Dually Weighted Stirling-type Sequences

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Abstract

We introduce a generalization of the Stirling numbers called the \( U \)-Stirling numbers which are inspired by the symmetric function form of the \( p,q \)-binomial coefficients. A number of properties are derived, including recurrence relations, generating functions, orthogonality relations and convolution formulas. The results are used to derive properties of some special cases and provide alternative proofs of some known results. We also obtain combinatorial interpretations in terms of certain colored permutations and partitions.

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1 Introduction

Let \( \mathbb{N} := \{0,1,2,\ldots\} \). For \( n,k \in \mathbb{N} \) such that \( n \geq k \), we define the \( q \)-analogues of \( n, n! \) and \( \binom{n}{k} \) by
\[
\begin{align*}
[n]_q &= 1 + q + q^2 + \cdots + q^{n-1}, \\
[0]_q &= 0, \\
[n]_q! &= [n]_q[n-1]_q \cdots [2]_q[1]_q, \\
[0]_q! &= 1, \quad \text{and} \\
\binom{n}{k}_q &= \frac{[n]_q!}{[k]_q! [n-k]_q!}. 
\end{align*}
\]
The corresponding \( p,q \)-analogues are defined by \( [n]_{pq} = p^{n-1} + p^{n-2}q + \cdots + q^{n-1} \).
\( p^{n-3}q^2 + \cdots + q^{n-1} \), \([0]_{pq} = 0\), \([n]_{pq}! = [n]_{pq}[n-1]_{pq}\cdots[2]_{pq}[1]_{pq}\), \([0]_{pq}! = 1\), and \([n]_{pq} = [n]_{pq}[n-1]_{pq}\cdots[2]_{pq}[1]_{pq}\). We also denote by \(c(n, k)\) and \(S(n, k)\) the Stirling number of the first kind and Stirling number of the second kind, respectively, and their \(p, q\)-analogues by \(c_{p,q}[n, k]\) and \(S_{p,q}[n, k]\). A number of authors have shown that these sequences have analogous symmetric function forms (see for example [6],[8],[9]). Specifically, if \(e_t(x_0, x_1, \ldots, x_r)\) denote the \(t\)-th elementary symmetric function and \(h_t(x_0, x_1, \ldots, x_r)\) the homogeneous symmetric function on the set \(\{x_0, x_1, \ldots, x_r\}\), then

\[
\begin{align*}
\binom{n}{k} &= e_{n-k}(1, 1, \ldots, 1) = h_{n-k}(1, 1, \ldots, 1) \\
\binom{n}{k}_{p,q} &= h_{n-k}(1, q^1, \ldots, q^k) \\
c(n, k) &= e_{n-k}(0, 1, 2, \ldots, n-1) \\
c_{p,q}[n, k] &= e_{n-k}([0]_{pq}, [1]_{pq}, [2]_{pq}, \ldots, [n-1]_{pq}) \\
S(n, k) &= h_{n-k}(0, 1, 2, \ldots, k) \\
S_{p,q}[n, k] &= h_{n-k}([0]_{pq}, [1]_{pq}, [2]_{pq}, \ldots, [k]_{pq}).
\end{align*}
\]

Using \([n]_{pq} = p^{n-1}[n]_{q/p}\) and hence, \([n]_{pq}! = p^{(n-2)}[n]_{q/p}!\), we can show that \([n]_{pq} = p^{k(n-k)}[n]_{q/p,1}\). Using the symmetric function expression for the \(q\)-binomial coefficients, we have

\[
\binom{n}{k}_{p,q} = h_{n-k}(p^k, p^{k-1}q, p^{k-2}q^2, \ldots, q^k). \tag{1}
\]

Medicis and Leroux introduced a class of generalized Stirling numbers called \(\mathcal{U}\)-Stirling numbers [8], which, in terms of symmetric functions, may be expressed as

\[
\begin{align*}
c^{\mathcal{U}}(n, k) &= e_{n-k}(w_0, w_1, \ldots, w_{n-1}) \\
S^{\mathcal{U}}(n, k) &= h_{n-k}(w_0, w_1, \ldots, w_k)
\end{align*}
\]

where \(w\) is a weight function from \(\mathbb{N}\) to a commutative ring \(K\). A number of other generalizations of the Stirling numbers that arise in different settings are also special cases of the \(\mathcal{U}\)-Stirling numbers. Andrews and Littlejohn’s Legendre Stirling numbers [1], for example, are obtained with \(w_i = i(i+1)\cdots(i+m)\). Miceli’s polystirling numbers [11] are obtained by letting \(w_i = p(i)\), where \(p\) is a polynomial in \(i\) with coefficients from \(\mathbb{N}\). Note that while the \(q\)-binomial coefficients and \(p, q\)-Stirling numbers are special cases of \(\mathcal{U}\)-Stirling numbers, it appears that deriving the \(p, q\)-binomial coefficients is not as straightforward as those of the sequences we just mentioned.

## 2 \(\mathcal{V}\)-Stirling Numbers

In this section, we introduce a dually weighted Stirling-type sequence which generalizes the \(\mathcal{U}\)-Stirling numbers and are inspired by the homogeneous symmetric function form (1) of the \(p, q\)-binomial coefficients.
Definition 2.1. Let \( V = (v, w) \), where \( v \) and \( w \) are weight functions from \( \mathbb{Z} \) to a commutative ring \( K \) and let \( \alpha, \beta \in \mathbb{Z} \). We define the \( V \)-Stirling numbers of the first kind and second kind, respectively, as

\[
\begin{align*}
C_{\alpha,\beta}[n, k] & := e_{n-k}(v_{\alpha+n-1}w_\beta, v_{\alpha+n-2}w_{\beta+1}, \ldots, v_{\alpha}w_{\beta+n-1}) \\
S_{\alpha,\beta}[n, k] & := h_{n-k}(v_{\alpha+k}w_\beta, v_{\alpha+k-1}w_{\beta+1}, \ldots, v_{\alpha}w_{\beta+k}).
\end{align*}
\]

for \( n, k \in \mathbb{N} \). If \( n, k < 0 \), we set \( C_{\alpha,\beta}[n, k], S_{\alpha,\beta}[n, k] := 0 \).

