Supermigrative semi-copulas and triangular norms

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Abstract

A semi-copula $S : [0, 1]^2 \rightarrow [0, 1]$ is called supermigrative if it is commutative and satisfies $S(ax, y) \geq S(x, ay)$ for all $a \in [0, 1]$ and for all $x, y \in [0, 1]$ such that $y \leq x$. In this paper, the class of supermigrative semi-copulas is investigated, by focusing, in particular, on the subclass of continuous triangular norms. Some interesting connections with the theory of copulas are also underlined.

1. Introduction

A triangular norm, t-norm for short, is a non-decreasing, commutative and associative binary operation on $[0, 1]$, with neutral element 1. After their introduction in the theory of probabilistic metric spaces [30], t-norms have been largely studied in multi-valued logic [18,19,21], where they extend the Boolean conjunction from the domain $\{0, 1\}^2$ to $[0, 1]^2$, and in the theory of functional equations and inequalities [1,28,29]. Nowadays, a special subclass of t-norms, namely that one formed by associative copulas [26], has been also used for constructing special stochastic models [25,27].

A generalization of a t-norm is the concept of semi-copula, introduced in order to investigate some notions of bivariate ageing for pairs of exchangeable random variables [4]. A semi-copula $S$ is a mapping from $[0, 1]^2$ to $[0, 1]$ that is non-decreasing in each variable and satisfies $S(x, 1) = S(1, x) = x$ for every $x \in [0, 1]$, but it may be neither associative nor commutative [11,15]. As shown in [9], the class of semi-copulas constitutes the lattice completion of the class of t-norms, in the sense that every semi-copula may be represented as the pointwise supremum and infimum of a suitable subset of t-norms. Recently, semi-copulas have been also considered in some problems arising in multi-valued logic and fuzzy measures [7,16,20,22].

In [4, Section 3], Bassan and Spizzichino have introduced several sub-classes of commutative semi-copulas, each of them characterized by the fact that their elements satisfy a given functional inequality. In particular, a property that attracts our interest is given by the following definition.

Definition 1.1. A semi-copula $S : [0, 1]^2 \rightarrow [0, 1]$ is called supermigrative if it is commutative and satisfies

$$S(ax, y) \geq S(x, ay)$$

for all $a \in [0, 1]$ and for all $x, y \in [0, 1]$ such that $y \leq x$. 

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Note that every semi-copula \( S \) satisfies \( S(x, 0) = S(0, x) = 0 \) for every \( x \in [0, 1] \) and, hence, inequality (1.1) can be considered only for \( x \in [0, 1] \) and \( y > 0 \). Moreover, since we are assuming \( S \) commutative, in the previous definition we can also consider \( y < x \). These conventions will be used in the sequel.

The term “supermigrative” is used here for the first time, in order to underline the connection of this property with the concept of migrativity. We recall that a binary operation \( S \) on \( [0, 1] \) is called \( \alpha \)-migrative if, for every \( x, y \) in \( [0, 1] \) and for a given \( \alpha \in (0, 1) \), \( S(\alpha x, y) = \alpha S(x, y) \) (see [13, 17] for more details); moreover, it is called migrative if it is \( \alpha \)-migrative for each \( \alpha \in [0, 1] \) (see [5]).

Following [5], we can intuitively interpret supermigrativity as a property of the aggregation process of two inputs into a single output. Basically, supermigrativity refers to the fact that, when the intensity of one input is reduced to 100 - \( \alpha \) per cent, global evaluation will be greater when the greater input is being reduced.

This paper aims at investigating the class of supermigrative semi-copulas, with emphasis on the most relevant sub-classes of t-norms and copulas. We recall that a semi-copula \( S \) is a copula if it is supermodular, i.e.

\[
S(x_1, y_1) + S(x_2, y_2) \geq S(x_1, y_2) + S(x_2, y_1)
\]

for every \( x_1, x_2, y_1, y_2 \) in \( [0, 1] \) with \( x_1 \leq x_2 \) and \( y_1 \leq y_2 \). Copulas are well-known in a statistical context, due the fact that they can be used for the constructions of suitable statistical models [10, 26]. In fact, given a copula \( C \) and two univariate distribution functions \( F_1 \) and \( F_2 \), a joint distribution function \( H \) can be constructed by means of the formula \( H(x, y) = C(F_1(x), F_2(y)) \).

