A_1) \setminus (A_0 \cup A_1)$, and no vertex $x \in D_1 = D \setminus (A_0 \cup A_1)$ is forced by $\bar{\pi}$ so that, if $d(x)$ is infinite, either $d_{A_0}(x) = d(x)$ or $d_{A_1}(x) = d_{A_0}(x) = d(x)$.

In order to prove Theorem 2 we first prove that, for $k < \omega$, the following assertion $\mathcal{F}_k$ holds.

$\mathcal{F}_k$: Let $m_0 < m_1 < \cdots < m_k$ be infinite regular cardinals. Let $\pi = [A, B]$ be a partial partition of any graph $G = (V, E)$, let $a \in D = V \setminus (A \cup B)$ be an element that is not forced by $\pi$, and let $V' = \{x \in D : d(x) \in \{m_0, \ldots, m_k\}\}$. Then there is a partition of $G$, $\pi' = [A', B']$ extending $\pi$ such that $a \in A'$ and $\pi'$ is satisfactory for every $x \in V'$.

Without loss of generality, we may assume that $x = e$ so that no vertex $x \in D$ is forced by $\pi$. Let $W_0 = \{x \in D : x \notin V' \text{ or } d_D(x) < d(x)\}$. Note that any extension $[A', B']$ of $\pi$ will be satisfactory for an element $x \in (A' \cup B') \cap (W_0 \cap V')$ irrespective of the side in which $x$ is placed.

Let $M = \{x \in V' \setminus W_0 : d(x) = m_k\}$, $N = V' \setminus (M \cup W_0)$. Let $\mathcal{F}$ denote the set of all partitions $[F_0, F_1]$ such that $F_0 \subseteq M$, $F_1 \subseteq N$ and

$$d_{F_{i-1}}(x) = d(x) \quad \text{for} \quad x \in F_i \quad (i = 0 \text{ or } 1).$$

Clearly $\rho^* = [F^*_0, F^*_1]$ is the largest member of $\mathcal{F}$, where $F^*_i = \bigcup \{F_i : [F_0, F_1] \in \mathcal{F}\}$ ($i \in \{0, 1\}$). Let $\bar{\rho} = [\bar{F}_0, \bar{F}_1]$ be the partial partition of $G \upharpoonright D$ forced by $\rho^*$. Put $D_1 = D \setminus (\bar{F}_0 \cup \bar{F}_1)$, $W_1 = \{x \in D_1 \setminus W_0 : d_{D_1}(x) < d(x)\}$, $W = W_0 \cup W_1$. Note that any extension of $[\bar{F}_0, \bar{F}_1]$ which includes an element $x$ of $V' \cap W_1$ ($= W_1$) is satisfactory for $x$.

Claim 1. If $S \subseteq D_1 \cap M$ and $|S| < m_k$, then the set $T = \{x \in D_1 \cap N : d_S(x) = d(x)\}$ has cardinality $|T| < m_k$.

Proof. If $k = 0$ this is obvious since, in this case, $N = \emptyset$. Suppose that $k > 0$. Let $S_0 = \{x \in S : d_T(x) < m_k\}$, $T_0 = \{y \in T : \{x, y\} \in E \text{ for some } x \in S_0\}$. Then $|T_0| < m_k$ since $m_k$ is regular. Since $[S \setminus S_0, T \setminus T_0] \in \mathcal{F}$, it follows that $S = S_0$, $T = T_0$, and so $|T| = |T_0| < m_k$.

Let $\mathcal{C}$ be the set of connected components of $G \upharpoonright D_1 \cap N$, and for $C \in \mathcal{C}$, let $L(C) = C \cup \{y \in D_1 : \{x, y\} \in E \text{ for some } x \in C\}$. Then $|L(C)| < m_k$ for $C \in \mathcal{C}$. Let $\mathcal{E}$ denote the set of all partial partitions $[X, Y]$ of $D_1$ such that
(i) \(|X \cup Y| < m_k\),
(ii) \(L(C) \subseteq X \cup Y\) whenever \(C \in \mathcal{C}\) and \(C \cap (X \cup Y) \neq \emptyset\),
(iii) \([X, Y]\) is satisfactory for all \(x \in (X \cup Y) \cap N \setminus W\).