We get the \( U \)-Stirling numbers when \( w \equiv 1 \) and \( \alpha = 0 \). On the other hand, \( V \)-Stirling number of the second kind reduces to the \( p, q \)-binomial coefficients when \( v(i) = p^i \), \( w(i) = q^i \) and \( \alpha = \beta = 0 \). In general, however, both \( V \)-Stirling numbers reduce to some factor of \( [n]_{p,q} \) involving powers of \( p \) and \( q \)

\[
\begin{align*}
S_{\alpha,\beta}[n, k] & = p^{\alpha(n-k)}q^{\beta(n-k)} \binom{n}{k}_{p,q} \\
C_{\alpha,\beta}[n, k] & = p^{\alpha(n-k)} + \binom{n-k}{3} q^{\beta(n-k)} + \binom{n-k}{2} \binom{n}{k}_{p,q}.
\end{align*}
\]

Barry introduced the sequence A080251 in OEIS \([2]\), which we call the \( b \)-Stirling number of the second kind, via the following generating function

\[
\sum_{n \geq k} S_b(n, k)x^n = \frac{x^k}{(1 - T_{k-2,0}x)(1 - T_{k-2,1}x) \cdots (1 - T_{k-2,k-1}x)(1 - T_{k-2,k}x)}
\]

where

\[
T_{k,j} = \left\{ \begin{array}{ll} 
\left\lfloor \frac{j+2}{2} \right\rfloor (k - j + \left\lfloor \frac{j+3}{2} \right\rfloor), & \text{if } k \geq j; \\
0, & \text{if } k < j.
\end{array} \right.
\]

We define the \( b \)-Stirling number of the first kind as

\[
\sum_{k \geq 0} c_b(n, k)x^k = (x + T_{n-3,0})(x + T_{n-3,1}) \cdots (x + T_{n-3,n-2})(x + T_{n-3,n-1})
\]

Observe that if \( j \) is even, \( \left\lfloor \frac{j+2}{2} \right\rfloor + \left( k - 2 - j + \left\lfloor \frac{j+3}{2} \right\rfloor \right) = \frac{j+2}{2} + k - 2j + \frac{j+3}{2} + 1 = k \) and if \( j \) is odd, \( \left\lfloor \frac{j+2}{2} \right\rfloor + \left( k - 2 - j + \left\lfloor \frac{j+3}{2} \right\rfloor \right) = \frac{j+1}{2} + k - 2j + \frac{j+3}{2} = k \). Hence \( \{T_{k-2,j}|0 \leq j \leq k\} \) is the multiset consisting of all products \( ab \) where \( 1 \leq a, b \leq k, a + b = k \). Equivalently, the union of multisets \( \{0, 0\} \cup \{T_{k-2,j}|0 \leq j \leq k\} \) equals the multiset consisting of all products \( ab \) where \( 0 \leq a, b \leq k, a + b = k \). Hence, the denominator on the RHS of (4) becomes

\[
(1 - T_{k-2,0}x)(1 - T_{k-2,1}x) \cdots (1 - T_{k-2,k-1}x)(1 - T_{k-2,k}x) = (1 - v_kw_0)(1 - v_{k-1}w_1) \cdots (1 - v_0w_k)
\]

where \( v(i) = w(i) = i \). Using a similar observation for the RHS of (5), we see that

\[
c_b(n, k) = c_{0,0}[n, k] \quad \text{and} \quad s_b(n, k) = S_{0,0}[n, k] \quad \text{where} \quad V = (i, i).
\]

Let us consider the following definition which is necessary in interpreting \( V \)-Stirling numbers via arrays called \( B \)-tableaux.
Definition 2.2.

(1) Define $T_{\alpha,\beta}[r, s]$ to be the set of $2 \times s$ arrays satisfying the following four conditions:

(a) The entries on the first row are from the set $\{\alpha, \alpha + 1, \ldots, \alpha + r\}$.
(b) The entries on the first row are non-increasing from left to right.
(c) The entries on the second row are from the set $\{\beta, \beta + 1, \ldots, \beta + r\}$.
(d) The sum of the entries in each column is $\alpha + \beta + r$.

The elements of $T_{\alpha,\beta}[r, s]$ and $T_{\alpha,\beta}[r, s]$ are called $B$-tableaux.

(2) Define $T_{\alpha,\beta}[r, s]$ as the subset of $T_{\alpha,\beta}[r, s]$ consisting of $B$-tableaux whose first row entries are distinct.

Remark 2.3.

(1) Conditions (2) and (4) above imply that the entries on the second row of a $B$-tableau are non-decreasing. Note that we allow a $B$-tableau to be a $2 \times 0$ array, in which case we say that the $B$-tableau has zero columns. That is, for any $r \geq 0$, $T_{\alpha,\beta}[r, 0] = T_{\alpha,\beta}[r, 0] = \{\zeta\}$, where $\zeta$ is a $B$-tableau with zero columns.

(2) The sets $T_{\alpha,\beta}[r, s]$ and $T_{\alpha,\beta}[r, s]$ are both empty if $r$ or $s$ is negative.

(3) A $B$-tableau with $s$ columns needs at least $s$ possible choices of entries for the first row so that these entries are distinct. For example, we can construct a $B$-tableau with 4 columns and whose first row entries are distinct from $\{0, 1, 2, 3\}$ but not from $\{0, 1, 2\}$. This shows that if $r < s - 1$, then the set of possible entries in a $B$-tableau is not enough to ensure that the entries on the first row are distinct. If this is the case, then $T_{\alpha,\beta}[r, s]$ is empty.

Definition 2.4. Two $B$-tableaux are compatible if the sum of the constant column sum of both $B$-tableaux are equal. Given two compatible $B$-tableaux $\phi$ and $\phi'$, denote by $\phi \boxtimes \phi'$ the $B$-tableau obtained by juxtaposing $\phi$ and $\phi'$, and then rearranging the columns so that entries on the first row are non-increasing from left to right. Let $\mathcal{T}$ and $\mathcal{T}'$ be two sets of $B$-tableaux such that every $B$-tableau in $\mathcal{T}$ is compatible with every $B$-tableau in $\mathcal{T}'$. Define $\mathcal{T} \boxtimes \mathcal{T}'$ to be the multiset

$$\mathcal{T} \boxtimes \mathcal{T}' = \{\phi \boxtimes \phi' : \phi \in \mathcal{T}, \phi' \in \mathcal{T}'\}$$

with $\emptyset \boxtimes \mathcal{T} = \mathcal{T} \boxtimes \emptyset = \emptyset$.