Here, a key observation is that supermigrative copulas can be used, together with a suitable univariate survival function in order to introduce models that describe the notion of “increasing failure rate” (IFR) for a bivariate random vector of exchangeable lifetimes. Specifically, given a supermigrative copula and a log-concave survival function \( C \), the joint survival function \( H(x, y) = C(G(x), G(y)) \) is Schur-concave and, hence, suitable for describing the IFR property of a pair of lifetimes (we refer the reader to [3, 4] for an in-depth discussion on these results and their motivations). Thus, supermigrative copulas represent an essential tool for introducing parametric stochastic models for the multivariate ageing.

The paper is so organized. In Section 2, we define and study basic properties of supermigrative semi-copulas. Then, the supermigrativity of continuous t-norms is considered (Section 3). Section 4 concludes.

### 2. The class of supermigrative semi-copulas

First of all, we give some examples of supermigrative semi-copulas in the sense of Definition 1.1.

It is easy to show that the minimum t-norm \( T_M(x, y) = \min(x, y) \) and the product t-norm \( T_P(x, y) = xy \) are supermigrative, instead the Łukasiewicz t-norm \( T_L(x, y) = \max\{x + y - 1, 0\} \) is not supermigrative. In fact, \( T_L(\alpha x, y) < T_L(x, \alpha y) \) for \( x = \frac{1}{2}, y = \frac{1}{2} \) and \( \alpha = \frac{1}{2} \).

Note that, if a semi-copula \( S \) is supermigrative, then \( S(x, y) \geq T_P(x, y) \) (it suffices to use inequality (1.1) when \( x = 1 \)). From a statistical point of view this means that, if a copula \( S \) is supermigrative, then it is positively quadrant dependent [26]. In other words, a supermigrative copula can be used for modelling the positive association among two random variables.

Moreover, if \( S_1 \) and \( S_2 \) are supermigrative semi-copulas, then \( \lambda S_1 + (1 - \lambda) S_2 \) is a supermigrative semi-copula for every \( \lambda \in [0, 1] \). In particular, every copula of type \( \lambda T_P + (1 - \lambda) T_M \) (which is not a t-norm) is supermigrative. These considerations can be generalised in the following way.

**Proposition 2.1.** Let \( S_1, S_2 \) be supermigrative semi-copulas. Let \( A : [0, 1]^2 \rightarrow [0, 1] \) be a mapping that is non-decreasing in each variable with \( A(x, x) = x \) for every \( x \in [0, 1] \). Then \( S : [0, 1]^2 \rightarrow [0, 1] \) defined by \( S(x, y) = A(S_1(x, y), S_2(x, y)) \) is a supermigrative semi-copula.

**Proof.** First of all, note that it is easily proved that \( S \) is semi-copula. In fact, it is non-decreasing in each variable, because it is a composition of non-decreasing mappings, and, since \( A(x, x) = x \) for every \( x \in [0, 1] \), it satisfies \( S(x, 1) = x = S(1, x) \) for all \( x \in [0, 1] \). Moreover, for every \( x \in [0, 1] \) and for every \( x, y \in [0, 1], y < x \), we have

\[
S(\alpha x, y) = A(S_1(\alpha x, y), S_2(\alpha x, y)) \geq A(S_1(x, y), S_2(x, y)) = S(x, y),
\]

that is \( S \) is supermigrative. \( \square \)

An equivalent formulation of the supermigrativity is given here.

**Proposition 2.2.** Let \( S \) be a semi-copula. Then \( S \) is supermigrative if, and only if,

\[
S(u, v + kv) \geq S(u + ku, v)
\]

for all \( 0 < v < u < 1 \) and for every \( k \in [0, \frac{1}{u-v}] \).

**Proof.** Suppose that \( S \) is supermigrative. Fixed \( u, v \in [0, 1] \) such that \( v < u \), and \( k \in [0, \frac{1}{u-v}] \), we set \( x = \frac{1}{1+k}, y = \frac{x}{2} \) and \( y = \frac{x}{2} \). Trivially, we get \( 0 < x < 1 \) and \( 0 < y < x \). Further, it is not difficult to see that \( x < 1 \) is equivalent to \( k \leq \frac{1}{u-v} \). Since \( S(\alpha x, y) \geq S(x, y) \), we obtain easily Eq. (2.1).

