Rigorous numerics in Floquet theory:
computing stable and unstable bundles of periodic orbits

Roberto Castelli *  Jean-Philippe Lessard†

Abstract
In this paper, a new rigorous numerical method to compute fundamental matrix
solutions of non-autonomous linear differential equations with periodic coefficients is
introduced. Decomposing the fundamental matrix solutions $\Phi(t)$ by their Floquet
normal forms, that is as product of real periodic and exponential matrices $\Phi(t) =
Q(t)e^{Rt}$, one solves simultaneously for $R$ and for the Fourier coefficients of $Q$ via a
fixed point argument in a suitable Banach space of rapidly decaying coefficients. As
an application, the method is used to compute rigorously stable and unstable bundles
of periodic orbits of vector fields. Examples are given in the context of the Lorenz
equations and the $\zeta^3$-model.

Keywords
Rigorous numerics · Floquet theory · Fundamental matrix solutions ·
Contraction mapping theorem · Periodic orbits · Tangent bundles

Mathematics Subject Classification (2000)
37B55 · 37M99 · 37C27 · 65G99 · 34D05

1 Introduction
In his seminal work [1] of 1883, Gaston Floquet studied linear non-autonomous differential
equations of the form
\[ \dot{y} = A(t)y, \]  
where $A(t)$ is a $\tau$-periodic continuous matrix function of $t$. The main result of [1] is now
presented, and its proof can be found for instance in [2].

Theorem 1.1. [Floquet, 1883] Let $A(t)$ be a $\tau$-periodic continuous matrix function and
denote by $\Phi(t)$ a fundamental matrix solution of (1). Then $\Phi(t + \tau)$ is also a fundamental
matrix solution, $\Phi(t + \tau) = \Phi(t)\Phi^{-1}(0)\Phi(\tau)$, and there exist a real constant matrix $R$ and
a real nonsingular, continuously differentiable, $2\tau$-periodic matrix function $Q(t)$ such that
\[ \Phi(t) = Q(t)e^{Rt}. \]

*Corresponding author. BCAM - Basque Center for Applied Mathematics, Bizkaia Technology
Park, 48160 Derio, Bizkaia, Spain. Phone: (+34) 946 567 842. Fax: (+34) 946 567 843. Email: rcastelli@bcamath.org.
†BCAM - Basque Center for Applied Mathematics, Bizkaia Technology Park, 48160 Derio, Bizkaia, Spain.
Email: lessard@bcamath.org.
Decomposition (2) is called a Floquet normal form for the fundamental matrix solution \( \Phi(t) \). The real time-dependent change of coordinates \( z = Q^{-1}(t)y \) transforms system (1) into a linear constant coefficients system of the form \( \dot{z} = Rz \). A stability theorem demonstrates that the stability of the zero solution of (1) can be determined by the eigenvalues of the so-called monodromy matrix \( \Phi(\tau) \). As mentioned in [2], while the stability theorem is very elegant, in applied problems it is usually impossible to compute the eigenvalues of the monodromy matrix. An even more challenging and central problem is the computation of the fundamental matrix solutions. The goal of the present work is to address this major difficulty by introducing a new rigorous numerical method to compute explicitly Floquet normal forms as in (2), hence providing a direct way to obtain fundamental matrix solutions of (1). Before proceeding with a general presentation of the rigorous computational method, let us introduce some motivations.

First of all, we are not aware of any method to construct rigorously Floquet normal forms as introduced in Theorem 1.1. Since this fundamental decomposition was introduced more than 125 years ago, we believe that developing a rigorous computational method leading to an explicit construction of Floquet normal forms is an important contribution to the field of differential equations.

The second motivation is directly linked to the study of dynamical systems. Indeed, equations of the form (1) arise naturally when studying stability properties of time-periodic solutions of differential equations \( \dot{y} = g(y) \), where \( g : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a smooth map. Assume that \( \Gamma \) is a \( \tau \)-periodic orbit of \( \dot{y} = g(y) \) parameterized by \( \gamma(t) \in \mathbb{R}^n \ (t \in [0, \tau]) \), and define the \( \tau \)-periodic matrix function \( A(t) = \nabla g(\gamma(t)) \), where \( \nabla g \) is the Jacobian matrix. Consider \( \Phi(t) \) the principal fundamental matrix solution of \( \dot{y} = A(t)y = \nabla g(\gamma(t))y \), that is the unique fundamental matrix solution so that \( \Phi(0) = I \), and assume that a Floquet normal form \( \Phi(t) = Q(t)e^{Rt} \) has been computed. Theorem 3.7 shows how the information from the Floquet normal form can directly be used to compute important dynamical properties of \( \Gamma \). More explicitly, it is demonstrated that the stability of the periodic orbit \( \Gamma \) can be determined by the eigenvalues of \( R \) while the stable and unstable tangent bundles of \( \Gamma \) can be retrieved from the action of \( Q(t) \) (with \( t \in [0, \tau] \)) on the eigenvectors of \( R \).

A final motivation comes from the fact that computing stable and unstable bundles of periodic orbits is an important step toward computing rigorous parameterization of invariant manifolds of periodic orbits. In fact, one of our future goal consists of combining the ideas of [3] to rigorously parameterize invariant manifolds of periodic orbits, and then to use that information to solve rigorously, following similar ideas than the ones presented in [4], a projected boundary value problem whose solutions would correspond to cycle-to-cycle connections and to point-to-cycle connections. Note that the approach of using projected boundary value problems to compute (non rigorously) cycle-to-cycle connections and to point-to-cycle connections has been adopted by several authors (e.g. see [5], [6], [7]).

Let us now introduce the ideas behind the rigorous method to compute Floquet normal forms. Rather than jumping immediately into a deep mathematical description of the method, we present the general ideas and we refer to Section 2 for a more detailed presentation.

