Categorical equivalences for $\sqrt{7}$ quasi-MV algebras

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Abstract
In previous investigations into the subject ([4], [9], [1]), $\sqrt{7}$ quasi-MV algebras have been mainly viewed as preordered structures w.r.t. the induced preorder relation of their quasi-MV term reducts. In this paper, we shall focus on a different relation which partially orders cartesian $\sqrt{7}$ quasi-MV algebras. We shall prove that: a) every cartesian $\sqrt{7}$ quasi-MV algebra is embeddable into an interval in a particular Abelian $\ell$-group with operators; b) the category of cartesian $\sqrt{7}$ quasi-MV algebras isomorphic with the pair algebras over their own polynomial MV subreducts is equivalent both to the category of such $\ell$-groups (with strong order unit), and to the category of MV algebras. As a byproduct of these results we obtain a purely group-theoretical equivalence, namely between the mentioned category of $\ell$-groups with operators and the category of Abelian $\ell$-groups (both with strong order unit).

1 Introduction
$\sqrt{7}$ quasi-MV algebras were introduced in [4] as the variety generated by an algebra over the complex numbers which is the abstract counterpart, in an appropriate sense, of the algebra whose universe is the set of all qumixes (i.e. density operators) of $\mathbb{C}^2$ and whose operations correspond to some of the most significant quantum logical gates: see [8] for an introduction to the basic concepts of quantum computation, and [4], [6] and especially [5], for a thorough motivational introduction to our approach. Subsequent papers elaborated by our research group ([9], [1]) tried to investigate in greater detail the structure theory and the algebraic properties of this variety. Two problems, however, remained open, namely:

- While $\sqrt{7}$ quasi-MV algebras bear a close resemblance to Chang’s MV algebras, as their denomination explicitly suggests, the latter - thanks to Daniele Mundici’s Gamma functor ([2], Chapters 2 and 7) - admit of a telling and mathematically powerful representation in terms of intervals in
Abelian lattice-ordered groups with strong order unit, whereas the status of the former remained unclear;

- Although it had been observed that two different preorderings, a weak and a strong one, could be associated with any $\sqrt{7}$ quasi-MV algebra, and although it was clear that the latter had a greater significance in quantum computational terms, only the former had been investigated from an algebraic viewpoint.

The present paper intends to amend both flaws at one fell swoop. By focussing on the strong order in $\sqrt{7}$ quasi-MV algebras, rather than on the weak one, we will be in a position to provide also in the present framework a partial analogue to Mundici’s Gamma functor: more precisely, we will prove that any cartesian (i.e. any "well-behaved") $\sqrt{7}$ quasi-MV algebra can be represented as a subalgebra of an interval in a particular Abelian $\ell$-group expanded by additional projection and rotation operators. This representation extends to a proper categorical equivalence for a "very well-behaved" subclass of cartesian $\sqrt{7}$ quasi-MV algebras, i.e. what we call pair algebras. In this way, we establish a somewhat surprising connection between algebras from quantum computation and group theory. Finally, we show that pair algebras can be identified, modulo categorical equivalence, with MV algebras, hence also with Abelian $\ell$-groups.

Reasons of space prevent us from giving a detailed presentation of the conceptual and notational preliminaries needed to make this paper self-contained. The reader who is not acquainted with our approach, therefore, should previously consult the papers referenced in our bibliography, which are all available either in print or online. In particular, it will suffice to read either [4] or [9] to fully understand both the content and the notation of this article.

2 The strong order in $\sqrt{7}$ quasi-MV algebras

A $\sqrt{7}$ quasi-MV algebra (henceforth, $\sqrt{7}$ qMV algebra for short) can be preordered not only by the induced preorder of its own qMV term reduct, i.e. by the relation $\leq$ which holds between $a$ and $b$ just in case $1 = a' \oplus b$, henceforth called weak order, but also by the relation of strong order hereafter defined.

**Definition 1** Let $A$ be a $\sqrt{7}$ qMV algebra. The relation $\preceq^A$ of strong order is defined in such a way that, for any $a, b \in A$,

$$a \preceq^A b \text{ iff } a \leq^A b \text{ and } \sqrt{7} a \preceq^A \sqrt{7} b.$$  

As usual, superscripts will be omitted whenever possible. Remark that, in the standard algebra $S_r$, $(a, b) \preceq^S_r (c, d)$ iff $a \leq^R c$ and $b \leq^R d$, where $\leq^R$ denotes the usual ordering on the real numbers. Thus, in $S_r$, the strong order coincides with the restriction to $[0, 1] \times [0, 1]$ of the componentwise lattice order on the complex numbers. We state without a proof the following obvious
Lemma 2 Let $A$ be a $\sqrt{7}$ qMV algebra. Then (i) $\preceq$ is a preordering, but not necessarily a partial ordering on $A$; (ii) if $a, b \in A$, then $a \preceq b$ implies $a \leq b$, while the converse need not hold.

Two questions naturally arise. Can we give necessary and sufficient conditions for a given $\sqrt{7}$ qMV algebra to be, respectively, a) partially ordered by $\preceq$ and b) lattice ordered by $\preceq$? The former question is easy to answer: the defining quasiequation of cartesian $\sqrt{7}$ qMV algebras, i.e.

$$a \oplus 0 \approx b \oplus 0 \& \sqrt{7}a \oplus 0 \approx \sqrt{7}b \oplus 0 \Rightarrow a \approx b,$$

boils down to the fact that $\preceq$ is a partial order on these algebras. Thus, a $\sqrt{7}$ qMV algebra is partially ordered by $\preceq$ if and only if it is cartesian. The latter question will be here given only partial answers. Whenever a given $\sqrt{7}$ qMV algebra is lattice ordered by $\preceq$, we denote the greatest lower bound and the least upper bound of two elements $a$ and $b$, respectively, by $a \land b$ and $a \lor b$. Remark that, unlike the corresponding operations in MV algebras, here $\land$ and $\lor$ need not be term operations, whence they are not necessarily preserved by $\sqrt{7}$ qMV homomorphisms. In particular, the canonical embedding

$$h(a) = \left(a \oplus 0, \sqrt{7}a \oplus 0\right)$$

of a cartesian $\sqrt{7}$ qMV algebra $A$ into the pair algebra $\mathcal{P}(R_A)$ over its polynomial MV subreduct $R_A$ of regular elements need not preserve existing binary meets or binary joins. The next lemma characterises the class of lattice ordered $\sqrt{7}$ qMV algebras where meets and joins are actually preserved.