**Claim 2.** Let \([X, Y] \in \mathcal{E}, K \subseteq D_1, |K| < m_k\). Then there is an extension \([X', Y']\) of \([X, Y]\) such that \([X', Y'] \in \mathcal{E}\) and \(K \subseteq X' \cup Y'\).

**Proof.** If \(k = 0\) this is obvious since \(N = \emptyset\) in this case. Suppose that \(k > 0\). For \(x \in K \cap N\), let \(C_x \in \mathcal{C}\) be the component containing \(x\). Then \(L = K \cup \{L(C_x); x \in K \cap N\}\) has cardinality \(|L| < m_k\). The vertices of \(L \cap N \setminus W\) have degrees \(m_0, m_1, \ldots, m_{k-1}\) in the graph \(G \uparrow X \cup Y \cup L\). It follows from the induction hypothesis \(S_{k-1}\) that there is a partition \([X', Y']\) of \(X \cup Y \cup L\) which extends \([X, Y]\) and is satisfactory for every \(x \in (L \cap N) \setminus (W \cup X \cup Y)\). Since \([X, Y] \in \mathcal{E}\), it follows that \([X', Y']\) is satisfactory for all \(x \in (X' \cup Y') \cap N \setminus W\), and hence \([X', Y'] \in \mathcal{E}\). 

To complete the proof of \(S_k\) we consider separately the following three cases.

**Case 1.** \(|D_1| < m_k\).

Since \([\emptyset, \emptyset] \in \mathcal{E}\), it follows by Claim 2 that there is a partition \([X, Y] \in \mathcal{E}\) such that \(X \cup Y = D_1\). Thus \([X, Y]\) is satisfactory for every \(x \in D_1 \cap N \setminus W\). If \(a \in \bar{F}_i\), then put \(A' = A \cup \bar{F}_i \cup X, B' = B \cup \bar{F}_i \cup Y\); if \(a \notin \bar{F}_0 \cup \bar{F}_i\), then we may assume that \(a \in X\), and we define \(A' = A \cup \bar{F}_0 \cup X, B' = B \cup \bar{F}_i \cup Y\). It is now a simple matter to verify that, in either case, \([A', B']\) is a partition of \(G\) having the required properties.

**Case 2.** \(|D_1| = m_k\).

We consider first the simple case \(k = 0\). Note that, for this case \(N = \emptyset\), \(D_1 = D\), and \(m_0\) may be a singular cardinal. Let \(x_\xi (\xi < m_0)\) be any sequence with \(x_0 = a\) such that \(|\{\xi; x_\xi = x\}| = m_0\) for each \(x \in D\). Construct an increasing sequence of partial partitions \([A_\xi, B_\xi] (\xi < m_0)\) of \(G\) as follows. For limit \(\xi \) or \(\xi = 0\), put \(A_\xi = A \cup \bigcup \{A_\eta; \eta < \xi\}, B_\xi = B \cup \bigcup \{B_\eta; \eta < \xi\}\). Now suppose that \(\xi = \eta + 1\) is a successor ordinal. In this case, if \(x_\xi \notin A_\eta \cup B_\eta\), then put \(A_\xi = A_\eta \cup \{x_\xi\}, B_\xi = B_\eta\); if \(x_\xi \in (A_\eta \cup B_\eta) \setminus V'\), then put \(A_\xi = A_\eta, B_\xi = B_\eta\); if \(x_\xi \in (A_\eta \cup B_\eta) \cap V'\), let \(\xi < m_0\) be the least ordinal such that \(x_\xi \notin A_\eta \cup B_\eta\) and \(\{x_\eta, x_\xi\} \in E\), and now define \(A_\xi, B_\xi\) so that \(A_\xi \cup B_\xi = A_\eta \cup B_\eta \cup \{x_\xi\}\) and so that \(x_\eta\) and \(x_\xi\) are on different sides of the partition. Clearly \([A_{m_0}, B_{m_0}]\) is a partition which satisfies the conclusion of \(S_0\).