The first step is to substitute the Floquet normal form \( \Phi(t) = Q(t)e^{Rt} \) in the differential equation (1). From this, it follows that \( (R, Q(t)) \) is a solution of the differential equation with periodic coefficients \( \dot{Q} = A(t)Q - QR \). On the converse, if a real constant matrix \( R \) and a \( 2\tau \)-periodic matrix function \( Q(t) \) solve

\[
\begin{align*}
\dot{Q} &= A(t)Q - QR \\
Q(0) &= I,
\end{align*}
\]

(3)
then the matrix function $\Phi(t) := Q(t)e^{Rt}$ is the principal fundamental solution of (1). Therefore, the problem of computing fundamental matrix solutions in the form $\Phi(t) = Q(t)e^{Rt}$ reduces to find $(R, Q(t))$ satisfying (3). The next step is to introduce a nonlinear operator $f$ (see Section 2.1 for details) whose zeros are in one-to-one correspondence with the solutions of (3). Letting $x = (R, Q_0, Q_1, Q_2, \ldots)$, where the $Q_k$’s are the Fourier coefficients of $Q(t)$, the problem of computing Floquet normal forms $\Phi(t) = Q(t)e^{Rt}$ is then equivalent to find $x$ such that $f(x) = 0$. By the a priori knowledge of the smoothness of $Q(t)$, the Fourier coefficients $Q_k$’s decay fast, meaning that the solutions of $f(x) = 0$ live in a suitable Banach space $\Omega^s$ of rapidly decaying coefficients. To prove existence, in a constructive way, of solutions of the infinite dimensional nonlinear operator equation $f(x) = 0$, we use rigorous numerics. To be more precise, the goal of rigorous numerics is to construct algorithms that provide an approximate solution to the problem together with precise bounds within which the exact solution is guaranteed to exist in the mathematically rigorous sense. It is worth mentioning that by now, the use of rigorous numerical methods is a standard approach to study differential equations and dynamical systems (e.g. see [8], [9], [10], [11], [12], [13], [14], [15], [16], [17]).

Based on the previous discussion, the next step consists of computing a numerical approximation $\bar{x}$ of $f(x) = 0$ and to demonstrate that close to $\bar{x}$, there exists a genuine solution $x^*$ of $f(x) = 0$, corresponding to the wanted explicit Floquet normal form of the principal fundamental matrix solution $\Phi(t)$ of (1). However, since the operator $f$ is infinite dimensional, a finite dimensional approximation of $f$ must be introduced in order to compute an approximation $\bar{x}$. This is done in Section 2.2. Once $\bar{x}$ is computed, a Newton-like operator $T : \Omega^s \rightarrow \Omega^s$ defined by $T(x) = x - Af(x)$ is introduced, where $A$ is an injective linear operator which acts as an approximation for $Df(\bar{x})^{-1}$. Since $A$ is injective, the fixed points of $T$ and the zeros of $f$ are in one-to-one correspondence. The next step is to consider small balls $B_{\bar{x}}(r) \subset \Omega^s$ centered at the numerical approximation $\bar{x}$, and to solve for $r$ for which $T : B_{\bar{x}}(r) \rightarrow B_{\bar{x}}(r)$ is a contraction (see Section 2.3). The rigorous verification that $T$ is a contraction on $B_{\bar{x}}(r)$ is done via the use of the so-called radii polynomials which provide, in the context of differential equations, an efficient means of determining a domain on which the contraction mapping theorem is applicable. The notion of the radii polynomials was originally introduced in [18] and [19] to prove existence of equilibria of PDEs. It was later on adapted to prove existence of equilibria of high-dimensional PDEs (e.g. see [20], [21], [22], [23]), of periodic orbits of delay equations and PDEs (e.g. see [24], [25], [26], [27]) and connecting orbits of ODEs [4]. We refer to [28] for a more extensive and general exposure of the radii polynomials.

In this work, we present a general formulation of the radii polynomials adapted to the context of computing rigorously Floquet normal forms (see Section 2.4 for more details). We present the explicit bounds in Section 2.5 that lead directly to their construction. Note that these bounds ensure that the truncation error terms, inevitably introduced by computing on a finite dimensional projection, are controlled. It is also important to mention that in the computation of the bounds, the floating point errors are controlled by using interval arithmetic [29]. In fact, all rigorous computations were performed in Matlab with the interval arithmetic package Intlab [30].

The paper is organized as follows. In Section 2, we introduce the rigorous numerical method to compute Floquet normal forms $\Phi(t) = Q(t)e^{Rt}$ of fundamental matrix solutions of systems of the form (1). In Section 3, we demonstrate how to use the information from Floquet normal forms to compute stable and unstable bundles of periodic orbits of vector field and how to determine the stability properties of periodic orbits. The main result of this section is Theorem 3.7. Finally in Section 4, we present some applications, where we
construct rigorously tangent stable and unstable bundles of some periodic orbits of the Lorenz equations and of the \( \zeta^3 \) model.

2 Rigorous computation of Floquet normal forms

In this section, we introduce the rigorous numerical method to compute Floquet normal forms \( \Phi(t) = Q(t)e^{Rt} \) of fundamental matrix solutions of systems of the form (1). As already mentioned in Section 1, the first step is to introduce a nonlinear operator \( f \) whose zeros are in one-to-one correspondence with the solutions of (3).

2.1 Set-up of the operator equation \( f(x) = 0 \)

In the following \( \text{Mat}(n, \mathbb{R}), \text{Mat}(n, \mathbb{C}) \) denote the space of \( n \times n \) matrices respectively with real and complex entries. The assumption on \( Q(t) \) to be real and \( 2\tau \)-periodic allows to consider the expansion

\[
Q(t) = Q_0 + \sum_{k \in \mathbb{Z} \setminus \{0\}} (Q_{k,1} + iQ_{k,2}) e^{ik\frac{2\pi}{2\tau}t},
\]

where the Fourier coefficients \( Q_0, Q_{k,1} \in \text{Mat}(n, \mathbb{R}) \) satisfy \( Q_{-k,1} = Q_{k,1} \) and \( Q_{-k,2} = -Q_{k,2} \) for any \( k \geq 1 \). Being \( \tau \)-periodic, the matrix-valued function \( A(t) \) is also \( 2\tau \)-periodic, thus it makes sense to consider the expansion

\[
A(t) = \sum_{k \in \mathbb{Z}} A_k e^{ik\frac{2\pi}{2\tau}t},
\]

where \( A_0 \in \text{Mat}(n, \mathbb{R}) \), while the matrices \( A_k \in \text{Mat}(n, \mathbb{C}) \) satisfy \( A_{-k} = C(A_k) \), for any \( k \geq 1 \). Here \( C(A) \) stands for the matrix whose entries are the complex conjugates of the entries of \( A \). It has to be remarked that the assumption for \( A \) to be \( \tau \)-periodic implies that \( A_k = 0 \) for \( k \) odd and \( A_{2l} = \hat{A}_l \) where \( \hat{A}_l \) is the \( l \)-th Fourier coefficient of \( A(t) \) in the basis \( \{e^{ik\frac{2\pi}{2\tau}t}\}_k \).