Lemma 3 Let $A$ be a lattice ordered $\sqrt{7}$ qMV algebra. The following are equivalent:

- for every $a, b$ in $A$:
  
  a. $(a \land b) \oplus 0 = (a \oplus 0) \land (b \oplus 0)$
  
  b. $\sqrt{7}(a \land b) \oplus 0 = \left(\sqrt{7}a \oplus 0\right) \land \left(\sqrt{7}b \oplus 0\right)$;

- The canonical embedding $h$ preserves $\land^A$ and $\lor^A$.

Proof. If $h$ preserves lattice operations, then both conditions are clearly satisfied. Conversely, let $a, b \in A$. By (a) and the fact that $h$ is an embedding,

$$h ((a \oplus^A 0) \land^A (b \oplus^A 0)) = h ((a \land^A b) \oplus^A 0) = h (a \land^A b) \oplus^A \mathcal{P}(R_A) (0, k).$$

However, $a \oplus^A 0$ and $b \oplus^A 0$ are regular elements of $A$, whence $(a \oplus^A 0) \land^A (b \oplus^A 0) = (a \oplus^A 0) \land^A (b \oplus^A 0)$. The operation $\oplus^A$, being a term operation, is
preserved by the embedding. Thus,
\[
h ( (a \oplus^A 0) \otimes^A (b \oplus^A 0) ) = h(a \oplus^A 0) \otimes^{P(R_A)} h(b \oplus^A 0)
\]
\[
= \langle a \oplus^A 0, k \rangle \wedge^{P(R_A)} \langle b \oplus^A 0, k \rangle
\]
\[
= \langle (a \oplus^A 0) \wedge^A (b \oplus^A 0), k \rangle
\]
\[
= \left( h (a) \wedge^{P(R_A)} h (b) \right) \oplus^{P(R_A)} \langle 0, k \rangle.
\]
Summing up, \( h(a \wedge^A b) \oplus^{P(R_A)} \langle 0, k \rangle = (h (a) \wedge^{P(R_A)} h (b)) \oplus^{P(R_A)} \langle 0, k \rangle \). Similarly, \( \sqrt{\mathcal{P}(R_A)} h(a \wedge^A b) \oplus^{P(R_A)} \langle 0, k \rangle = \sqrt{\mathcal{P}(R_A)} (h (a) \wedge^{P(R_A)} h (b)) \oplus^{P(R_A)} \langle 0, k \rangle \). Consequently, since \( \mathcal{P}(R_A) \) is a cartesian \( \sqrt{\text{qMV}} \) algebra, \( h(a \wedge^A b) = h (a) \wedge^{P(R_A)} h (b) \).

**Corollary 4** If \( A \) is a lattice ordered \( \sqrt{\text{qMV}} \) algebra satisfying the equations in Lemma 3, then: a) \( A \) has a minimum \( \bot = 0 \wedge \sqrt{1} \) and a maximum \( \top = 1 \vee \sqrt{0} \), w.r.t. \( \preceq \); b) for such elements, the following conditions hold:

1. \( \bot \oplus 0 = \sqrt{\top} \bot \oplus 0 = 0 \),
2. \( \top \oplus 0 = \sqrt{\top} \top \oplus 0 = 1 \).

**Proof.** a) follows from \( 0 \wedge \sqrt{1} \leq 0 \leq a \) and \( \sqrt{\mathcal{I}} (0 \wedge \sqrt{1}) \leq \sqrt{\mathcal{I}} \leq \sqrt{\mathcal{I}} a \).

As regards b), we prove condition 1. In fact, \( \bot \oplus 0 = (0 \wedge \sqrt{1}) \oplus 0 = 0 \wedge (\sqrt{1} \oplus 0) = 0 \wedge k = 0 \), while \( \sqrt{\bot} \oplus 0 = \sqrt{\mathcal{I}} (0 \wedge \sqrt{1}) \oplus 0 = k \wedge 0 = 0 \).

Recall from [4] that a cartesian \( \sqrt{\text{qMV}} \) algebra \( A \) is called a pair algebra in case it is isomorphic to \( \mathcal{P}(R_A) \). The next theorem characterises pair algebras within the class of cartesian \( \sqrt{\text{qMV}} \) algebras.

**Theorem 5** If \( A \) is a cartesian \( \sqrt{\text{qMV}} \) algebra, the following conditions are equivalent:

1. \( A \) is a pair algebra;
2. \( A \) is lattice ordered and the equivalent conditions of Lemma 3 are satisfied.

**Proof.** (1 \( \rightarrow \) 2). Let \( \langle a, b \rangle, \langle c, d \rangle \in A \); then \( \langle a, b \rangle \wedge^A \langle c, d \rangle = \langle a \otimes^{R_A} c, b \otimes^{R_A} d \rangle \). It is immediate to see that the conditions of Lemma 3 are satisfied in this case.

(2 \( \rightarrow \) 1). We have to show that for any \( a, b \in \mathcal{R} (A), \langle a, b \rangle \in h (A) \). Since \( A \) is lattice ordered, \( \left( a \wedge \sqrt{\mathcal{T}} \right) \vee \left( \sqrt{b^\prime} \wedge \sqrt{\bot} \right) \in A \), and consequently \( h \left( a \wedge \sqrt{\mathcal{T}} \right) \vee h \left( \sqrt{b^\prime} \wedge \sqrt{\bot} \right) = h \left( \left( a \wedge \sqrt{\mathcal{T}} \right) \vee \left( \sqrt{b^\prime} \wedge \sqrt{\bot} \right) \right) \in h (A) \). Now,
\[
h \left( a \wedge \sqrt{\mathcal{T}} \right) = \langle a, k \rangle \wedge \sqrt{\mathcal{T}} \langle 1, k \rangle \vee \sqrt{\mathcal{T}} \langle 0, k \rangle \]
\[
= \langle a, k \rangle \wedge \sqrt{\mathcal{T}} \langle 1, k \rangle \vee \langle k, 1 \rangle \]
\[
= \langle a, k \rangle \wedge \sqrt{\mathcal{T}} \langle 1, 1 \rangle \]
\[
= \langle a, k \rangle \wedge \langle 1, 0 \rangle = \langle a, 0 \rangle .
\]
Similarly, $h \left( \sqrt{b'} \wedge \sqrt{\bot} \right) = \langle 0, y \rangle$. Thus, $\langle a, 0 \rangle \vee \langle 0, b \rangle = \langle a, b \rangle \in h(A)$. □

3 Pair algebras are equivalent to Abelian PR-groups with strong unit

Daniele Mundici established a well-known equivalence between the categories of MV algebras and Abelian $\ell$-groups with strong unit via an invertible functor (the Gamma functor: [2]). A partial analogue of Mundici’s Gamma functor turns out to be available in the present framework too, its upshot being a categorical equivalence between pair algebras and a special category of Abelian $\ell$-groups with additional operators. A more general categorical equivalence regarding quasi-MV algebras has been investigated in [3]; in spite of this, we believe that the present, less comprehensive result may be of independent interest both in view of the greater mathematical significance of the target category, and in consideration of its purely group-theoretical applications mentioned below.