On the other hand, suppose that \( S \) satisfies (2.1). For all \( x \in [0, 1] \), and for all \( x, y \in [0, 1] \) such that \( y < x \), we set \( u = \alpha x, v = \alpha y \) and \( k = (1 - \alpha)/\alpha \). Then, since (2.1), we obtain that \( S \) is supermigrative. \( \square \)
By using this last result, supermigrativity can be also rewritten in the following equivalent way.

**Corollary 2.3.** Let $S$ be a semi-copula. Then $S$ is supermigrative if, and only if,

$$S(u, \beta v) \geq S(\beta u, v)$$

for all $0 \leq u < v \leq 1$ and for every $\beta \in \mathbb{I}.$

Several copulas are supermigrative, as the following examples show.

**Example 2.4.** Let $S$ be a semi-copula. Then $S$ is supermigrative if, and only if, decreasing on $[0,1].$ Consider the function $C_r$ given, for every $x, y \in [0,1], by$

$$C_r(x,y) = \min(x,y) f(\max(x,y)).$$

It was proved in [6] (see also [8.12.23]) that $C_r$ is a copula, but, in general, not a triangular norm (consider, for example, $f(t) = \sqrt{t}$). Moreover, after some calculations, one can easily check that $C_r$ is supermigrative.

**Example 2.5.** For every $x \in [0,1],$ let us consider the copula

$$C_s(x,y) = xy + axy(1-x)(1-y),$$

which is a member of the Farlie–Gumbel–Morgenstern family [26]. After some calculations, one can easily check that every $C_s$ is supermigrative.

However, there exist supermigrative semi-copulas that are not copulas, as shown here.

**Example 2.6.** Let $S$ be the commutative semi-copula given, for every $x, y \in [0,1], x \geq y,$ by

$$S(x,y) = \begin{cases} \frac{y-1}{x} & y \geq \frac{1}{1-x}, \\ xy, & \text{otherwise}. \end{cases}$$

Then, it is easy to show that $S$ is continuous and supermigrative. However, $S$ is not 1-Lipschitz. In fact, given $x, y_1, y_2 \in [0,1]$ such that $x \geq \max(y_1, y_2), \min(y_1, y_2) \geq \frac{1}{1-x},$ we have that $|S(x,y_1) - S(x,y_2)| = \frac{1}{2}|y_1 - y_2|.$ Thus, $S$ is not a quasi-copula and, hence, not a copula [26].

Now, we will show that the supermigrativity of commutative semi-copulas may be characterized by means of another property that gives a geometrical interpretation of its meaning.

We denote by $S$ any commutative semi-copula and by $(p, p')$ any couple of points of $\mathcal{A} = \{(z, w) \in [0,1]^2 | z \geq w\},$ with $p = (x, y)$ and $p' = (x', y').$ For every $(p, p') \in \mathcal{A},$ we denote by $\sigma_3(p, p')$ the difference among the value of $S$ at the points $p$ and $p'$, respectively, that is

$$\sigma_3(p, p') = S(x,y) - S(x',y').$$

We are interested on checking whether the sign of $\sigma_3$ is positive on the pairs $(p, p') \in \mathcal{A}$ satisfying $x' \geq x.$ These pairs will be called *admissible*.

The following result gives another characterization of the supermigrative property.