After substituting the expansions (4) and (5) in problem (3), the latter system of ODEs moves into an equation \( F(t) = 0 \), where \( F(t) \) is a \( 2\tau \)-periodic matrix function. By a subsequent projection of \( F(t) \) in the Fourier basis \( \{e^{ik\frac{2\pi}{2\tau}t}\} \), it follows that solving (3) is equivalent to solve for the unknowns

\[
R, Q_0 \in \text{Mat}(n, \mathbb{R}) \quad \text{and} \quad Q_k := (Q_{k,1}, Q_{k,2}) \in \text{Mat}(n, \mathbb{R})^2
\]

the infinite dimensional algebraic system

\[
f(R, Q_0, \ldots, Q_k, \ldots) = 0 \tag{6}
\]

\[
f = (f_*, f_0, f_1, \ldots, f_k, \ldots)
\]

defined by

\[
f_* := Q_0 + 2 \sum_{k \geq 1} Q_{k,1} - I
\]

\[
f_0 := Q_0R - (A \cdot Q)_0
\]

\[
f_k := \begin{bmatrix} f_{k,1} \\ f_{k,2} \end{bmatrix} = \begin{bmatrix} -k\frac{2\pi}{2\tau}Q_{k,1} + Q_{k,1}R - (A \cdot Q)_{k,1} \\ k\frac{2\pi}{2\tau}Q_{k,1} + Q_{k,2}R - (A \cdot Q)_{k,2} \end{bmatrix}, \quad k \geq 1
\]
where \((A \cdot Q)_{k,1}, (A \cdot Q)_{k,2}\) denote respectively the real and imaginary part of the convolution

\[
 (A \cdot Q)_k := \sum_{k_1+k_2=k} A_{k_1} (Q_{k_1,1} + iQ_{k_1,2}).
\]

Note that \(f_*, f_0 \in \text{Mat}(n, \mathbb{R})\) and \(f_k \in \text{Mat}(n, \mathbb{R})^2\) for every \(k \geq 1\).

The problem (6) consists of: i) a system of \(n^2\) real scalar equations for \(f_* = 0\) representing the initial condition \(Q(0) = I; ii) n^2\) real scalar equations for system \(f_0 = 0\) that reproduces \(<F(t), 1 > = 0; iii) 2n^2\) real scalar equations for each \(f_k = 0 (k \geq 1)\). Note that \(f_{k,1}, f_{k,2}\) are the real and complex part of the equation \(<F(t), e^{ik \frac{2\pi}{T} t} > = ik \frac{2\pi}{T} Q_k + Q_k R - (A \cdot Q)_k\).

Here, \(<\cdot, \cdot>\) represents the inner product in \(L^2 \left([0, \frac{2\pi}{T}]\right)\).

Before proceeding with the analysis of the system \(f = 0\) given by (6), let us introduce some notation that will be adopted throughout the paper.

**Notation**

Let \(A, B\) be matrices with entries \(A = \{a_{i,j}\}, B = \{b_{i,j}\}\) and \(A = (A_1, \ldots, A_n), B = (B_1, \ldots, B_n)\) be vectors of matrices. Denote by

i) \(|A| = \{|a_{i,j}|\}\) the matrix of absolute values, where \(|\cdot|\) denotes both the real and complex absolute value, according with \(a_{i,j}\). For vectors \(|A| = (|A_1|, \ldots, |A_n|)\); \(|A|_\infty = \max_i \{|a_{i,j}|\}\) and \(|A|_\infty = \max\{|A_1|_\infty, \ldots, |A_n|_\infty|\}\)

ii) \(A \leq_{cw} B\) means \(a_{i,j} \leq b_{i,j}\) for any \(i, j\). In case \(b\) is a scalar, \(A \leq_{cw} B\) means \(a_{i,j} \leq b\). In case of vectors \(A \leq_{cw} B\) and \(A \leq_{cw} b\) extends as \(A_k \leq_{cw} B_k\) and \(A_k \leq_{cw} b\), for any \(k = 1 \ldots n\). The same for \(\geq_{cw}, >_{cw}, <_{cw}\).

iii) \(|A|_\infty\) is the standard infinity norm of a matrix: \(|A|_\infty = \max_i \sum_j |a_{i,j}|\);

iv) \(I\) denotes the identity \(n \times n\) matrix, \(I_n\) is the \(n \times n\) matrix whose entries are all 1.