We begin by introducing a class of Abelian $\ell$-groups endowed with two operators of projection and rotation.

**Definition 6** An Abelian projection-rotation group (for short, PR-group) is an algebra

$$G = (G, \wedge, \vee, +, -, P, R, 0),$$

of type $(2, 2, 2, 1, 1, 0)$, such that:

- $\langle G, \wedge, \vee, +, -, 0 \rangle$ is an Abelian $\ell$-group;
- The following additional equations are satisfied:
  
1. $P - x \approx -Px$
2. $P(x + y) \approx Px + Py$
3. $PPx \approx Px$
4. $P(x \wedge y) \approx Px \wedge Py$
5. $P(x \vee y) \approx Px \vee Py$
6. $R(x + y) \approx Rx + Ry$
7. $RRx \approx -x$
8. $PRPx \approx 0$
9. $PR(x \wedge y) \approx PRx \wedge PRy$
10. $PR(x \vee y) \approx PRx \vee PRy$
11. $x \approx Px - RPRx$

The definition of strong order unit which is usually given for lattice ordered groups can be adapted in such a way as to guarantee an appropriate interaction with the operators $P$ and $R$. 
Definition 7 Let \( G = \langle G, \wedge, \vee, +, -, P, R, 0 \rangle \) be an Abelian PR-group. A positive element \( u \in G^+ \) is said to be a strong order unit of \( G \) just in case it has the following properties:

U1 for all \( a \in G \), there exists a nonnegative integer \( n \) s.t. \( a \leq nu \);

U2 \( Pu \leq u \);

U3 for all \( a \in G \), if \( -u \leq a \leq u \), then \( -u \leq Ra \leq u \).

Definition 6 is motivated by a desire to abstract over some properties of the \( \ell \)-group of the complex numbers, once the latter is endowed with suitable projection and rotation operators as in the following Example.

Example 8 Let \( \langle \mathbb{C}, \wedge, \vee, +, -, 0 \rangle \) be the \( \ell \)-group of the complex numbers, and let \( P \) and \( R \) be, respectively, the projection operator onto the X axis (i.e., the operator which extracts the real part out of any complex number) and the \( \frac{\pi}{2} \) clockwise rotation operator (i.e., multiplication by \(-i\)). In full:

\[
P \langle a, b \rangle = \langle a, 0 \rangle
\]
\[
R \langle a, b \rangle = \langle b, -a \rangle
\]

It is readily checked that \( \langle \mathbb{C}, \wedge, \vee, +, -, P, R, 0 \rangle \) is an Abelian PR-group, and that \((1, 1)\) is a strong order unit for that group.

For good measure, we list hereafter a few more examples of Abelian PR-groups.

Example 9 Let \( \langle B, \wedge, \vee, +, -, 0 \rangle \) be the \( \ell \)-group of all bounded functions of one real variable, with pointwise defined operations, and let \( Pf \) and \( Rf \) be defined in such a way that, for any \( a \in \mathbb{R} \),

\[
Pf(a) = \begin{cases} f(a), & \text{if } a > 0; \\ 0, & \text{if } a \leq 0. \end{cases}
\]
\[
Rf(a) = \begin{cases} f(-a), & \text{if } a > 0; \\ -f(-a), & \text{if } a \leq 0. \end{cases}
\]

It is readily checked that \( \langle B, \wedge, \vee, +, -, P, R, 0 \rangle \) is an Abelian PR-group, and that the constant function \( 1 \) is a strong order unit for that group. Remark that \( PB \) and \( PRB \) are isomorphic copies of the subgroups of functions with respective supports in \( \mathbb{R}^+ \) and \( \mathbb{R}^- \cup \{0\} \).

Example 10 Let \( V \) be a finite-dimensional inner product vector lattice with underlying \( \ell \)-group \( \langle V, \wedge, \vee, +, -, 0 \rangle \), and let \( W \) be a closed subspace of such, isomorphic to \( W^\perp \) via the mapping \( f \). If \( P_W \) is the projection operator associated with \( W \), and \( R_W \) is the operator defined by

\[
R_W(a + b) = (f^{-1}(b) - f(a))
\]

the structure \( \langle V, \wedge, \vee, +, -, P_W, R_W, 0 \rangle \) is an Abelian PR-group.
The arithmetical properties of the next Lemma will prove useful in what follows. Throughout the rest of this paper, we will denote by \( \leq \) the induced lattice order of an Abelian PR-group and we will use without an explicit mention the Abelian \( \ell \)-group properties of these structures.

**Lemma 11** Let \( G = (G, \wedge, \vee, +, - , P, R, 0) \) be an Abelian PR-group with strong order unit \( u \). (i) \( -u \leq P - u = P - u \). (ii) \( P0 = P(0 - 0) = P0 - P0 = 0 \). The proof of (iii) is similar. (iv) Immediate from Axiom 7. (v) By Axiom 4, \( a \land b = a \) implies \( Pa \land Pb = P(a \land b) = Pa \), and similarly for the other claim. (vi) This follows directly from Axioms 2, 1, 6, 4, 5, 9, 10 and from the previous items.

We now proceed to define an analogue of Mundici's Gamma functor for Abelian PR-groups with strong unit. The idea, given a PR-group \( G \), is to cut out the interval in \( G \) in between the strong unit \( u \) and its inverse, turning it into a cartesian \( \sqrt{7} \) qMV algebra whose strong order coincides, when restricted to such an interval, with the original order of \( G \).