Remark 2.5. It follows from Definition 2.4 that for any two compatible $B$-tableaux $\phi$ and $\phi'$, $\phi \boxtimes \phi' = \phi' \boxtimes \phi$. Hence for any two sets of compatible $B$-tableaux $\mathcal{T}$ and $\mathcal{T}'$, we have $\mathcal{T} \boxtimes \mathcal{T}' = \mathcal{T}' \boxtimes \mathcal{T}$. Also note that if $\phi = \zeta$ has zero columns, $\phi \boxtimes \phi' = \phi'$.

Definition 2.6. Let $\rho$ be a $B$-tableau with $s$ columns. For a pair $\mathcal{V} = (v, w)$ of weight functions, the weight of $\rho$ with respect to $\mathcal{V}$ (or the $\mathcal{V}$-weight of $\rho$) is defined by

$$\mathcal{V}_{\text{wt}}(\rho) = \prod_{j=0}^{s} v(\rho_{1,j})w(\rho_{2,j})$$
where \( \rho_{1,j} \) (respectively, \( \rho_{2,j} \)) is the \( j \)th column entry of the first row (respectively, second row) of \( \rho \).

**Remark 2.7.** For any two compatible \( B \)-tableaux \( \phi \) and \( \phi' \), \( V_{\text{wt}}(\phi)V_{\text{wt}}(\phi') = V_{\text{wt}}(\phi \boxtimes \phi') \). If a \( B \)-tableau \( \zeta \) has zero columns, then \( V_{\text{wt}}(\zeta) = 1 \).

The following relation will be useful when we derive orthogonality relations and convolutions formulas. Given two sets of compatible \( B \)-tableaux \( T_1 \) and \( T_2 \), we have

\[
\left( \sum_{\phi \in T_1} V_{\text{wt}}(\phi) \right) \left( \sum_{\psi \in T_2} V_{\text{wt}}(\psi) \right) = \left( \sum_{\phi \in T_1} \left( \sum_{\psi \in T_2} V_{\text{wt}}(\phi)V_{\text{wt}}(\psi) \right) \right) \nonumber \\
= \left( \sum_{\phi \in T_1} \left( \sum_{\psi \in T_2} V_{\text{wt}}(\phi)V_{\text{wt}}(\psi) \right) \right) \nonumber \\
= \sum_{\phi \in T_1, \psi \in T_2} V_{\text{wt}}(\phi)V_{\text{wt}}(\psi) \nonumber \\
= \sum_{\rho \in T_1 \boxtimes T_2} V_{\text{wt}}(\rho) \nonumber 
\]

### 3 Recurrence Relations

The \( V \)-Stirling numbers have the following initial values: \( c^V_{\alpha,\beta}[0,k] = c^V_{\alpha,\beta}[0,k] = \delta_{0,k} \), \( S^V_{\alpha,\beta}[n,0] = (v_\alpha w_\beta)^n \) and \( c^V_{\alpha,\beta}[n,0] = v_{\alpha+n-1}w_\beta v_{\alpha+n-2}w_\beta+1 \cdots v_\alpha w_\beta+n-1 \). Using these initial values together with recurrence relations we can compute any value of a \( V \)-Stirling number.

**Theorem 3.1.** The \( V \)-Stirling numbers satisfy the following recurrence relations

1. **Triangular Recurrence Relations**

   \[ c^V_{\alpha,\beta}[n,k] = c^V_{\alpha,\beta+1}[n-1,k-1] + v_{\alpha+n-1}w_\beta c^V_{\alpha,\beta+1}[n-1,k] \]  
   \[ S^V_{\alpha,\beta}[n,k] = S^V_{\alpha,\beta+1}[n-1,k-1] + v_{\alpha+k}w_\beta S^V_{\alpha,\beta+1}[n-1,k] \]  

2. **Vertical Recurrence Relations**

   \[ c^V_{\alpha,\beta}[n+1,k+1] = \sum_{j=k}^{n} v_{\alpha+n}w_\beta v_{\alpha+n-1}w_\beta+1 \cdots v_{\alpha+j+1}w_\beta+n-j-1 c^V_{\alpha,\beta+n-j+1}[j,k] \]  
   \[ S^V_{\alpha,\beta}[n+1,k+1] = \sum_{j=k}^{n} (v_{\alpha+k+1}w_\beta)^{n-j} S^V_{\alpha,\beta+1}[j,k] \]
3. Horizontal Recurrence Relations

\[ c_{\alpha,\beta}^V[n, k] = \sum_{j=k}^{n} (-1)^{j-k} (v_{\alpha+n}w_{\beta-1})^{n-j} c_{\alpha,\beta-1}^V[n+1, j+1] \]  
\[ S_{\alpha,\beta}^V[n, k] = \sum_{j=0}^{n-k} (-1)^j v_{\alpha+k+1}w_{\beta-1}v_{\alpha+k+2}w_{\beta-2} \cdots v_{\alpha+k+j}w_{\beta-j} S_{\alpha,\beta-j-1}^V[n+1, k+j+1] \]  
\[ \text{Proof.} \text{ The triangular and vertical recurrence relations may be derived by partitioning the corresponding sets of tableaux. For example, (6) follows from the fact that} \]
\[ Td_{\alpha,\beta}[n-1, n-k] = Td_{\alpha,\beta+1}[n-2, n-k] \cup \{ \rho \} \times Td_{\alpha,\beta+1}[n-2, n-k-1] \]  
where \( \rho = \left[ \begin{array}{c} \alpha + n - 1 \\ \beta \end{array} \right] \) since RHS (resp. LHS) of (6) is simply the sum of the \( V \)-weight of the tableaux in RHS (resp. LHS) of (12). In this case, we partitioned \( Td_{\alpha,\beta}[n-1, n-k] \) into tableaux whose first column is \( \rho \), i.e., the first column entry is \( \alpha + n - 1 \) and tableaux whose first column entry is less than \( \alpha + n - 1 \). To prove (10), we partition every tableau in the multiset \( \bigcup_{j=k}^{n} \{ \rho_j \} \times Td_{\alpha,\beta-1}[n, n-j] \) into pairs consisting of identical tableaux and assign the weight \( (-1)^j V_{wt}(\phi) \) to every tableau \( \phi \in \{ \rho_j \} \times Td_{\alpha,\beta-1}[n, n-j] \). Again, (11) is similarly proved.