Coming back to the analysis of system (6) let us define the space

\[
 X = \left\{ x = (x_0, x_1, \ldots, x_k, \ldots), x_0 = (R, Q_0) \in \text{Mat}(n, \mathbb{R})^2, x_k = Q_k = (Q_{k,1}, Q_{k,2}) \in \text{Mat}(n, \mathbb{R})^2, k \geq 1 \right\}.
\]

Note that \(f : X \rightarrow X\). Later on the problem of solving \(f = 0\) will be transformed into a fixed point problem for an operator \(T\) that requires the choice of a suitable Banach subspace of \(X\) where to investigate the existence of solutions. To define the proper Banach space, let us first introduce the weigh function

\[
w_k = \begin{cases} |k| & k \neq 0 \\ 1 & k = 0 \end{cases}
\]

and given \(x = (R, Q_0, Q_{1,1}, Q_{1,2}, \ldots, Q_{k,1}, Q_{k,2}, \ldots) \in X\), let us define the \(s\)-norm of \(x\) in \(X\) by

\[
\|x\|_s := \sup_{k \geq 0} \{|x_k|_\infty w_k^s| \} = \sup \left\{ |R|_\infty, |Q_0|_\infty, \sup_{k \geq 1} \{|Q_{k,1}|_\infty w_k^s, |Q_{k,2}|_\infty w_k^s| \right\}.
\]

According with the \(s\)-norm, let us define the space \(\Omega^s\) of sequences in \(X\) with algebraically decaying tails

\[
\Omega^s = \{ x \in X : \|x\|_s < \infty \}.
\]

For any \(s > 0\) the space \(\Omega^s\) endowed with the \(s\)-norm is a Banach space and the inclusion \(\Omega^s \supset \Omega^{s+1}\) holds. The introduction of \(\Omega^s\) is motivated by the fact that a periodic solution
Lemma 2.1. Assume \( \|A\|_s < \infty \) for \( s^* \geq 2 \). Then \( f \) maps \( \Omega^s \) in \( \Omega^{s-1} \), for any \( 2 \leq s \leq s^* \).

Proof. Let \( 2 \leq s \leq s^* \) and suppose \( x \in \Omega^s \). Then \( |A_k|_\infty < C_s w_k^{-s} \) and, from Lemma 2.1 in [20], \( |(A \cdot Q)_k|_\infty \leq \frac{C_s^2}{w_k^s} \). Thus \( |f_k(x)|_\infty \leq C_3 k |Q_k|_\infty + C_4 |Q_k|_\infty + C_2 w_k^{-s} < C w_k^{-s+1} \), for suitable constants \( C, C_i \). This shows that \( f(x) \in \Omega^{s-1} \).

Thus we will look for solutions of the system (6) within the space \( \Omega^s \) for some \( s \geq 2 \). The idea is to reformulate the zero finding problem \( f(x) = 0 \) as a fixed point problem for a suitable operator \( T \) defined in \( \Omega^s \) and to verify the hypothesis of the contraction mapping theorem in order to conclude about the existence of a fixed point. More explicitly, the idea is to prove the existence of a ball \( B_\varepsilon(x) \) in \( \Omega^s \) around a finite dimensional approximate solution \( \bar{x} \) on which the operator \( T \) is a contraction. The proof will follow by verifying a finite number of polynomial inequalities: the so-called \textit{radii polynomials}. Their computation will result from rigorous numerical computations and analytic estimates. The next step is to compute a finite dimensional approximate solution \( \bar{x} \). For this, one needs to introduce a finite dimensional projection of \( f(x) = 0 \) given by (6).

### 2.2 Finite dimensional projection

As mentioned earlier, the first step involved in the computational method is to consider a finite dimensional projection and to compute an approximate numerical solution of (6).

For \( m > 1 \) consider the finite dimensional space \( X^m = \prod_{k=1}^m \text{Mat}(n, \mathbb{R})^2 \) and define the projections

\[
\Pi_m : X \rightarrow X^m \\
x \mapsto \Pi_m(x) = x^m = (R, Q_0, \ldots, Q_{m-1})
\]

\[
\Pi_\infty : x \mapsto (Q_m, Q_{m+1}, \ldots)
\]

so that \( x = (x^m, \Pi_\infty(x)) \). Denote with \( 0^\infty := \Pi_\infty(0) \). Moreover let us define the restricted map

\[
f^{(m)} : X^m \rightarrow X^m \\
x^m \mapsto \Pi_m f(x^m, 0^\infty)
\]

Note that for any \( x \in X \) the sequence \( (x^m, 0^\infty) \in X \) and the finite dimensional projection \( \Pi_m \) applied to \( f(x) \) reads as \( \Pi_m f(x) = (f_\ast, f_0, \ldots, f_{m-1})(x) \). Since \( X^m \) is isomorphic to
$\mathbb{R}^{m2n^2}$, one can think of $f^{(m)} : \mathbb{R}^{m2n^2} \to \mathbb{R}^{m2n^2}$. Suppose that using a Newton-like iterative algorithm, one computed
\[
\bar{x} = (\bar{R}, Q_0, \ldots, Q_{m-1})
\]
an approximate zero of $f^{(m)}$, that is $f^{(m)}(\bar{x}) \approx 0$. For simplicity the same notation $\bar{x}$ is used to identify the above vector in $X^m$ and the sequence $(\bar{x}, 0^\infty)$ in $X$. As already mentioned at the end of Section 2.1, the idea is to consider a ball $B_2(r) \in \Omega$ centered at the approximate solution $\bar{x}$ and to show the existence of a contraction mapping $T$ acting on $B_2(r)$. Hence, let us now introduce the fixed point operator $T$.

### 2.3 The fixed point operator $T(x) = x$

In this section, we first define an operator $T$ on $\Omega$ whose fixed points correspond to solutions of $f(x) = 0$ and then, we introduce some computable conditions from which one can conclude about the existence of fixed point of $T$. To begin with, suppose to have chosen a representation of the matrices $\text{Mat}(n, \mathbb{R})$ as vector in $\mathbb{R}^{n^2}$ and have extended it to an isomorphism between the space of sequences of $N$ matrices $\text{Mat}(n, \mathbb{R})$ to $\mathbb{R}^{Nn^2}$. Note that $X^m$ is isomorphic to $\mathbb{R}^{m2n^2}$.

In the sequel, consider a vector $V = [v_1, \ldots, v_{N2n^2}] \in \mathbb{R}^{N2n^2}$. We denote by $V_k \in \mathbb{R}^{2n^2}$, $k = 0, \ldots, N-1$ the vector with $2n^2$ components $V_k = [v_{k2n^2+1}, v_{k2n^2+2}, \ldots, v_{(k+1)2n^2}]$. The reason of this choice of notation is the following: suppose that $V$ is the vector representation of the sequence $x = (R, Q_0, Q_1, \ldots, Q_{N-1}) \in X^N$ for a positive $N$, then $V_k$ represents the couple $(R, Q_0)$ when $k = 0$ and $Q_k = (Q_{k,1}, Q_{k,2})$ for $k \geq 1$.