**Definition 12** Let \( G = (G, \wedge^G, \vee^G, +^G, -^G, P^G, R^G, 0^G) \) be an Abelian PR-group with strong order unit \( u \). The interval algebra \( \Gamma_i(G, u) \) in \( G \) is the algebra

\[
\left\langle [-u, u], \otimes^{\Gamma_i(G,u)}, \sqrt{\gamma}^{\Gamma_i(G,u)}, \sqrt{\gamma}^{\Gamma_i(G,u)}, \sqrt{\gamma}^{\Gamma_i(G,u)}, \sqrt{\gamma}^{\Gamma_i(G,u)} \right\rangle
\]

of type \( (2,1,0,0,0) \), such that, omitting unnecessary superscripts,

\[
\begin{align*}
a \otimes b &= P(a + b + u) \land Pu; \\
\sqrt{\gamma}^a &= Ra; \\
1 &= Pu; \\
k &= 0^G.
\end{align*}
\]

**Theorem 13** Let \( G = (G, \wedge^G, \vee^G, +^G, -^G, P^G, R^G, 0^G) \) be an Abelian PR-group with strong order unit \( u \). i) The interval algebra \( \Gamma_i(G, u) \) in \( G \) is a cartesian \( \sqrt{7} \) qMV algebra; ii) \( \leq^{\Gamma_i(G,u)} \leq^G [-u,u] \).

**Proof.** i) 0 lies in the interval because \( u \) has to be a positive element. \( Pu \leq u \) by U2, while \( 0 \leq u \) implies \( -u \leq 0 \) by Lemma 11(ii)-(v). By U2 and Lemma 11(iv), \( -u \leq P - u \), while \( -u \leq u \) entails \( P - u \leq Pu \leq u \) by U2 and Lemma 11(v). Obviously \( P(a + b + u) \land Pu \leq Pu \leq u \). On the other hand, \( -u \leq a, b \) implies \( -u - u \leq a + b \), whence \( -u \leq a + b + u \) and \( -u \leq P - u \leq P(a + b + u) \). We already checked that \( -u \leq Pu \), so \( -u \leq P(a + b + u) \land Pu \leq u \). U3 ensures closure w.r.t. rotations, whence \( \Gamma_i(G, u) \) is a total algebra.

Now we are left with the task of checking the \( \sqrt{7} \) qMV axioms one by one. Axiom SQ1 presupposes a check over all the defining equations (A1-A7) of quasi-MV algebras, which we carry out below.
(Ad A1). We have that:

\[
a \oplus (b \oplus c) = P(a + (P(b + c + u) \land Pu) + u) \land Pu = (Pa + P(P(b + c + u) \land Pu) + Pu) \land Pu = (Pa + Pu + P(b + c + u)) \land (Pa + Pu + Pu) \land Pu = (P(a + b + c + u) + Pu) \land Pu
\]

Likewise, \((a \oplus c) \oplus b = (P(a + c + b + u) + Pu) \land Pu\), and our claim follows.

(Ad A2). By Axiom 7, \(\sqrt{7}\sqrt{7}\sqrt{7}a = RRRRa = - - a = a\).

(Ad A3). By Axioms 2 and 3 \(a \oplus Pu = P(a + Pu + u) \land Pu = (Pa + Pu + Pu) \land Pu = Pu\), for \(-Pu \leq Pa\).

(Ad A4). We have that:

\[
(a' \oplus b') \oplus b = P(RR(P(RRa + b + u) \land Pu) + b + u) \land Pu = P((-P(-a + b + u) \land Pu) + b + u) \land Pu = P((P(a - b - u) \lor P - u) + b + u) \land Pu = ((P(a - b - u) \lor P - u) + Pb + Pu) \land Pu = ((P(a - Pb - Pu) \lor -Pu) + Pb + Pu) \land Pu = (Pa \lor Pb) \land Pu = Pa \lor Pb
\]

Likewise, \((b' \oplus a)' \oplus a = Pb \lor Pa\), and our claim follows.

(Ad A5). We have that:

\[
(a \oplus 0)' = RR((Pa + P - u + Pu) \land Pu) = -((Pa + P - u + Pu) \land Pu) = -Pa - Pu = Pa \lor -Pu = Pa \lor Pu = P(RRa + P - u + u) \land Pu = Pa \lor Pu = a' \oplus 0
\]

(Ad A6). We have that:

\[
(a \oplus b) \oplus 0 = P((P(a + b + u) \land Pu) + P - u + u) \land Pu = ((P(a + b + u) \land Pu) - Pu + Pu) \land Pu = P(a + b + u) \land Pu = a \oplus b
\]

(Ad A7). By Axioms 7 and 1, \(\sqrt{7}\sqrt{7}P - u = RRP - u = -P - u = Pu\).

(Ad SQ2). By Axiom 7, \(\sqrt{7}\sqrt{7}0 = RR0 = -0 = 0\).

(Ad SQ3). We have that:

\[
\sqrt{7}(a \oplus b) \oplus 0 = P(R(P(a + b + u) \land Pu) + P - u + u) \land Pu = (PR(P(a + b + u) \land Pu) - Pu + Pu) \land Pu = PRP(a + b + u) \land PRPu \land Pu = 0 \land Pu = 0
\]

Df. 7, Lemma 11(ii)-(v)
Thus, $\Gamma_i(G, u)$ is a $\sqrt{\Gamma}$ qMV algebra. As to the cartesian quasiequation, suppose that $a \oplus 0 = b \oplus 0$, i.e. $Pa \wedge Pu = Pb \wedge Pu$, and that $\sqrt{\Gamma}a \oplus 0 = \sqrt{\Gamma}b \oplus 0$, i.e. $PRa \wedge Pu = PRb \wedge Pu$. Using Axiom 4 it follows that $a = P(a \wedge u) = P(b \wedge u) = Pb$, and similarly, by Axiom 9, $PRa = PRb$, whence $-RPRa = -RPRb$. Then, by Axiom 11,

$$a = Pa - RPRa = Pb - RPRb = b.$$

ii) Left to the reader. \(\blacksquare\)

To invert the preceding functor, the first ingredient we need is the following definition.

**Definition 14** Let $G = (G, \land, \lor, +, -, 0)$ be an Abelian $\ell$-group. Its Gaussian square is the structure

$$G(G) = \left( G^2, \land^G, \lor^G, +^G, -^G, 0^G \right),$$

where:

- $\langle G^2, \land^G, \lor^G, +^G, -^G, 0^G \rangle$ is the direct product $G 	imes G$;
- for $a, b \in G$, $P a^G = \langle a, 0^G \rangle$ and $P b^G = \langle b, -^G a \rangle$.