Since the \( V \)-Stirling numbers are symmetric with respect to \((v, \alpha)\) and \((w, \beta)\), every identity is equivalent to another identity which is obtained by transferring the increments of \( \alpha \) to \( \beta \) and vice versa. We no longer state these alternative identities in the results that follow. As an example, we have the other horizontal recurrence relation,

\[ c_{\alpha,\beta}^V[n, k] = \sum_{j=k}^{n} (-1)^{j-k} (v_{\alpha+n}w_{\beta})^{n-j} c_{\alpha-1,\beta}^V[n+1, j+1] . \]  
If \( V = (i, 1) \) and \( \alpha = \beta = 0 \), then this identity reduces to

\[ c(n, k) = \sum_{j=k}^{n} (-1)^{j-k} n^{n-j} c(n+1, j+1) . \]  
Let \( c_r(n, k) \) and \( S_r(n, k) \) denote the non-central Stirling numbers with parameter \( r \) [7, 10]. They are obtained by letting \( V = (i, 1) \) and \( \alpha = r \). Equation (13) then reduces to the equivalent identity

\[ c(n, k) = \sum_{j=k}^{n} (-1)^{j-k} (-1)^{n-j} c_{-1}(n+1, j+1) . \]  
Using Carlit’z identity (Theorem 2.2, [8]), we may write \( c_r(n, k) \) as

\[ c_r(n, k) = \sum_{t=k}^{n} \binom{t}{k} (-r)^{t-k} c(n, t) \]  
6
from which we get
\[
c(n, k) = \sum_{j,t} (-1)^{n-k+t-1} \binom{t}{j+1} t^{t-j-1} c(n+1, t).
\]

In the case of \(p, q\)-binomial coefficients, the identities in Theorem 3.1 reduce to the following:
\[
\begin{align*}
\binom{n}{k}_{p,q} &= q^{n-k} \binom{n-1}{k-1}_{p,q} + p^k \binom{n-1}{k}_{p,q} \\
\binom{n+1}{k+1}_{p,q} &= \sum_{j=k}^n p^{(k+1)(n-j)} q^{j-k} \binom{j}{k}_{p,q} \\
\binom{n+1}{k+1}_{p,q} &= \sum_{j=0}^{n-k} (-1)^j p^{(j+1)(n-1-k)} q^{j+k+1} \binom{n+1}{k+j+1}_{p,q},
\end{align*}
\]

These identities are also found Theorems 1 and 2 in [3].

The recurrence relations of the \(V\)-Stirling numbers are named analogously after the recurrence relations of the classical Stirling numbers that they generalize. For instance, consider the array of Stirling numbers of the first kind \(c(n, k)\) where the rows correspond to \(n\) and the columns correspond to \(k\). The triangular recurrence relations \(c(n, k)\) are so named because they allow us to compute \(c(n, k)\) from the values of \(c(n-1, n-k)\) and \(c(n-1, k)\) which are both on the row above \(c(n, k)\). However, in the case of the \(V\)-Stirling numbers, the recurrence relations are no longer “geometrically” triangular. Since these numbers involve four parameters, \(\alpha, \beta, n,\) and \(k,\) a table of \(V\)-Stirling numbers can be thought of as a four-dimensional array. If we fix one of these parameters, say \(\alpha\), we can create a sequence of arrays for different values of \(\beta\). To compute, for example, \(c^\gamma_{\alpha, \beta}[n, k]\), using the triangular recurrence relation (6), one needs the values of \(c^\gamma_{\alpha, \beta+1}[n-1, k-1]\) and \(c^\gamma_{\alpha, \beta+1}[n-1, k]\). A different situation occurs in the case where \(S^\gamma_{\alpha, \beta}[n, k]\) may be written in terms of \(S^\gamma_{\gamma, \gamma}[n, k]\) for some integer \(\gamma\). For example, if \(V = (p^\gamma, q^\gamma)\), then \(S^\gamma_{\alpha, \beta}[n, k] = p^{\alpha(n-k)} q^{\beta(n-k)} \binom{n}{k}_{p,q}\) may be written in terms of \(S^\gamma_{\alpha, \beta}[n, k] = \binom{n}{k}_{p,q}\).

4 Generating Functions

The generating functions are known to be useful in obtaining combinatorial interpretations of some combinatorial objects and their properties.

We will need the following notation. For \(n > 0,\)
\[
\begin{align*}
[x]^{(n)}_{\alpha, \beta} &:= (x - v_{\alpha+n-1} w_{\beta-n+1})(x - v_{\alpha+n-2} w_{\beta-n+2}) \cdots (x - v_{\alpha} w_{\beta}) ,
\end{align*}
\]
and
\[
\begin{align*}
[x]^{[n]}_{\alpha, \beta} &:= (x - v_{\alpha+n-1} w_{\beta+1})(x - v_{\alpha+n-2} w_{\beta+2}) \cdots (x - v_{\alpha} w_{\beta+n}) ,
\end{align*}
\]
and
\[
\begin{align*}
[x]^{(0)}_{\alpha, \beta} &:= 1,
\end{align*}
\]
and
\[
\begin{align*}
[x]^{[0]}_{\alpha, \beta} &:= 1
\end{align*}
\]
Theorem 4.1. The $V$-Stirling numbers satisfy the following generating functions

\begin{equation}
\sum_{k=0}^{n} c_{\alpha,\beta}^{V}[n,k]x^{k} = (x + v_{\alpha+n-1}w_{\beta})(x + v_{\alpha+n-2}w_{\beta+1}) \cdots (x + v_{\alpha}w_{\beta+n-1}) \tag{15}
\end{equation}

\begin{equation}
\sum_{n\geq k} S_{\alpha,\beta}^{V}[n,k]x^{n} = \frac{x^{k}}{(1 - xv_{\alpha+k}w_{\beta})(1 - xv_{\alpha+k-1}w_{\beta+1}) \cdots (1 - xv_{\alpha}w_{\beta+k})} \tag{16}
\end{equation}

\begin{equation}
x^{n} = \sum_{k=0}^{n} S_{\alpha,\beta-k}^{V}[n,k][x]_{\alpha,\beta}^{(k)} \tag{17}
\end{equation}

\begin{equation}
x^{n} = \sum_{k=0}^{n} S_{\alpha,\beta}[n,k][x]_{\alpha,\beta}^{[k]} \tag{18}
\end{equation}