Denote by $D f^{(m)}(\bar{x})$ the Jacobian of $f^{(m)}$ with respect to $x^m$ evaluated at $\bar{x}$, that is
\[
D f^{(m)} := D f^{(m)}(\bar{x}) = \frac{\partial (f^*, f_0, f_1, \ldots, f_{m-1})}{\partial (R, Q_0, \ldots, Q_{m-1})}(\bar{x}) \in \text{Mat}(2n^2 m, \mathbb{R}).
\]

For clarity and completeness,

\[
D f^{(m)} = \begin{bmatrix}
\frac{\partial f_1}{\partial R} & \frac{\partial f_1}{\partial Q_0} & \frac{\partial f_1}{\partial Q_1} & \frac{\partial f_1}{\partial Q_2} & \cdots & \frac{\partial f_1}{\partial Q_{m-1}} & \frac{\partial f_1}{\partial Q_{m-2}} \\
\frac{\partial f_2}{\partial R} & \frac{\partial f_2}{\partial Q_0} & \frac{\partial f_2}{\partial Q_1} & \frac{\partial f_2}{\partial Q_2} & \cdots & \frac{\partial f_2}{\partial Q_{m-1}} & \frac{\partial f_2}{\partial Q_{m-2}} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
\frac{\partial f_{m-1}}{\partial R} & \frac{\partial f_{m-1}}{\partial Q_0} & \frac{\partial f_{m-1}}{\partial Q_1} & \frac{\partial f_{m-1}}{\partial Q_2} & \cdots & \frac{\partial f_{m-1}}{\partial Q_{m-1}} & \frac{\partial f_{m-1}}{\partial Q_{m-2}} \\
\end{bmatrix}
\]

where for $k, j = 1, \ldots, m - 1$
\[
\frac{\partial f_k}{\partial R} = \begin{bmatrix}
\frac{\partial f_{k,1}}{\partial Q_{j,1}} & \frac{\partial f_{k,1}}{\partial Q_{j,2}} \\
\frac{\partial f_{k,2}}{\partial Q_{j,1}} & \frac{\partial f_{k,2}}{\partial Q_{j,2}} \\
\end{bmatrix}, \quad \frac{\partial f_k}{\partial Q_j} = \begin{bmatrix}
\frac{\partial f_{k,1}}{\partial Q_{j,1}} & \frac{\partial f_{k,1}}{\partial Q_{j,2}} \\
\frac{\partial f_{k,2}}{\partial Q_{j,1}} & \frac{\partial f_{k,2}}{\partial Q_{j,2}} \\
\end{bmatrix},
\]

and each $\frac{\partial f_{k,l}}{\partial Q_{j,l}} \in \text{Mat}(n^2, \mathbb{R})$ denotes the Jacobian matrix of the components of $f_{k,j}$ with respect to the components of $Q_{j,l}$.

Suppose to have numerically computed $D f^{(m)}$ and denote by $A_m \in \text{Mat}(2n^2 m, \mathbb{R})$ an invertible numerical approximation of $(D f^{(m)})^{-1}$
\[
A_m : D f^{(m)} \approx I
\]
and for $k \geq m$, define
\[
A_k := \frac{\partial f_k}{\partial Q_k} \in \text{Mat}(2n^2, \mathbb{R}).
\]
Lemma 2.2. Recall (5) and (10), and assume that $\|A\|_s < \infty$ for some $s \geq 2$. Then there exist two constants $K$ and $C_A$ such that for any $k \geq K$ the linear operator $\Lambda_k$ is invertible and $\|\Lambda_k^{-1}\|_\infty < \frac{C_A}{k}$. The constants $K$ and $C_A$ depend on $\|A\|_s$, the period $\tau$ and $|\bar{R}|_\infty$.

Proof. The real and imaginary parts of $(A \cdot \bar{Q})_k$ can be written explicitly as

$$
(A \cdot \bar{Q})_{k,1} = (\text{Re}(A_0) + \text{Re}(A_{2k}))Q_{k,1} + \text{Im}(A_{2k})Q_{k,2} + W_1
$$

$$
(A \cdot \bar{Q})_{k,2} = \text{Im}(A_{2k})Q_{k,1} + (\text{Re}(A_0) - \text{Re}(A_{2k}))Q_{k,2} + W_2
$$

where $W_1$ and $W_2$ do not depend on $Q_{k,1}$ and $Q_{k,2}$. Thus, looking at the definition of $f_k$ in (7), it follows that $\Lambda_k$ is of the form

$$
\Lambda_k = \begin{bmatrix}
\lambda_{1,1} & -k\frac{2\pi}{\tau}I_{n^2} + \lambda_{1,2} \\
k\frac{2\pi}{\tau}I_{n^2} + \lambda_{2,1} & \lambda_{2,2}
\end{bmatrix},
$$

where the entries of $\lambda_{1,1}$ and $\lambda_{2,2}$ are linear combination of the entries of $\bar{R}$, $A_0$, $A_{2k}$ so that $|\lambda_{1,1}|_\infty, |\lambda_{2,2}|_\infty < |\bar{R}|_\infty + |A_0|_\infty + |A_{2k}|_\infty$ holds. Also, $\lambda_{2,1} = \lambda_{1,2}$ only depend on $A_{2k}$. By a row permutation, the invertibility of $\Lambda_k$ is equivalent to the invertibility of

$$
\hat{\Lambda}_k = \begin{bmatrix}
k\frac{2\pi}{\tau}I_{n^2} + \lambda_{2,1} & \lambda_{2,2} \\
\lambda_{1,1} & -k\frac{2\pi}{\tau}I_{n^2} + \lambda_{1,2}
\end{bmatrix}.
$$

Since $|\lambda_{1,1}|_\infty, |\lambda_{2,2}|_\infty < |\bar{R}|_\infty + |A_0|_\infty + |A_{2k}|_\infty$ and $|\lambda_{1,2}|_\infty < |A_{2k}|_\infty$, the assumption $\|A\|_s < \infty$ implies that the $|\lambda_{i,j}|_\infty$ are uniformly bounded in $k$, and moreover $|\lambda_{1,2}|_\infty$ is decreasing. Thus there exists $K$ such that $\hat{\Lambda}_k$ is diagonally dominant for any $k \geq K$. This is enough to conclude that $\hat{\Lambda}_k$ is invertible for any $k \geq K$.