Mimicking the embedding of the $\ell$-group of the real numbers into the $\ell$-group of the complex numbers, we immediately have that:

**Lemma 15** (i) The Gaussian square $G(G)$ of an Abelian $\ell$-group $G$ is an Abelian PR-group; (ii) if the former has a strong unit $u^G$, the latter has a strong unit $\langle u^G, u^G \rangle$; (iii) $G$ is embeddable into the appropriate reduct of $G(G)$.

**Proof.** (i)-(ii). By properties of direct products, $G \times G$ is an Abelian $\ell$-group with strong unit $\langle u^G, u^G \rangle$. We now check the remaining equations, dropping superscripts for the sake of clarity and denoting by $a, b, c...$ arbitrary elements of $G$.

(Ad Axiom 1). $P(\langle a, b \rangle) = P(\langle -a, b \rangle) = \langle -a, 0 \rangle = -\langle a, 0 \rangle = -P \langle a, b \rangle$.
(Ad Axiom 2). $P(\langle a, b \rangle + \langle c, d \rangle) = P \langle a + c, b + d \rangle = \langle a + c, 0 \rangle$, and $P \langle a, b \rangle + P \langle c, d \rangle = \langle a, 0 \rangle + \langle c, 0 \rangle = \langle a + c, 0 \rangle$.
(Ad Axiom 3). $PP \langle a, b \rangle = P \langle a, 0 \rangle = \langle a, 0 \rangle = P \langle a, b \rangle$.
(Ad Axioms 4 and 5). $P(\langle a, b \rangle \wedge \langle c, d \rangle) = P \langle a \wedge c, b \wedge d \rangle = \langle a \wedge c, 0 \rangle = \langle a, 0 \rangle \wedge \langle c, 0 \rangle = P \langle a, b \rangle \wedge P \langle c, d \rangle$, and similarly for join.
(Ad Axiom 6). $R \langle a, b \rangle + \langle c, d \rangle = R \langle a + c, b + d \rangle = \langle b + d, -a - c \rangle$, and $R \langle a, b \rangle + R \langle c, d \rangle = \langle b, -a \rangle + \langle d, -c \rangle = \langle b + d, -a - c \rangle$.
(Ad Axiom 7). $RR \langle a, b \rangle = R \langle -a, b \rangle = \langle -a, -b \rangle = -\langle a, b \rangle$.
(Ad Axiom 8). $PRP \langle a, b \rangle = PR \langle a, 0 \rangle = P \langle 0, -a \rangle = \langle 0, 0 \rangle$.
(Ad Axioms 9 and 10). $PR \langle a, b \rangle \wedge \langle c, d \rangle = PR \langle a \wedge c, b \wedge d \rangle = P \langle b \wedge d, -a \wedge c \rangle = \langle b \wedge d, 0 \rangle = PR \langle a, b \rangle \wedge PR \langle c, d \rangle$, and similarly for join.
(Ad Axiom 11).

\[ P \langle a, b \rangle - RPR \langle a, b \rangle = \langle a, 0 \rangle - RP \langle b, -a \rangle = \langle a, 0 \rangle - R \langle b, 0 \rangle = \langle a, 0 \rangle - \langle 0, -b \rangle = \langle a, b \rangle \]

(iii) The mapping \( \varphi(a) = \langle a, 0 \rangle \) gives, as expected, the required embedding.

Recall from [2] that, whenever \( M \) is an MV algebra, the structure \( \Xi(M) \) of all equivalence classes\(^1\) of pairs of good sequences of \( M \), endowed with appropriate operations, is an Abelian \( \ell \)-group with strong order unit \([\langle 1 \rangle, \langle 0 \rangle]\). This applies in particular to the MV algebra \( R_Q \) of regular elements of a cartesian \( \sqrt{7} \) qMV algebra. As a consequence of the previous lemma, we have then:

**Corollary 16** Let \( Q = (Q, \oplus^Q, \sqrt{Q}, 0^Q, 1^Q, k^Q) \) be a cartesian \( \sqrt{7} \) qMV algebra. The Gaussian square \( \Xi_i(Q) = \mathcal{G}(\Xi(R_Q)) \) is an Abelian PR-group with strong unit \( \langle \overline{w}, \overline{u} \rangle \).

There is a striking disanalogy between Mundici’s functor and our own: while every MV algebra is isomorphic to an interval in an Abelian lattice ordered group, the best we can do for a cartesian \( \sqrt{7} \) qMV algebra, in general, is to provide an embedding thereof into an interval in an Abelian PR-group with strong unit. More precisely, what we prove is that every pair algebra is actually an interval in an Abelian PR-group, whence our claim follows in view of the embedding theorem for cartesian \( \sqrt{7} \) qMV algebras.

**Theorem 17** Let \( Q = (Q, \oplus^Q, \sqrt{Q}, 0^Q, 1^Q, k^Q) \) be a cartesian \( \sqrt{7} \) qMV algebra. Then \( Q \) is embeddable into \( \Gamma_i(\Xi_i(Q)) \).

**Proof.** As remarked above, it suffices to show that \( \mathcal{P}(R_Q) \) is isomorphic to \( \Gamma_i(\Xi_i(Q)) \). Let

\[ \psi(a, b) = \langle \varphi(a), \varphi(b) \rangle, \]

where \( \varphi(a) = \langle (a), (0) \rangle \) is Mundici’s isomorphism from the MV algebra \( R_Q \) to the interval \( \Gamma(\Xi(R_Q)) \) in Chang’s \( \ell \)-group \( \Xi(R_Q) \). Since \( \varphi \) is an isomorphism, \( \psi \) is clearly both one-one and onto. We are left with the task of checking that it preserves operations. Nullary operations are easily seen to be OK.