**Proposition 2.7.** Let $S$ be a commutative semi-copula. Then the following statements are equivalent:

(a) $S$ is supermigrative,

(b) for every admissible pair $(p, p')$,

$$\sigma_3(p, p') = 0 \implies \sigma_3(p, p') \geq 0.$$

**Proof.** Assume that $S$ is supermigrative. Given an admissible pair $(p = (x, y), p' = (x', y'))$, suppose that $\sigma_3(p, p') = 0.$ The claim is trivial if $x' = 0.$ Otherwise, according to our assumptions, there exists $x \in [0,1]$ such that

$$x = x \cdot x' \quad \text{and} \quad y = x \cdot y.$$

Therefore, we get

$$S(x, y) = S(x \cdot x', y) = S(x', y) = S(x', y'),$$

where the inequality is due to the supermigrativity of $S.$ Thus, $\sigma_3(p, p') \geq 0.$

Conversely, assume that $S$ satisfies (b). Fixed any $x \in [0,1]$ and any $x, y \in [0,1]$ such that $x > y,$ suppose that $x x \geq y$ and let $(p_x, p_y)$ be the admissible pair given by

$$p_x = (x, y) \quad \text{and} \quad p' = (x, y').$$

Since $\sigma_3(p_x, p_y) = 0,$ we have that $\sigma_3(p_x, p_y') \geq 0,$ which is equivalent to (a). Finally, in the case $x x < y,$ by the commutativity of $S,$ we may repeat the above argument only changing the components of $p_x.$
As a consequence of Proposition 2.7, we have the following result.

**Corollary 2.8.** Let $S$ be a commutative semi-copula. Then the following statements are equivalent:

(a) $S$ is supermigrative,
(b) for every admissible pair $(p, p')$, $\sigma_{T_S}(p, p') > 0$ implies $\sigma_{f}(p, p') > 0$.

**Proof.** Assume that $S$ is supermigrative. Given an admissible pair $(p = (x, y), p' = (x', y'))$, suppose that $\sigma_{T_S}(p, p') > 0$. Since $x > 0$, we set $y := \frac{x}{y}$, and $p = (x, y)$. Note that $p \in \Delta$, and the evident fact that $y > y$, force $p \in \Delta$, hence trivially $(p, p')$ is an admissible pair. Then, $\sigma_{T_S}(p, p') > \sigma_{T_S}(p, p') = 0$, which leads to $\sigma_{f}(p, p') > 0$ by Proposition 2.7. Finally, observe that $\sigma_{f}(p, p') > \sigma_{f}(p, p')$ is a straightforward consequence of $y > y$. Conversely, the proof follows directly from Proposition 2.7. ∎

**Remark 2.9.** We would like to emphasize that, if $\sigma_{f}(p, p') > 0$ for some admissible pair $(p, p')$ and a supermigrative semi-copula $S$, then we cannot deduce that $\sigma_{T_S}(p, p') > 0$. In fact, $T_M(\frac{1}{2}, \frac{1}{2}) > T_M(\frac{1}{2}, \frac{1}{2})$, but $T_P(\frac{1}{2}, \frac{1}{2}) < T_P(\frac{1}{2}, \frac{1}{2})$.

As immediate consequence of Proposition 2.7, we have the following relation among supermigrativity and Schur-concavity [1,14]. We recall that a commutative semi-copula $S$ is Schur-concave if $S(x, y) > S(x', y')$ for every $x, y, x', y' \in [0, 1]$ with $x > y, x' > y', x' > x$ and $x + y = x' + y'$. No semi-copula is Schur-concave if $S(x, y) > S(x', y')$ for every $x, y, x', y' \in [0, 1]$ with $x > y, x' > y', x' > x$ and $x + y = x' + y'$.

**Corollary 2.10.** Every supermigrative semi-copula is Schur-concave.

**Proof.** Let $x, y, x', y' \in [0, 1]$ with $x > y, x' > y', x' > x$ and $x + y = x' + y'$. Then $(p = (x, y), p' = (x', y'))$ constitutes an admissible pair. Moreover, it holds:

$$\sigma_{T_S}(p, p') = xy - x'y' = x(y - y') + y'(x - x') = (x - x')(y' - x) > 0,$$

where the last equality follows from the fact that $y - y' = -(x - x')$. From Corollary 2.8 it follows that $S(x, y) > S(x', y')$, which is the desired assertion. ∎

Note that there exist Schur-concave semi-copulas that are not supermigrative (for example, $T_L$).