Proof. The generating functions (15) and (16) follow directly from the definition of the $V$-Stirling numbers. On the other hand, we can prove (17) by induction. Assume that (17) holds for some $n \in \mathbb{N}$ such that $n > 0$. Using the triangular recurrence (7),

\begin{equation}
\sum_{k=0}^{n+1} S_{\alpha,\beta-k}[n+1,k][x]_{\alpha,\beta}^{(k)}
\end{equation}

\begin{equation}
= \sum_{k=0}^{n} S_{\alpha,\beta-(k+1)+1}[n,k][x]_{\alpha,\beta}^{(k+1)} + \sum_{k=0}^{n} v_{\alpha+k}w_{\beta-k}S_{\alpha,\beta-k}[n,k][x]_{\alpha,\beta}^{(k)}
\end{equation}

\begin{equation}
= \sum_{k=0}^{n} S_{\alpha,\beta-k}[n,k][x - v_{\alpha+k}w_{\beta-k}][x]_{\alpha,\beta}^{(k)} + \sum_{k=0}^{n} v_{\alpha+k}w_{\beta-k}S_{\alpha,\beta-k}[n,k][x]_{\alpha,\beta}^{(k)}
\end{equation}

\begin{equation}
= \sum_{k=0}^{n} S_{\alpha,\beta-k}[n,k] \left( x[x]_{\alpha,\beta}^{(k)} \right)
\end{equation}

\begin{equation}
= x^{n+1} ss
\end{equation}

The other identity (18) is proved similarly. \hfill \Box

Let $V = (p^{1}, q^{1})$, $\alpha = \beta = 0$. Then, using (15), we obtain

\begin{equation}
\sum_{k=0}^{n} (pq)^{(n-k)} \binom{n}{k}_{p,q} x^{k} = (x + p^{n-1})(x + p^{n-2}q) \cdots (x + q^{n-1})
\end{equation}

By replacing $p$ with $p/q$ and $q$ by 1, and multiplying both sides by $q(n)_{2}$, we get

\begin{equation}
\sum_{k=0}^{n} p^{(n-k)} q^{-((n-k)+(k)+k(n-k))} \binom{n}{k}_{p/q,1} x^{k} = (q^{n-1}x + p^{n-1})(q^{n-1}x + p^{n-2}) \cdots (x + 1)
\end{equation}

\begin{equation}
\sum_{k=0}^{n} p^{(n-k)} q^{(k)} \binom{n}{k}_{p,q} x^{k} = (p^{n-1} + xq^{n-1})(p^{n-2} + xq^{n-2}) \cdots (1 + x)
\end{equation}

which is exactly Theorem 3 in [3].
On the other hand, (16) reduces to

\[
\sum_{n \geq k} \binom{n}{k}_{p,q} x^n = \frac{x^k}{(1 - xp^k)(1 - xq^k)}
\]

\[
(-x)^n = \sum_{k=0}^{n} q^{k(k-n)} \frac{n}{k}_{p,q} \left( -x - \left( \frac{p}{q} \right)^{k-1} \right) \left( -x - \left( \frac{p}{q} \right)^{k-2} \right) \cdots \left( -x - \left( \frac{p}{q} \right)^0 \right)
\]

\[
= \sum_{k=0}^{n} (-1)^k q^{k(k-n)} \binom{n}{k}_{p,q} \left( xq^{k-1} + p^{k-1} \right) \left( xq^{k-2} + p^{k-2} \right) \cdots \left( xq^0 + p^0 \right)
\]

from which we get

\[
q^n x^n = \sum_{k=0}^{n} (-1)^{n-k} q^{\frac{n-k}{2}} \binom{n}{k}_{p,q} \left( xq^{k-1} + p^{k-1} \right) \left( xq^{k-2} + p^{k-2} \right) \cdots (x+1)
\]

which is found in Theorem 5 of [3].

5 Orthogonality and Inverse Relations

The binomial coefficients and Stirling numbers satisfy the orthogonality relations

\[
\sum_{k=m}^{n} (-1)^{n-k} \binom{n}{k} \binom{k}{m} = \delta_{n,m}
\]

\[
\sum_{k=m}^{n} (-1)^{n-k} c(n,k) S(k,m) = \delta_{n,m}
\]

The corresponding relations for the \( V \)-Stirling numbers are given in the next theorem.

**Theorem 5.1.** Let \( m \leq n \). Then the following orthogonality relations hold

\[
\sum_{k=m}^{n} (-1)^{n-k} c_{\alpha,\beta+n}^V[n, k] S_{\alpha,\beta+n}^V[k, m] = \delta_{n,m}
\]  \hspace{1cm} (19)

\[
\sum_{k=m}^{n} S_{\alpha,\beta}^V[n, k] (-1)^{k-m} c_{\alpha,\beta+1}^V[k, m] = \delta_{n,m}
\]  \hspace{1cm} (20)

Equivalently, the following matrices are inverses of each other

\[
\left( (-1)^{n-k} c_{\alpha,\beta+n}^V[n, k] \right), \left( S_{\alpha,\beta+n}^V[n, k] \right)
\]  \hspace{1cm} (21)

\[
\left( (-1)^{n-k} c_{\alpha,\beta+1}^V[n, k] \right), \left( S_{\alpha,\beta}^V[n, k] \right)
\]  \hspace{1cm} (22)
Proof. The generating functions (17) and (15) (where in the latter we replaced $x$ with $-x$ and $\beta$ with $\beta + n - 1$) implies that $\langle (-1)^{n-k}c_{\alpha,\beta-n+1}[n,k]\rangle$ and $S_{\alpha,\beta-k}[n,k]$ are change of basis matrices between $\{1, x, x^2, \ldots\}$ and $\{1, [x]^{(1)}_{\alpha,\beta}, [x]^{(2)}_{\alpha,\beta}, \ldots\}$. The other generating function (18) together with (15) implies the other pair of inverse matrices. 