\(^1\)Modulo the relation \( \langle (a, b), (c, d) : a + d = b + c \rangle \).
As regards truncated sum,
\[
\psi \left( (a, b) \oplus^\mathcal{P}(\mathcal{R}_Q) (c, d) \right) = \psi \langle a \oplus^\mathcal{R}_Q c, k \rangle = \langle \varphi(a) \oplus^\mathcal{P}(\mathcal{R}_Q) \varphi(c), 0 \rangle = \langle u \wedge^\mathcal{X}(\mathcal{R}_Q) \varphi(a) + \mathcal{X}(\mathcal{R}_Q) \varphi(c), 0 \rangle, 0 \rangle = \langle \varphi(a) + \mathcal{X}(\mathcal{R}_Q) \varphi(c), 0 \rangle \wedge^\mathcal{X}(\mathcal{Q}) \langle u, 0 \rangle \rangle^\mathcal{X}(\mathcal{Q}) \mathcal{P} \langle u, u \rangle = \langle \varphi(a), \varphi(b) \rangle + \mathcal{X}(\mathcal{Q}) \langle \varphi(c), \varphi(d) \rangle + \mathcal{X}(\mathcal{Q}) \langle u, u \rangle = \langle \varphi(a), \varphi(b) \rangle \oplus^\mathcal{P}(\mathcal{Q}) \langle \varphi(c), \varphi(d) \rangle = \psi \langle a, b \rangle \oplus^\mathcal{P}(\mathcal{Q}) \psi \langle c, d \rangle.
\]

As regards square root of the inverse,
\[
\psi \left( \sqrt{\mathcal{P}(\mathcal{R}_Q)} (a, b) \right) = \psi \langle b, a^\mathcal{R}_Q \rangle = \langle \varphi(b), \varphi(a^\mathcal{R}_Q) \rangle = \langle \varphi(b), \varphi(a)^{\mathcal{X}(\mathcal{R}_Q)} \rangle = \langle \varphi(b), -\mathcal{X}(\mathcal{R}_Q) \varphi(a) \rangle = \mathcal{R}^\mathcal{X}(\mathcal{Q})(\varphi(a), \varphi(b)) = \sqrt{\mathcal{P}(\mathcal{Q})} \langle \varphi(a), \varphi(b) \rangle = \sqrt{\mathcal{P}(\mathcal{Q})} \psi \langle a, b \rangle.
\]

We now start working out the details of the categorical equivalence between pair algebras and Abelian PR-groups. As a first step, we must rigorously define the categories we want to prove equivalent.

**Definition 18** By \( \mathcal{P} \) we mean the category whose objects are cartesian \( \sqrt{\mathcal{Q}} \) qMV algebras \( \mathcal{Q} \) s.t. \( \mathcal{Q} \cong^\mathcal{P}(\mathcal{R}_Q) \), and whose arrows are \( \sqrt{\mathcal{Q}} \) qMV algebra homomorphisms. By \( \mathcal{G} \) we mean the category whose objects are Abelian PR-groups with strong order unit, and whose arrows are unital (i.e. unit-preserving) PR-group homomorphisms.

Next, what we must show is that the functors \( \Xi_i \) and \( \Gamma_i \) defined above are mutually inverse. The proof of Theorem 17 implies that every pair algebra \( \mathcal{Q} \) is isomorphic to \( \Gamma_i(\Xi_i(\mathcal{Q})) \). The dual statement holds for Abelian PR-groups with strong unit:
Theorem 19 Let $G = (G, \wedge, \vee, +, -, P, R, 0)$ be an Abelian PR-group with strong order unit $u$. Then $G$ is isomorphic to $\Xi_i(\Gamma_i(G))$.

Proof. Let $\varphi : G \rightarrow \Xi_i(\Gamma_i(G))$ be given by

$$\varphi(a) = \langle Pa, PRa \rangle.$$ 

The function is clearly well-defined. It is one-one; suppose in fact that $\varphi(a) = \varphi(b)$, i.e. $\langle Pa, PRa \rangle = \langle Pb, PRb \rangle$; then $\langle Pa, -RPRa \rangle = \langle Pb, -RPRb \rangle$ and thus $a = Pa - RPRa = Pb - RPRb = b$. We now check that it is onto. Let $(a, b) \in \Xi_i(\Gamma_i(G))$. Then, by the definitions of these functors, there exist $c, d \in G$ s.t. $a = Pc$ and $b = Pd$, since belonging to $R(\Gamma_i(G))$ means, for an element $a$, just $a = P(a + u) \wedge Pu$. Let $f = Pc + RP - d$. Then:

$$\varphi(f) = \varphi(Pc + RP - d) = \langle P(Pc + RP - d), PR(Pc + RP - d) \rangle = \langle PPc + PRP - d, PRPc + PRRP - d \rangle = \langle Pc + 0, 0 + P - P - d \rangle = \langle Pc, Pd \rangle = (a, b).$$

It is easily checked that $\varphi$ preserves 0 and $u$.

(Ad +). Using Axioms 2, 6 and the definitions,

$$\varphi(a +^G b) = \langle P(a +^G b), PR(a +^G b) \rangle = \langle Pa +^G Pb, PRa +^G PRb \rangle = \langle Pa, PRa \rangle + \Xi_i(\Gamma_i(G)) \langle Pb, PRb \rangle = \varphi(a) + \Xi_i(\Gamma_i(G)) \varphi(b).$$

(Ad −). Using Axiom 1, Lemma 11.(iv) and the definitions,

$$\varphi(-^G a) = \langle P -^G a, PR -^G a \rangle = \langle -^G Pa, -^G PRa \rangle = -\Xi_i(\Gamma_i(G)) \langle Pa, PRa \rangle = -\Xi_i(\Gamma_i(G)) \varphi(a).$$

(Ad $\wedge$ and $\vee$). We confine ourselves to $\wedge$. Using Axioms 4 and 9 and the definitions,

$$\varphi(a \wedge^G b) = \langle P(a \wedge^G b), PR(a \wedge^G b) \rangle = \langle Pa \wedge^G Pb, PRa \wedge^G PRb \rangle = \langle Pa, PRa \rangle + \Xi_i(\Gamma_i(G)) \langle Pb, PRb \rangle = \varphi(a) + \Xi_i(\Gamma_i(G)) \varphi(b).$$
(Ad P). Using Axioms 3 and 8 and the definitions,

\[
\varphi(P^G a) = \langle P^G P^G a, P^G R^G P^G a \rangle \\
= \langle P^G a, 0 \rangle \\
= P^\Xi_i(\Gamma_i(G)) \langle P^G a, P^G R^G a \rangle \\
= P^\Xi_i(\Gamma_i(G)) \varphi(a)
\]