### 3. Supermigrative continuous triangular norms

By considering the supermigrativity in the class of continuous t-norms, firstly, we focus on the important subset of Archimedean t-norms. We recall that a continuous t-norm $T : [0, 1]^2 \rightarrow [0, 1]$ is Archimedean if, and only if, there exists a continuous and decreasing function $f : [0, 1] \rightarrow [0, \infty], f(1) = 0$, such that

$$T(x, y) = f^{-1}(\min(f(x) + f(y), f(0))) \quad \text{for all } x, y \in [0, 1].$$

In particular, a continuous Archimedean t-norm is strict if $f(0) = +\infty$, otherwise it is nilpotent. For more details about t-norms, see [21] and the references therein.

Note that, among continuous Archimedean t-norms, we can investigate the supermigrative property just for strict t-norms.

**Proposition 3.1.** Let $T$ be a continuous Archimedean t-norm. If $T$ is supermigrative, then $T$ is strict.

**Proof.** Suppose, ab absurdò, that $T$ is a nilpotent and supermigrative t-norm. Let $f$ be an additive generator of $T$ and fix $x \in [0, 1]$. Set $x_0 = f^{-1}(f(0) - f(x))$; immediately, we have that $T(x, x_0) = 0$. Moreover, the strict monotonicity of $f$ forces $x_0 \in [0, 1]$. Because $T$ is supermigrative, we have

$$0 = T(x, x_0) \geq 2x_0 > 0,$$

which is a contradiction. ∎

Now, our purpose is to discuss the conditions under which the additive generator of a strict t-norm $T$ ensures that it is supermigrative. These conditions are given in [4, Proposition 6.1] (see also [2]).

**Proposition 3.2.** Let $T$ be a strict t-norm generated by $f$. Then, $T$ is supermigrative if, and only if, $f^{-1}$ is log-convex.

An important class of copulas is formed by the so-called Archimedean ones. It is known that a copula $C$ is Archimedean if, and only if, it is a continuous Archimedean t-norm whose additive generator $f$ is convex (and so, also $f^{-1}$ is convex). Because log-convexity implies convexity [24], we have the following result.

**Corollary 3.3.** If $T$ is a continuous Archimedean t-norm that is supermigrative, then $T$ is a copula.

Note that there are (strict) Archimedean copulas (and thus t-norms) that are not supermigrative, even if they are greater than $T_P$ in the pointwise order. Consider, for example, [4, Proposition 6.3], and the results by [2].
Now, knowing that every continuous t-norm \( T \) is representable as an ordinal sum of continuous Archimedean t-norms, we wonder whether ordinal sums of supermigrative continuous Archimedean t-norm are supermigrative. To this end, we give the following representation for an ordinal sum of strict t-norms (for the concept of ordinal sums see [21] and the references therein).

**Proposition 3.4.** Let \((T_i)_{i \in \mathcal{I}}\) be a family of strict t-norms such that each \( T_i \) is additively generated by \( f_i \). Let \(( |a_i, b_i| )_{i \in \mathcal{I}}\) be a family of pairwise disjoint open subintervals of \([0, 1]\), where \( \mathcal{I} \) is a finite, or countable, index set. Then, the following function \( T : [0, 1]^2 \to [0, 1] \) is a t-norm:

\[
T(x, y) = \begin{cases} 
    h_i^{-1}(h_i(x) + h_i(y)), & \text{if } (x, y) \in |a_i, b_i|^2; \\
    T_M(x, y), & \text{otherwise,}
\end{cases}
\]

(3.1)

where, for each \( i \in \mathcal{I} \), \( h_i : [a_i, b_i[ \to [0, +\infty] \) is defined by \( h_i = f_i \circ \varphi_i \), where \( \varphi_i : [a_i, b_i[ \to [0, 1] \) is given by \( \varphi_i(z) = \frac{z - a_i}{b_i - a_i} \).

The t-norm \( T \) defined by Eq. (3.1) is called the **ordinal sum** of the summands \( (a_i, b_i, T_i) \), and we shall write \( T = ((a_i, b_i, T_i))_{i \in \mathcal{I}} \). In particular, if any summand \( T_i \) is additively generated by \( f_i \), we write \( T = ((a_i, b_i, T(f_i)))_{i \in \mathcal{I}} \).

**Proposition 3.5.** Let \( T = ((a_i, b_i, T_i(f_i)))_{i \in \mathcal{I}} \) be the ordinal sum given by Eq. (3.1). Suppose that \( T_i \) is supermigrative for all \( i \in \mathcal{I} \). Then \( T \) is supermigrative.