When $V = (p^i, q^i)$, $\alpha = \beta = 0$, the orthogonality relation (20) above becomes

$$
\sum_{k=m}^{n} (-1)^{k-m} p^{(k-m)/2} q^{(k-m)/2} \binom{n}{k} \binom{k}{m} = \delta_{n,m}
$$

and

$$
\sum_{k=m}^{n} (-1)^{k-m} p^{(k-m+1)/2} q^{(k-m+1)/2} \binom{n}{k} \binom{k}{m} = \delta_{n,m}
$$

By multiplying both sides by $q^{(k-m+1)/2 - (n-k)/2}$, we obtain

$$
\sum_{k=m}^{n} (-1)^{k-m} p^{(k-m)/2} q^{(k-m+1)/2} \binom{n}{k} \binom{k}{m} = \delta_{n,m}
$$

since for $m = n$, $q^{(n-m+1)/2 - (n-k)/2} = 1$. For an alternative derivation, see Theorem 4 of [3].

Given any two sequences $(a_n)_{n \geq 0}$ and $(b_n)_{n \geq 0}$ from a commutative ring $K$ with unity, the orthogonality relations give us the following inverse relations

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} c_{\alpha,\beta-n+1}[n,k]b_k \iff b_n = \sum_{k=0}^{n} S_{\alpha,\beta-k}[n,k]a_k$$

and

$$a_n = \sum_{k=0}^{n} S_{\alpha,\beta}[n,k]b_k \iff b_n = \sum_{k=0}^{n} (-1)^{n-k} c_{\alpha,\beta+1}[n,k]a_k$$

Variants of these relations also exist. For instance,

$$\sum_{k=m}^{n} (-1)^{n-k} c_{\alpha,\beta+m+1}[n+\gamma,k+\gamma]S_{\alpha,\beta+n}[k+\gamma,m+\gamma] = \delta_{n,m}$$

Let $r \geq n$. By taking the transpose of the matrices in (21),

$$a_n = \sum_{k=0}^{r} (-1)^{n-k} c_{\alpha,\beta-k+1}[k,n]b_k \iff b_n = \sum_{k=0}^{r} S_{\alpha,\beta-k}[k,n]a_k$$

Some of Riordan’s [12] method in obtaining alternative forms of the binomial may also be applied with $V$-Stirling numbers.

6 Combinatorial Interpretations

Let $\mathbb{N}[i]$ be the set of polynomials in $i$ with coefficients from $\mathbb{N}$. Throughout this section, we will assume that $V = (v, w)$, where $v, w \in \mathbb{N}[i]$, unless stated otherwise. We will derive here
a general method for obtaining combinatorial interpretations for the \( V \)-Stirling numbers in terms of certain colored partitions and permutations. Another approach can be found in [8] which may also be applied to some of the particular cases considered here.

Consider a \( B \)-tableau \( \phi \) whose \( j \)th first row and second row entries are \( \phi_{1,j} \) and \( \phi_{2,j} \), respectively. A 0,1-tableau of shape \( \phi \) is an array of top and left-justified boxes such that the length of the \( j \)th column is \( \phi_{1,j} \) and exactly one box in each column is filled with 1 while the rest are filled with 0’s. If \( V = (i, 1) \), then number of 0,1-tableaux of shape \( \phi \) is \( V_{\text{wt}}(\phi) \). Note that we require each column to be of positive length, so that if \( \phi \) contains a first row entry equal to 0, then the number of 0,1-tableaux of shape \( \phi \) is 0.

We now introduce a generalization of the 0,1-tableau which we call the 0,1\( _V \)-tableau, where \( V = (v, w) \), \( v, w \in \mathbb{N}[i] \). Specifically, for a \( B \)-tableau \( \phi \in T_{\alpha,\beta}[r, s] \), we define a 0,1\( _V \)-tableau of shape \( \phi \) as a rectangular array of boxes partitioned by a lattice path consisting of vertical (up to down) and horizontal (right to left) steps from the upper right hand to the lower left hand corner, such that

1. the length (i.e, the number of boxes) on the \( j \)th column above (respectively, below) the lattice path is \( \phi_{1,j} + 1 \) (respectively, \( \phi_{2,j} + 1 \)),
2. exactly two boxes in each column is filled with 1, one for the column of boxes below the lattice path and one for those below the lattice path. The rest of the boxes are filled with zeroes, and
3. if 1 is placed on the first (respectively, last) row, then 1 is assigned \( v(0) \) (respectively \( w(0) \)) colors while if 1 is placed on other rows, then it is assigned \( (v(i) - v(0))/i \) colors if it is above the lattice path, and \( (w(i) - w(0))/i \) colors if it is below the lattice path. If \( v(0) = 0 \) or \( w(0) = 0 \), then we assume that we cannot place a 1 on the first and last rows, respectively.

One observes that the number of 0,1\( _V \)-tableaux of shape \( \phi \) is \( V_{\text{wt}}(\phi) \). When \( v(i) = i \), we can ignore the first row of boxes since they cannot contain a 1 and the boxes below the path since only the last row can contain a 1. Hence we get the usual 0,1-tableau.