(Ad R). Using Axioms 1, 7 and the definitions,

\[
\varphi(R^G a) = \langle P^G R^G a, P^G R^G R^G a \rangle \\
= \langle P^G R^G a, P^G -G a \rangle \\
= \langle P^G R^G a, -G P^G a \rangle \\
= R^\Xi_i(\Gamma_i(G)) \langle P^G a, P^G R^G a \rangle \\
= R^\Xi_i(\Gamma_i(G)) \varphi(a).
\]

■

Finally, we have:

**Theorem 20** The categories $\mathbb{P}$ and $\mathbb{G}$ are equivalent.

**Proof.** As regards objects, our claim follows from Theorem 19 and from the proof of Theorem 17. It remains to take care of morphisms. Thus, let $f : G_1 \to G_2$ be a morphism in $\mathbb{G}$; we define

\[
\Gamma_i(f) = f \big[ [-G_1 u^{G_1}, u^{G_1}] \big].
\]

On the other hand, if $h : P_1 \to P_2$ is a morphism in $\mathbb{P}$, we define $\Xi_i(h)$ in such a way that, if $a, b \in \Xi(R(P_1))$,

\[
\Xi_i(h) \langle a, b \rangle = \langle \Xi(h[R_{P_1}](a), \Xi(h[R_{P_1}](b)) \rangle.
\]

Both $\Gamma_i$ and $\Xi_i$ are well-defined (up to isomorphism of the involved structures). We now check preservation of the various operations. It is easy to see that $\Gamma_i(f)$ preserves the various $\sqrt{7}$ qMV operations; for the sake of definiteness, we show how $\sqrt{7}$ is preserved.

\[
\Gamma_i(f) \left( \sqrt[7]{\Gamma_i(G_1)} a \right) = \Gamma_i(f) \left( R^{G_1} a \right) \\
= f \left( R^{G_1} a \right) \\
= R^{G_2} \left( f(a) \right) \\
= \sqrt[7]{\Gamma_i(G_2)} \left( \Gamma_i(f)(a) \right).
\]
As for $\Xi_i(h)$, we check group sum and the projection operator. As regards the former,

\[
\Xi_i(h) \left( (a, b) + \Xi(P_1) (c, d) \right) = \Xi_i(h) \left( a + \Xi(R_{P_1}) c, b + \Xi(R_{P_1}) d \right)
\]

\[
= \left( \Xi(h[R_{P_1}]) (a + \Xi(R_{P_1}) c), \Xi(h[R_{P_1}]) (b + \Xi(R_{P_1}) d) \right)
\]

\[
= \left( \Xi(h)(a) + \Xi(R_{P_2}) \Xi(h)(c), \Xi(h)(b) + \Xi(R_{P_2}) \Xi(h)(d) \right)
\]

\[
= \left( \Xi(h[R_{P_2}]) (a), \Xi(h[R_{P_2}]) (b) \right) + \Xi(P_2) \left( \Xi(h[R_{P_2}]) (c), \Xi(h[R_{P_2}]) (d) \right)
\]

\[
= \Xi_i(h) (a, b) + \Xi(P_2) \Xi_i(h) (c, d).
\]

As regards the latter,

\[
\Xi_i(h) \left( p^{\Xi_i(P_1)} (a, b) \right) = \Xi_i(h) \left( a, 0^{\Xi(R_{P_1})} \right)
\]

\[
= \left( \Xi(h[R_{P_1}]) (a), \Xi(h[R_{P_1}]) (0^{\Xi(R_{P_1})}) \right)
\]

\[
= \left( \Xi(h)(a), 0^{\Xi(R_{P_2})} \right)
\]

\[
= p^{\Xi_i(P_2)} \left( \Xi(h[R_{P_2}]) (a), \Xi(h[R_{P_2}]) (b) \right)
\]

\[
= p^{\Xi_i(P_2)} \Xi_i(h) ((a, b)).
\]

Next, we have to show that, for $h \in Hom(P_1, P_2)$, $\Gamma_i (\Xi_i(h)) = h$, i.e. that, for $(a, b) \in P \simeq P_1$, $h ((a, b)) = \Gamma_i (\Xi_i(h)) (\psi (a, b))$, where $\psi$ is the map of Theorem 17 - in this case, an isomorphism because of $P \simeq P (R_{P_1})$. Now, by definition

\[
\Gamma_i (\Xi_i(h)) (\psi (a, b)) = \Xi_i(h) \left( \left( -\Xi(R_{P_1}) u, -\Xi(R_{P_1}) u \right), (u, u) \right) \left( \langle \varphi(a), \varphi(b) \rangle \right),
\]

where, once again, $\varphi$ is like in Theorem 17. Since membership in the interval is guaranteed, all we need to do is to apply the definition of $\Xi_i(h)$ to get

\[
\Xi_i(h) \left( \left( -\Xi(R_{P_1}) u, -\Xi(R_{P_1}) u \right), (u, u) \right) \left( \langle \varphi(a), \varphi(b) \rangle \right) = \langle \Xi(h[R_{P_1}]) (\varphi(a)), \Xi(h[R_{P_1}]) (\varphi(b)) \rangle.
\]

By properties of Mundici’s $\Xi$ functor, this amounts to $\langle h[R_{P_1}] (a), h[R_{P_1}] (b) \rangle$, i.e. to $h ((a, b))$, as $a, b \in P_{R_{P_1}}$.

Finally, we must prove that, for $f \in Hom(G_1, G_2)$, $\Xi_i (\Gamma_i(f)) = f$. That much is proved similarly. ■

4 Pair algebras are equivalent to MV algebras

The aim of this section is showing that, from a categorical viewpoint, pair algebras are the same as MV algebras - hence, the same as Abelian $\ell$-groups with a strong unit. Hereafter, we denote by $MV$ the category whose objects are MV algebras and whose morphisms are MV homomorphisms. Also, we denote by $\overline{MV}$ the category whose objects are MV algebras containing an element
Remark 21 Every MV algebra $A$ with a fixed point $k = k'$ is isomorphic to the algebra of all pairs of the form $(a, k)$, for $a \in A$, where the operations are defined as follows:

- $(a, k) \oplus (b, k) = (a \oplus b, k)$,
- $(a, k)' = (a', k)$,
- $0 = (0, k)$,
- $k = (k, k)$,
- $1 = (1, k)$.

In what follows, therefore, we will sometimes identify $A$ with its corresponding pair algebra as in Remark 21.