We now assume that \(k > 0\), so that \(m_k\) is regular. Let \(x_\alpha (\alpha < m_k)\) be a 1–1 enumeration of the elements of \(D_1\). For \(x \in D_1 \cap M \setminus W\) and \(\alpha < m_k\), let \(\mathcal{E}(x, \alpha)\) denote the set of all partitions \([X, Y] \in \mathcal{E}\) such that \(x \in X \cup Y\) and,
for some $\xi > \alpha$, there is $x_\xi \in X \cup Y$ such that $\{x, x_\xi\} \in E$ and $x, x_\xi$ are on different sides of the partition $[X, Y]$.

**Claim 3.** If $[X, Y] \in \mathcal{E}$, $x \in D_1 \cap M \setminus W$, and $\alpha < m_k$, then there is an extension $[X', Y']$ of $[X, Y]$ such that $[X', Y'] \in \mathcal{E}(x, \alpha)$.

**Proof.** By Claim 2 we may assume that $x \in X \cup Y$ and without loss of generality we may suppose that $x \in X$. If $d_{D_1 \setminus N}(x) = m_k$, then there is $\xi > \alpha$ such that $\{x, x_\xi\} \in E$ and $x_\xi \notin X \cup Y \cup N$. Then $[X, Y \cup \{x_\xi\}] \in \mathcal{E}(x, \alpha)$. Therefore, we may assume that $d_{D_1 \setminus N}(x) < m_k$. Note that, since $[X, Y] \in \mathcal{E}$, if $y \in (D_1 \cap N) \setminus (X \cup Y)$ and $z \in X \cup Y$ is joined to $y$ by an edge, then $z \in D_1 \cap M$. Therefore, by Claim 1, the set $T = \{y \in D_1 \cap N: d_{X \cup Y}(y) = d(y)\}$ has cardinality $|T| < m_k$. Since $d_{\mathcal{N}}(x) = m_k$, it follows that there is some $\xi > \alpha$ such that $\{x, x_\xi\} \in E$ and $x_\xi \in D_1 \cap N \setminus T$. Since $d_{X \cup Y}(x_\xi) < d(x_\xi)$ it follows that $x_\xi$ is not forced by $[X, Y]$. Therefore, by the induction hypothesis applied to the graph $G \upharpoonright X \cup Y \cup L(C_{x_\xi})$, there is a partition of $X \cup Y \cup L(C_{x_\xi})$, $[X', Y']$, which extends $[X, Y]$, is satisfactory for all $y \in L(C_{x_\xi}) \setminus N \setminus W$ and is such that $x_\xi \in Y'$. Then $[X', Y'] \in \mathcal{E}(x, \alpha)$.

We now conclude the proof of $\mathcal{S}_k$ in Case 2 for $k > 0$ as follows. Let $t_\xi$ ($\xi < m_k$) be a 1–1 enumeration of $D_1 \cup ((D_1 \cap M \setminus W) \times m_k)$. By Claims 2 and 3 there is an increasing sequence of partial partitions $\pi_\xi = [X_\xi, Y_\xi]$ ($\xi < m_k$) such that (i) if $t_\xi \in D_1$, then $t_\xi \in X_\xi \cup Y_\xi$ and $[X_\xi, Y_\xi] \in \mathcal{E}$, (ii) if $\xi = (x, \alpha) \in (D_1 \cap M \setminus W) \times m_k$, then $[X_\xi, Y_\xi] \in \mathcal{E}(x, \alpha)$. Let $X = \bigcup \{X_\xi: \xi < m_k\}$, $Y = \bigcup \{Y_\xi: \xi < m_k\}$. Then $[X, Y]$ is a partition of $D_1$ which is satisfactory for every $x \in D_1 \setminus W$. Indeed, if $x = t_\xi \in D_1 \cap N \setminus W$, then $[X_\xi, Y_\xi]$ is satisfactory for $x$ and so also is $[X, Y]$. If $x \in D_1 \cap M \setminus W$, then for $m_k$ different $\xi < m_k$ we have $t_\xi = (x, \alpha_\xi)$ and so by the regularity of $m_k$ and the fact that $[X_\xi, Y_\xi] \in \mathcal{E}(x, \alpha_\xi)$, it follows that $x$ is joined to $m_k$ different vertices on the opposite side of the partition $[X, Y]$.