Denote by $a_{i,i}$ the diagonal elements of $\hat{\Lambda}_k$. Hence, if $\hat{\Lambda}_k$ is diagonally dominant, that is if $|a_{i,i}| > \sum_{j \neq i} |\hat{\Lambda}_k(i,j)|$ for any $i = 1, \ldots, 2n^2$, then using a result from [31], one gets the following bound

$$
\|\hat{\Lambda}_k^{-1}\|_\infty \leq \frac{1}{\max_i \left\{ |a_{i,i}| - \sum_{j \neq i} |\hat{\Lambda}_k(i,j)| \right\}}.
$$

Therefore, for $k \geq K$

$$
\|\Lambda_k^{-1}\|_\infty = \|\hat{\Lambda}_k^{-1}\|_\infty \leq \frac{C_A}{k}
$$

for a constant $C_A$ depending on $\tau$, $|\bar{R}|$, $|A_0|_\infty$ and $|A_{2k}|_\infty$. 

Suppose that we chose the finite dimensional parameter $m > K$ where $K$, as defined in Lemma 2.2, is such that $\Lambda_k$ is invertible for any $k \geq K$. A formal diagonal concatenation of the operator $A_m$ and the sequence $\Lambda_k^{-1}$, for $k \geq m$, produces the linear operator

$$
A : X \rightarrow X \\
x \mapsto Ax
$$

$$(Ax)_k := \begin{cases}
(A_m x^m)_k & k = 0, \ldots, m - 1 \\
\Lambda_k^{-1} Q_k & k \geq m.
\end{cases}
$$

We define the operator $T$ on $X$ as

$$
T(x) := x - Af(x),
$$

and denote $T_k(x) = (T(x))_k$. 

8
Lemma 2.3. Recall (5) and (10), and assume that \( \|A\|_s \leq \infty \) for \( s^* \geq 2 \). Then for any \( 2 \leq s \leq s^* \), \( T : \Omega^s \to \Omega^s \) and solutions of \( T(x) = x \) correspond to solutions of \( f(x) = 0 \).

Proof. From Lemma 2.1, given \( x \in \Omega^s \) it follows that \( f(x) \in \Omega^{s-1} \). The linear operator \( A \) maps \( \Omega^{s-1} \) in \( \Omega^s \). Indeed for \( k \geq m \), \( \|(Ax)_k\|_\infty = |\Lambda_k^{-1} x_k|_\infty \leq \|\Lambda_k^{-1}\|_\infty |x_k|_\infty \). Thus from Lemma 2.2 and assuming \( x \in \Omega^{s-1} \), it follows that \( \|(Ax)_k\|_\infty \leq C \frac{\|x\|_k}{w_k} < C \frac{\|x\|_k}{w_k} \) for positive constants \( C, C_1 \). This proves that \( T : \Omega^s \to \Omega^s \). Since \( A_m \) is invertible by assumption and \( \Lambda_k \) have been proved in Lemma 2.2 to be invertible for all \( k \geq m > K \), it follows that the linear operator \( A \) is invertible and therefore fixed points of \( T \) correspond to zeros of \( f(x) \).

By construction, when restricted to the finite dimensional reduction \( \Pi_m \Omega^s \), the operator \( T \) acts as \( T(x^m) = x^m - A_m f^{(m)}(x^m) \). Thus on \( \Pi_m \Omega^s \), \( T \) is close to the Newton operator: the only difference is that point where the derivative is computed does not change along the iteration process. Therefore we can consider \( T \) as an extension to a infinite dimensional space of a finite dimensional Newton-like operator.

The existence of a fixed point for the operator \( T \) will be assured by the Banach Fixed Point Theorem once the operator \( T \) has been proved to be a contraction on a suitable ball in \( \Omega^s \). The suitable ball on which \( T \) will be proved to be a contraction will be sought within the family of balls \( B_2(r) \in \Omega^s \)

\[
B_2(r) = \bar{x} + B(r)
\]

where \( B(r) \) is the ball of radius \( r \) in \( \Omega^s \) centered in the origin and \( r \) is treated as variable. Following the same approach as in different other papers (e.g. see [20], [16], [24], [26], [27], [25], [22], [23], [4], [18], [19]), we are going to construct a finite set of computable polynomials defined as realization of the hypotheses of the following theorem.

Suppose there exist two matrices sequences

\[
Y = (Y_0, Y_1, \ldots, Y_k, \ldots), \quad Z(r) = (Z_0, Z_1, \ldots, Z_k, \ldots)(r), \quad Y, Z \in X
\]

such that

\[
|T(x) - \bar{x}|_k \leq c w Y_k, \quad \sup_{b_1, b_2 \in B(r)} \left| DT(x + b_1)b_2 \right|_k \leq c w Z_k(r), \quad \forall k \geq 0. \tag{15}
\]

Theorem 2.4. Fix \( s \geq 2 \) and let \( Y \) and \( Z \) defined as in (15). If there exists \( r > 0 \) such that \( \| Y + Z \|_s < r \), then the operator \( T \) maps \( B_2(r) \) into itself and \( T : B_2(r) \to B_2(r) \) is a contraction. Thus, by the Banach Fixed Point Theorem, there exists an unique \( x^* \in B_2(r) \) solution of \( T(x^*) = x^* \) and therefore solution of \( f(x^*) = 0 \).

Proof. Two statements need to be proved:

i) \( T(B_2(r)) \subset B_2(r) \), that is \( \|T(x) - \bar{x}\|_s < r \) for all \( x \in B_2(r) \).

ii) \( T \) is a contraction, that is there exists \( \kappa \in (0,1) \) such that for every \( x, y \in B_2(r) \), one has that \( \|T(x) - T(y)\|_s \leq \kappa \|x - y\|_s \).