**Proof.** Let \((x, y)\) be in \([0, 1]^2\) such that \( x > y > 0 \), and let \( z \) be in \([0, 1]\). First, suppose that \((x, y)\) and \((x, yz)\) belong to \([a_i, b_i]^2\) for some \( i \in \mathcal{I} \). Then we have to prove that \( T(x, y) \geq T(x, yz) \), where \( z \in \left[ \frac{a_i}{y}, \frac{b_i}{y} \right] \). Since \( T_i \) is a strict t-norm and \( f_i \) is a strict generator, then \( h_i : [a_i, b_i[ \to [0, +\infty] \) admits an inverse and there exists \( y : [0, +\infty] \to \log(b_i) \). Moreover, \( h_i^{-1} \) is log-convex if, and only if, \( y^{-1} \) is convex, which is equivalent to the fact that, for every \( x, y \in [a_i, b_i] \), with \( x > y \), and for every \( z \in \left[ \frac{a_i}{y}, \frac{b_i}{y} \right] \),

\[
\gamma^{-1}(-\log(x) - \log(z)) = \gamma^{-1}(-\log(y) - \log(z)) = \gamma^{-1}(-\log(y)),
\]

that can be rewritten as

\[
h_i(xz) - h_i(x) \leq h_i(yz) - h_i(y).
\]

(3.2)

Eq. (3.2) is also equivalent to

\[
h_i^{-1}(h_i(xz) + h_i(yz)) \geq h_i^{-1}(h_i(x) + h_i(yz)).
\]

(3.3)

Thus, by (3.3), \( T(x, y) \geq T(x, yz) \), if, and only if, \( h_i^{-1} \) is log-convex. Now, explicit calculations show that \( h_i^{-1} \) is log-convex if, and only if, \( (h_i \circ \exp)(t) = f_i\left(\frac{e - \ln a_i}{e - \ln b_i}\right) \) is convex. Now, since \( f_i \circ \exp \) is convex (in fact, \( f_i^{-1} \) is log-convex), it follows that \( h_i \circ \exp \) is convex, and, hence, \( T(x, y) \geq T(x, yz) \).

Now, the thesis follows by considering that the remaining cases are easy consequences of the supermigrativity of \( T_M \) and the continuity of \( T \). \( \square \)

However, note that, if an ordinal sum of continuous Archimedean t-norms is supermigrative, then this does not imply that each summand is supermigrative, as shown by the following example.

**Example 3.6.** Consider the t-norm \( T \) defined by

\[
T(x, y) = \begin{cases} 
2xy, & \text{if } (x, y) \in [0, \frac{1}{2}]^2, \\
\frac{1}{2}, & \text{if } (x, y) \in \left[ \frac{1}{2}, 1 \right]^2 \text{ and } xy \leq \frac{1}{2}, \\
xy, & \text{if } (x, y) \in \left[ \frac{1}{2}, 1 \right]^2 \text{ and } xy > \frac{1}{2}, \\
T_M(x, y), & \text{otherwise.}
\end{cases}
\]

It is only a matter of calculation to see that \( T \) is an ordinal sum of type \( T = (0, \frac{1}{2}, T_P, (\frac{1}{2}, 1, T_1)) \), where \( T_1 \) is the continuous Archimedean t-norm generated by \( f_1(t) = \ln(4) - 2 \ln(1 + t) \). It is quite trivial to check that \( T \) is supermigrative unlike \( T_1 \); that is nilpotent and, hence, cannot be supermigrative.

4. Concluding remarks

Motivated by possible applications in the study of bivariate ageing and in the construction of stochastic models, we have studied the supermigrative property in the class of continuous semi-copulas. Several connections with the theory of copulas have been also underlined.

Similar methods can be used for investigating the concept of submigrative semi-copulas, obtained by reversing inequality (1.1). In this case, note that \( T_n \) is submigrative and, hence, there are Archimedean nilpotent submigrative semi-copulas.

Extensions of supermigrativity to the class of \( n \)-dimensional aggregation functions \((n \geq 3)\) should be also possible (for example, by using Proposition 2.7).
References