We now denote by \( T_{\alpha,\beta}^{0,1}_V[k, n - k] \) the set of 0,1\( _V \)-tableaux of shape \( \phi \) for some \( \phi \in T_{\alpha,\beta}[k, n - k] \). The set \( T^{0,1}_V[n - 1, n - k] \) is defined analogously. Let \( [n]_0 = [n] \cup \{0\} \). We consider first 0,1\( _V \)-tableaux where \( V = (v, 1) \); hence we can ignore the boxes below the lattice path and construct bijections involving the placement of 1 on the boxes above the lattice path. We define \( \text{Part}(n, k; v) \) to be the set of partitions of \( [n]_0 \) into \( (k + 1) \) blocks \( B_0, B_1, \ldots, B_k \) such that all block minima are not colored and if \( a_1 < a_2 < a_3 < \cdots < a_{n-k} \) are the elements of \( [n]_0 \) which are not subset minima, then \( a_j \) takes \( v(0) \) colors if \( a_j \in B_0 \) and \( (v(a_j) - v(0))/j \) colors if otherwise. We define the set of colored permutations \( \text{Perm}(n, k; v) \) similarly, except that if \( a_1 < a_2 < a_3 < \cdots < a_{n-k} \) are the elements of \( [n]_0 \) which are not cycle minima, then \( a_j \) takes \( (v(a_j) - v(0))/j \) colors if \( a_j \) is not in the zeroth cycle (instead of \( (v(a_j) - v(0))/j \)).

If \( V = (v, 1) \), \( v \in \mathbb{N}[i] \), then there exists a bijection between the set \( T_{\alpha,\beta}^{0,1}_V[k, n - k] \) and \( \text{Part}(n + \alpha, k + \alpha; v) \) and between \( T_{\alpha,\beta}^{0,1}_V[n - k, n - k] \) and \( \text{Perm}(n + \alpha, k + \alpha; v) \), with the added restriction that 1, 2, \ldots, \alpha are in distinct subsets or cycles. For a 0,1\( _V \)-tableau \( \phi_{0,1V} \) of shape \( \phi \in T_{\alpha,\beta}[k, n - k] \), we obtain the corresponding partition as follows. Label the
steps (except the last one, which we will ignore) by 0, 1, 2, . . . , α + β + k + 1, i.e., we speak of the zeroth, first, second, etc. steps on the lattice path. If the jth step on the lattice path is a vertical step, then j is a block minimum of the associated partition. On the other hand, if the jth step is a horizontal step which is below a column with a 1 on the rth box from the top, then we assign j to the rth block, whence j is assigned the same color as the 1 corresponding to it. On the other hand, if φ ∈ Td_{α,β}[n − 1, n − k], we obtain the corresponding permutation as follows. Remove a vertical step after every horizontal step on the lattice path associated with. If the jth step on the resulting lattice path is vertical, then j is a cycle minimum. Form the word \(σ_0 = (s_1)(s_2)(s_3)\cdots(s_{α+k})\), where the \(s_j\)'s are the cycle minima arranged in increasing order. For \(1 ≤ j ≤ n − 1 + α\), if \(φ_0,1_v\) contains a column of length \(j\) which is above the lattice path with a 1 on the rth box, then we insert the element \(j\) after the \(r\)th letter of \(σ_{j−1}\), where \(j\) is assigned the same color as the 1. The resulting permutation is represented by the word \(σ_{n−1+α}\). Note that when \(ν(0) = 0\), the zeroth cycle or blocks are empty, and hence, may be ignored.

As an illustration, let \(ν(i) = 2i + 4, w(i) = 1\) and \(φ_{0,1_v}\) and \(ψ_{0,1_v}\) be the 0, 1-v-tableaux in Figure 1, where \(φ_{0,1_v}\) is of shape \[
\begin{bmatrix}
3 & 1 & 0 \\
3 & 5 & 6
\end{bmatrix}
\] \(∈ T_{0,0}[5,3]\) and \(ψ_{0,1_v}\) is of shape \[
\begin{bmatrix}
3 & 1 & 0 \\
0 & 1 & 1
\end{bmatrix}
\] \(∈ T_{0,0}[5,3]\). The lattice path associated with \(φ_{0,1_v}\) is \(V V H H V V H V V\) so that the associated block minima are 0, 1, 4, 5, 7, 8. The second step on the lattice path is an H, which is below a column with a 1 on the second row. Hence, 21 is an element of the second block. The corresponding partition is \{0, 3\} \{1, 2\} \{4, 6\} \{5\} \{7\} \{8\} \(∈ \text{Part}(8,5; ν)\). For \(ψ_{0,1_v}\), the associated lattice path is \(V H H V H V\); hence the cycle minima are 0, 3, 5, 6 so that \(σ_0 = (0)(3)(5)(6)\), \(σ_1 = (0, 14)(3)(5)(6)\), \(σ_2 = (0, 14, 21)(3)(5)(6)\), \(σ_3 = (0, 14, 21, 3)(49)(5)(6)\) \(∈ \text{Perm}(6,3; ν)\).

We now use the approach described above to some special cases. We will sometimes only interpret the first kind in terms of permutations, but analogous interpretation applies for the second kind in terms of partitions.

1. Merris’s \(p\)-Stirling numbers [10] and Koutras’s non-central Stirling number with pa-
rameter $p \in \mathbb{N}$ [7] $c_p(n, k)$ and $S_p(n, k)$ are obtained by letting $V = (i, 1)$ and $\alpha = p$, or by letting $V = (i + p, 1)$ and $\alpha = 0$. We can therefore interpret, for example, $c_p(n, k)$ as the number of permutations of $[n]_0$ into $(k + 1)$ cycles such that the non-minimal elements of the zeroth cycle take $p$ colors, or as the number of permutations of $[n + \alpha]$ into $(k + \alpha)$ blocks such that 1, 2, 3, …, $\alpha$ are in distinct cycles. Alternatively, if $[m, n] = \{m, m + 1, \ldots, n\}$, then $c_p(n, k)$ is the number of permutations of $[a - 1, n]$ into $(k + \alpha)$ blocks such that the non-positive elements are in distinct cycles.