For a given \( k \geq 0 \) and any \( x, y \in B_2(r) \), the mean value theorem implies

\[
T_k(x) - T_k(y) = DT_k(z)(x - y)
\]
for some \( z \in \{tx + (1-t)y : t \in [0,1]\} \subset B_\varepsilon(r) \). Note that \( r \frac{(x-y)}{\|x-y\|} \in B(r) \) thus for (15)

\[
|T_k(x) - T_k(y)| = \left| DT_k(z) \frac{r(x-y)}{\|x-y\|} \right| \frac{1}{r} \|x-y\| \leq \frac{Z_k(r)}{r} \|x-y\|_s \tag{16}
\]

The triangular inequality applied component-wise gives

\[
|T_k(x) - \bar{x}k| \leq_{cw} |T_k(x) - T_k(\bar{x})| + |T_k(\bar{x}) - \bar{x}k| \leq_{cw} Y_k + Z_k(r)
\]

hence

\[
|T_k(x) - \bar{x}k|_{\infty} \leq |Y_k + Z_k(r)|_{\infty}.
\]

Therefore for any \( x \in B_\varepsilon(r) \)

\[
\|T(x) - \bar{x}\|_s = \sup_{k \geq 0} \{|T_k(x) - \bar{x}k|_{\infty} w_k^s\} \leq \sup_{k \geq 0} \{|Y_k + Z_k(r)|_{\infty} w_k^s\} = \|Y + Z(r)\|_s < r.
\]

This proves \( i \).

Again from (16), for any \( x, y \in B_\varepsilon(r) \),

\[
|T_k(x) - T_k(y)|_{\infty} \leq \frac{|Z_k(r)|_{\infty}}{r} \|x-y\|_s,
\]

thus

\[
\|T(x) - T(y)\|_s \leq \frac{\|Z(r)\|_s}{r} \|x-y\|_s. \tag{17}
\]

Note that all the entries of \( Y_k \) and \( Z_k(r) \) are non negative, thus \( |Z_k(r)|_{\infty} \leq |Y_k + Z_k(r)|_{\infty} \) and \( \|Z(r)\|_s \leq \|Y + Z(r)\|_s < r \). That implies that

\[
\kappa := \frac{\|Z(r)\|_s}{r} \in (0,1),
\]

and we can conclude the proof of \( ii \). An application of the Banach Fixed Point Theorem on the Banach space \( B_\varepsilon(r) \) gives the existence and unicity of a solution \( x^* \) of the equation \( T(x) = x \) in \( B_\varepsilon(r) \) and, from Lemma 2.3, of a solution of \( f(x) = 0 \).

\section{2.4 The radii polynomials}

As already mentioned in Section 1, the radii polynomials are a set of \( r \)-dependent polynomials \( p_k(r) \) defined in such a way that if \( r^* \) is a common solution of \( p_k(r^*) < 0 \), then the ball \( B_\varepsilon(r^*) \subset \Omega^* \) of radius \( r^* \) centered at the numerical approximation \( r^* \) contains a unique solution of \( f(x) = 0 \). This is due to the fact that by construction of the polynomials, one has that \( \|Y + Z(r^*)\|_s < r^* \), meaning that the hypotheses of Theorem 2.4 are satisfied. In terms of the components, the formula \( \|Y + Z(r)\|_s \leq r \) reads as

\[
|Y_k + Z_k(r)|_{\infty} - \frac{r}{w_k^s} < 0, \quad \forall k \geq 0. \tag{18}
\]

The latter consists of a system of infinitely many inequalities, which is then impossible to be verify directly with computations. In order to reduce (18) to a finite number of inequalities, suppose that, for a given \( M \), there exist \( Y_M \) and \( Z_M(r) \) such that

\[
|(T(\bar{x}) - \bar{x}k)|_{\infty} \leq \frac{M^*}{k^s} Y_M, \quad \sup_{b_1, b_2 \in B(r)} \left| DT(\bar{x} + b_1 b_2)_{k} \right|_{\infty} \leq \frac{M^*}{k^s} Z_M(r), \quad \forall k \geq M, \tag{19}
\]

and introduce the set of \( M + 1 \) radii polynomials as follows.
Definition 2.5. The radii polynomials are defined as
\[
p_k(r) := Y_k + Z_k(r) - \frac{r}{w_k^m}(I_n, I_n), \quad k = 0, \ldots, M - 1
\]
\[
p_M := Y_M + Z_M - \frac{r}{w_M^m}.
\]

Theorem 2.6. Consider \( M \) and let \( Y, Z \) such that \( Y_k, Z_k \) satisfy (15) for \( k = 0, \ldots, M - 1 \) while for \( k \geq M \) define
\[
Y_k := \frac{M^s}{k^s} Y_M[I_n, I_n], \quad Z_k(r) := \frac{M^s}{k^s} Z_M[I_n, I_n],
\]
where \( Y_M, Z_M \) satisfy the tail condition (19). If there exists \( r > 0 \) such that \( p_k(r) < c_w 0 \) for all \( k = 0, \ldots, M, \) then there exists a unique \( x^* \in B_2(r) \) such that \( T(x^*) = x^* \) and \( f(x^*) = 0. \)

Proof. Since by definition \( Y_k \geq c_w 0, Z_k \geq c_w 0, \) the relations \( p_k(r) < c_w 0 \) imply that \( |Y_k + Z_k(r)|_\infty < \frac{r}{w_k^m} \) for \( k = 0, \ldots, M - 1. \) For \( k \geq M, Y_k, Z_k \) satisfy (15) and from \( Y_M + Z_M(r) - \frac{r}{w_M^m} < 0, \) it follows that \( |Y_k + Z_k(r)|_\infty - \frac{r}{w_k^m} < 0. \) Indeed
\[
|Y_k + Z_k(r)|_\infty = \frac{M^s}{k^s} (Y_M + Z_M) < \frac{M^s}{k^s} \frac{r}{M^s}, \quad \forall k \geq M.
\]

Hence
\[
\|Y + Z\|_s = \sup_{k \geq 0} \{|Y_k + Z_k| w_k^s\} < r,
\]
and the result follows from Theorem 2.4.