We know that, if $A$ is an object of $\mathcal{MV}^\mathcal{F}$, the structure $\mathcal{P}(A)$ is a pair algebra. In other words, $\mathcal{P}$ associates to each object of $\mathcal{MV}^\mathcal{F}$ an object of $\mathcal{P}$. Thus, we refer to the pair structure $\mathcal{P}(A) = \langle A^2, \oplus^\mathcal{P}(A), \sqrt{\mathcal{P}(A)}, 0^\mathcal{P}(A), k^\mathcal{P}(A), 1^\mathcal{P}(A) \rangle$ by the label $\Pi(A)$.

Definition 22 Let $A, B \in \text{Ob}(\mathcal{MV}^\mathcal{F})$ and let $f : A \rightarrow B \in \text{Hom}(A, B)$. The morphism $\Pi(f) \in \text{Hom}(\Pi(A), \Pi(B))$ is defined as

$$\Pi(f) (\langle a, b \rangle) = \langle f(a), f(b) \rangle.$$  

Lemma 23 $\Pi$ is a functor from $\mathcal{MV}^\mathcal{F}$ to $\mathcal{P}$

Proof. We already remarked that $\Pi$ associates to each object $A \in \mathcal{MV}^\mathcal{F}$ an object in $\Pi(A) \in \mathcal{P}$. It remains to show that, for $h$ a morphism in $\mathcal{MV}^\mathcal{F}$, $\Pi(h)$ is a morphism in $\mathcal{F}$ such that the following two properties hold: $\Pi(id(A)) = id(\Pi(A))$ for every object $A \in \mathcal{MV}^\mathcal{F}$, and $\Pi(g \circ f) = \Pi(g) \circ \Pi(f)$, for all objects $X, Y, Z$ in $\mathcal{MV}^\mathcal{F}$ and all morphisms $f : X \rightarrow Y$ and $g : Y \rightarrow Z$. First of all we show that $\Pi(h)$ is a morphism in $\mathcal{P}$. Preservation of $\oplus$ is straightforward. As for $\sqrt{\mathcal{I}}$, let $\langle a, b \rangle \in \Pi(X)$; then

$$\sqrt{\Pi(Y)} \Pi(h) (\langle a, b \rangle) = \sqrt{\Pi(Y)} \langle h(a), h(b) \rangle = \langle h(b), h(a) \rangle^Y = \langle h(b), h(a)^X \rangle = \Pi(h) \left( \sqrt{\Pi(X)} \langle a, b \rangle \right).$$
Let us check that $\Pi$ preserves composition of morphisms.

$$\Pi(g \circ f)(\langle a, b \rangle) = \langle g \circ f(a), g \circ f(b) \rangle = \Pi(g)(\langle f(a), f(b) \rangle) = \Pi(g) \circ \Pi(f)(\langle a, b \rangle).$$

\[ \Box \]

**Remark 24** Upon recalling that every object $P \in \mathbb{P}$ is isomorphic to $\mathbb{P}(R_P)$, it is obvious that $\Pi$ is an essentially surjective forgetful functor, i.e. each object $P$ in category of pair algebras is isomorphic to an object of the form $\Pi(M)$, for $M$ in the category $\mathbb{MV}^\star$.

**Lemma 25** $\Pi$ is a full functor.

**Proof.** Our claim is that, if $P, Q \in \text{Ob}(\mathbb{P})$ and $h : P \to Q$ is a $\sqrt{\tau}$ qMV homomorphism, there exists a homomorphism $h^*$ in the category $\mathbb{MV}^\star$ s.t. $\Pi(h^*) = h$. If $\pi_i, i \in \{1, 2\}$ denotes the $i$-th projection function, we can define our $h^*$ as

$$h^*(a) = \pi_1 \circ h(\langle a, k \rangle),$$

It is immediate to check that $h^*$ is a homomorphism in $\mathbb{MV}^\star$. Moreover, suppose that $h(\langle a, b \rangle) = \langle c, d \rangle$. Then $h(\langle b, a' \rangle) = h(\sqrt{\tau} \langle a, b \rangle) = \sqrt{\tau} h(\langle a, b \rangle) = \langle d, c' \rangle$.

$$\Pi(h^*)(\langle a, b \rangle) = \langle h^*(a), h^*(b) \rangle = \langle \pi_1 \circ h(\langle a, k \rangle), \pi_1 \circ h(\langle b, k \rangle) \rangle = \langle \pi_1 \circ h(\langle a, b \rangle \oplus P \langle 0, k \rangle), \pi_1 \circ h((b, a') \oplus P \langle 0, k \rangle) \rangle = \langle \pi_1(h((a, b)) \oplus Q h(\langle 0, k \rangle)), \pi_1(h((b, a')) \oplus Q h(\langle 0, k \rangle)) \rangle = \langle \pi_1((c, d) \oplus Q \langle 0, k \rangle), \pi_1((d, c') \oplus Q \langle 0, k \rangle) \rangle = \langle c, d \rangle = h(\langle a, b \rangle)$$

\[ \Box \]

**Lemma 26** $\Pi$ is a faithful functor.

**Proof.** The fact that for any two objects $M$ and $N$ in $\mathbb{MV}^\star$, the map $\text{Hom}_{\mathbb{MV}^\star}(M, N) \to \text{Hom}_\mathbb{P}(\Pi(M), \Pi(N))$ induced by $\Pi$ is injective is a direct consequence of the pair algebra construction. In fact, let $h, f \in \text{Hom}_{\mathbb{MV}^\star}(M, N)$. Clearly $\Pi(h), \Pi(f) \in \text{Hom}_\mathbb{P}(\Pi(M), \Pi(N))$. If $\Pi(h) = \Pi(f)$, then $\Pi(h)(\mathcal{R}(\Pi(M))) = \Pi(f)(\mathcal{R}(\Pi(M)))$, i.e., for any $\langle a, k \rangle \in \mathcal{R}(\Pi(M))$, $\Pi(h)(\langle a, k \rangle) = \langle h(a), h(k) \rangle = \langle f(a), f(k) \rangle = \Pi(f)(\langle a, k \rangle)$. Now, since $\Pi(h)(\mathcal{R}(\Pi(M))) = h(M)$ and $\Pi(f)(\mathcal{R}(\Pi(M))) = f(M)$ up to isomorphism, then $h(M) = f(M)$.

It follows, by Lemmas 23, 25, 26, and our previous remarks, that

**Corollary 27** The categories of MV algebras and of pair algebras are equivalent.
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