2. We get Sun’s $p$-Stirling numbers [14] $c^p(n, k)$ and $S^p(n, k)$ when $V = (v(i), 1), v(i) = i^p$ and $\alpha = 0$. The number of 0, $1_V$-tableaux is the number of $p$-tuples of 0, $1_V$-tableaux of identical shape, where $V' = (i, 1)$. Hence, $c^p(n, k)$ counts the number of permutations of $[n]$ into $k$ cycles such that every non-minimal element $a_j$ is assigned $v(a_j)/j$ colors and also counts the number of $p$-tuples of permutations of $[n]$ having $k$ cycles such that the permutations have identical cycle minima. It is easy to see that the latter interpretation is equivalent to that of Sun’s, namely, that $c^p(n, k)$ counts the number of $p$ by $k$ matrices $M = [m_{ij}]$ such that $m_1m_2\cdots m_{ik}$ is permutation of $[n]$ written as a product of $k$ cycles arranged by increasing cycle minima, and such that the cycle minima of $m_{1j}, m_{2j}, \ldots, m_{jk}$ are identical for all $1 \leq j \leq k$. Sun called such a matrix a $k$-matrix permutation of $M(n, p)$.

3. Everett et. al.’s Jacobi Stirling numbers $Jc(n, k)$ and $JS(n, k)$ [4] are obtained by setting $V = (i(i + z), 1), z \in \mathbb{R}$ and $\alpha = 0$. The Jacobi Stirling numbers, in turn, reduce to the Legendre Stirling numbers $Lc(n, k)$ and $LS(n, k)$ when $z = 1$. Further generalizations of these sequences were given in OEIS, where $V = (\prod_{i=1}^m(i + a_i), 1)$, where $a_i \in \mathbb{N}$ (see for example, sequences A071951, A089504, A090215 and A090217 [2]).

Using the approach described above, if $V = (\prod_{i=1}^m(i + a_i), 1)$, then for a $B$-tableau $\phi$, $\mathcal{V}_{wt}(\phi)$ equals the number of 0, $1_V$-tableaux of shape $\phi$, which in turn, equals the number of $j$ tuples $(\phi_{0,1\nu_1}, \phi_{0,1\nu_2}, \ldots, \phi_{0,1\nu_j})$ where $\phi_{0,1\nu_i}$ is a 0, $1_V$-tableau where $\nu_i = (i + a_i, 1)$. Hence, $c^p_{\alpha, \beta}(n, k)$ counts the number of $j$ tuples $(\sigma_1, \sigma_2, \ldots, \sigma_j)$ whose positive minimal elements are identical and $\sigma_i$ is a permutation of $[-a_i - 1, n]$ into $(n + a_i)$ cycles such that the non-positive elements are in distinct cycles.

Let $V = (i(i + 1), 1)$ and $\alpha = 0$. Let $[\pm n]_0 := \{0, 1, -1, 2, -2, \ldots, n, -n\}$. We say that a subset forming a partition of $[\pm n]_0$ contains both copies of a non-zero integer $j$ if both $j$ and $-j$ are in that subset. Define $\text{Part}_\pm(n, k)$ as the set of partitions of $[\pm n]_0$ into $k + 1$ subsets $D_0, D_1, D_2, \ldots, D_k$ satisfying the following properties:

(a) $0 \in D_0$ and $D_0$ is not allowed to contain both copies of the same non-zero integer.
(b) The subsets $D_1, D_2, \ldots, D_k$ are non-empty and indistinguishable. The subset minimum of every subset $D_j$, where $1 \leq j \leq k$, is the positive integer $|m|$ such that $|m| \leq |m'|$ for all $m' \in D_j$. Every subset must contain both copies of its subset minimum but no other element occurs in two copies in that subset.

Gelineau and Zang [5] obtained the following combinatorial interpretation: $LS(n, k) = |\text{Part}_\pm(n, k)|$. It is straightforward to construct a bijection between our combinatorial
interpretation and those of Zhang and Gelineau. Let \((\pi_1, \pi_2)\) be a pair of partitions, where \(\pi_1\) is a partition of \([n]\) into \(k\) blocks \(B_1, B_2, \ldots, B_k\) arranged by increasing minimal element, and \(\pi_2\) is a partition of \([n]\) into \(k+1\) blocks \(B'_0, B'_1, B'_2, \ldots, B'_k\), also arranged by increasing minimal element. Assign the negative sign to the nonminimal elements of \(\pi_2\). Then, the partition with blocks \(B_0, B_1 \cup B'_1, B_2 \cup B'_2, \ldots, B_k \cup B'_k\) is an element of \(\text{Part}_{\pm}(n, k)\). This combinatorial interpretation can be easily generalized for \(\mathcal{V} = (\prod_{i=1}^m (i + a_i), 1)\).

4. Miceli’s poly-Stirling numbers [11] arise as a special case when \(v \in \mathbb{N}[i], \alpha = 0, w(i) = 1\). We have thus provided an alternative interpretation from those given in Miceli’s paper.

5. We have, so far, considered only \(\mathcal{V}\)-Stirling numbers where \(\mathcal{V} = (v, 1)\). However, we can also apply the same method of obtaining a permutation or partition to the boxes below the lattice path. Recall that the \(b\)-Stirling number of the first kind, defined by the generating function (REF) and obtained by letting \(\mathcal{V} = (i, i)\) and \(\alpha, \beta = 0\), can be combinatorially interpreted as the number of pairs of permutations \((\sigma_1, \sigma_2)\), where \(\sigma_1\) is a permutation of \([n]\) into cycles \(C_1, C_2, \ldots, C_k\) and \(\sigma_2\) is a permutation of \([n]\) into cycles \(C'_1, C'_2, \ldots, C'_k\), such that for \(1 \leq j \leq k\), the sum of the minimal elements of \(C_j\) and \(C'_{k-j+1}\) is \(n + 1\). Here, the cycle minima of \(C'_j\) are obtained by labelling the lattice path with \(0, 1, 2, \ldots\) starting from the lower left hand corner to the upper right hand corner.

As a final note, we can extend the approach we described here for non-negative weight functions which are not necessarily polynomials in \(i\). For instance, for all \(i \in \mathbb{N}\), we can fix a sum \(v(i) = c_{i,1} + c_{i,2} + \ldots + c_{j(i)}\) such that \(j(1) < j(2) < j(3) < \cdots\) and assign to 1 \(c_{i,r}\) colors if it is placed on the \(r\)th box of a column of length \(j(i)\). This will entail a more general class of colored permutations and partitions. Details are left to the reader.

References