Put $U_0 = \overline{F_0} \cup X$, $U_1 = \overline{F_1} \cup Y$. Then $[U_0, U_1]$ is a partition of $D$ which is satisfactory for every vertex $x \in D \setminus W_0$. For, if $x \in W_1$, then $d_{F_0}(x) = d_{F_1}(x) = d(x)$. Without loss of generality, we may assume that $a \in U_0$. Then $[A', B']$ is a partition of $G$ having the stated properties, where $A' = A \cup U_0$, $B' = B \cup U_1$.

**Case 3.** $|D_1| > m_k$.

Let $\mathcal{D}$ be the set of connected components of $G \upharpoonright V'$ and for each $C \in \mathcal{D}$, let $L(C) = \{y \in V: \{x, y\} \in E \text{ for some } x \in C\}$. Let $C_\xi$ ($\xi < \lambda$) be a 1–1 enumeration of $\mathcal{D}$. Since $|C_\xi| \leq m_k$, it follows from Cases 1 and 2 that there is a sequence of partial partitions $[A_\xi, B_\xi]$ ($\xi < \lambda$) such that $A \subseteq A_0 \subseteq A_1 \subseteq \cdots$, $B \subseteq B_0 \subseteq B_1 \subseteq \cdots$, $a \in A_\xi$ if $a \in C_\xi$, and $[A_\xi, B_\xi]$ is satisfactory for every $x \in V' \cap C_\xi$. The partition $[V \setminus \{B_\xi: \xi < \lambda\}, \bigcup \{B_\xi: \xi < \lambda\}] = [A', B']$ has the required properties.

This completes the proof that $\mathcal{S}_k$ holds for all $k < \omega$. 
We conclude the proof of Theorem 2 as follows. Let $F = \{ x \in V : d(x) < \omega \}$, $V' = V \setminus F$. Since $|F| < m_0$, it follows that $d_{V'}(x) = d(x) \in \{ m_0, \ldots, m_k \}$ for every $x \in V'$. By $\mathcal{R}_k$, it follows that there is a partition $\pi = [A, B]$ of $V'$ which is satisfactory for every $x \in V'$. Now by $\mathcal{R}_{|V|}$ it follows that there is a partition $\pi' = [A', B']$ of $V$ which extends $\pi$ and satisfies

$$a_{\pi'}(x) \geq b_{\pi'}(x)$$

for every $x \in V \setminus V'$. Thus $\pi'$ is an unfriendly partition of $G$.

5. Concluding Remarks

Call a cardinal $\kappa$ unfriendly if every graph $G = (V, E)$ of cardinality $|V| \leq \kappa$ has an unfriendly partition. Szalkai [5] observed the following compactness result: If $\kappa > \omega$ is a measurable cardinal and $\lambda$ is unfriendly for every $\lambda < \kappa$, then $\kappa$ is also unfriendly. Also a referee points out that “measurable” can be replaced by “weakly compact.” Unfortunately, we do not know if $\omega$ is unfriendly.

Since this paper was written Shelah (see [6]) has shown that $(2^\omega)^+ \omega$ is not unfriendly (and it is consistent that $\aleph_\omega$ is not). However, every graph can be partitioned into three sets so that each vertex is joined to at least as many vertices not in the same part as to vertices in the same part.

References

5. I. Szalkai, Written communication.