2.5 Construction of the bounds \( Y, Z \)

This section is devoted to the construction of the matrices \( Y_k, Z_k \) satisfying (15), and of the asymptotic bounds \( Y_M, Z_M \) satisfying (19). This construction provides the complete description of the radii polynomials introduced in Definition 2.5. With the aim of remaining as general as possible, the only constraint we assume on the \( \tau \)-periodic function \( \mathcal{A}(t) \) is that the vector of Fourier coefficients \( \mathcal{A} \) given in (5) satisfies \( \|A\|_s < \infty \) for \( s^* \geq 2. \) Nevertheless, further information on the coefficients \( \mathcal{A}_k \) may be useful to get sharper analytical estimates.

In what follows, the growth rate parameter \( s \) has been fixed so that \( 2 \leq s \leq s^* \), the finite dimensional parameter \( m \) has been chosen greater than \( K, \) where \( K \) is a lower bound given by Lemma 2.2 and the computational parameter \( M \) has been chosen so that \( M > m. \) Moreover, assume that one computed \( \Lambda_k^{-1} \) for \( k = m, \ldots, M - 1. \) Note that in some cases, it will be possible to achieve this task analytically, but in other cases, only an interval enclosure using rigorous numerics will be possible. Also, recalling Lemma 2.2, denote by \( C_\Lambda \) a computable constant such that
\[
\|\Lambda_k^{-1}\|_\infty \leq \frac{C_\Lambda}{k}, \quad \text{for} \quad k \geq m.
\]

2.5.1 The bound \( Y \)

By definition, \( T(\bar{x}) - \bar{x} = -\mathcal{A}(\bar{x}), \) thus define \( Y \) as
\[
Y_k = \begin{cases} 
|\mathcal{A}_m f^{(m)}(\bar{x})_k|, & k = 0, \ldots, m - 1 \\
|\Lambda_k^{-1} f_k(\bar{x})|, & k = m, \ldots, M - 1.
\end{cases}
\]
The tail bound $Y_M$ is defined so that $Y_M \geq |\Lambda_k^{-1} f_k(\bar{x})|\,\|\bar{x}\|_{\infty}$, for any $k > M$. We now introduce a coarse bound $Y_M$ based on the relation $|\Lambda_k^{-1} f_k(\bar{x})|\,\|\bar{x}\|_{\infty} \leq \|\Lambda_k^{-1}\|\,|f_k(\bar{x})|\,\|\bar{x}\|_{\infty}$. Since $\tilde{Q}_{k,1} = \tilde{Q}_{k,2} = 0$ for any $k \geq m$, it follows that

$$f_k(\bar{x}) = \begin{bmatrix} -\langle A, Q \rangle_{k,1} \\ -\langle A, Q \rangle_{k,2} \end{bmatrix} = \sum_{k_1+k_2=k} \begin{bmatrix} -Re\left(\Lambda_{k_1}(\tilde{Q}_{k_2,1} + i\tilde{Q}_{k_2,2})\right) \\ -Im\left(\Lambda_{k_2}(\tilde{Q}_{k_1,2} + i\tilde{Q}_{k_2,2})\right) \end{bmatrix}, \quad \forall k \geq M. \quad (23)$$

Now, using the fact that $|\Lambda_k| \leq \|A\|_{\|\cdot\|_s},\, w_k^{-s}$, both $|f_k(\bar{x})|$ and $|f_k(\bar{y})|$ are component-wise bounded by

$$\left|\sum_{k_1+k_2=k} A_{k_1}(\tilde{Q}_{k_2,1} + i\tilde{Q}_{k_2,2})\right| \leq \sum_{k_1+k_2=k, |k_2| < m} |A_{k_1}| |\tilde{Q}_{k_2,1} + i\tilde{Q}_{k_2,2}| \leq |A_k| |\tilde{Q}_0| + \sum_{l=1}^{m-1} |A_{k-l} + A_{k+l}| |\tilde{Q}_l_1 + i\tilde{Q}_l_2| \leq |A_k| |\tilde{Q}_0| + \sum_{l=1}^{m-1} w_k^l \left( \frac{1}{w_{k+l}} + \frac{1}{w_{k-l}} \right) W_n |\tilde{Q}_l_1 + i\tilde{Q}_l_2|.$$ 

For $k \geq M$ the bounds $w_k^l \leq 1$ and $w_k^l \left( \frac{1}{w_{k+l}} + \frac{1}{w_{k-l}} \right) \leq 1 + (1 - \frac{l}{M})^{-s}$ hold, thus one computes the matrix

$$W = I_n |\tilde{Q}_0| + \sum_{l=1}^{m-1} \left( 1 + \left( 1 - \frac{l}{M} \right)^{-s} \right) W_n |\tilde{Q}_l_1 + i\tilde{Q}_l_2|$$

so that

$$|f_k(\bar{x})| \leq k^{-s} \|A\|_{\|\cdot\|_s} |W|_{\infty}, \quad \text{for } k \geq M.$$ 

Finally, using $\|\Lambda_k^{-1}\|_{\infty} \leq \frac{C_A}{M}$, define

$$Y_M := \frac{1}{M^{s+1}} \|A\|_{\|\cdot\|_s} C_A |W|_{\infty}.$$

### 2.5.2 The bound $Z$

To construct the bound $Z$ so that

$$\sup_{b_1, b_2 \in B(r)} \left| \left\langle DT(\bar{x} + b_1) b_2 \right\rangle \right| \leq \|D T(\bar{x} + b_1) b_2 \|, \quad \forall k \geq 0,$$

it is convenient to factor the points $b_1, b_2 \in B(r)$ as $b_1 = ru, b_2 = rv$ with $u, v \in B(1)$, to expand in the variable $r$ and finally to uniformly bound the expression using the fact that $u, v \in B(1)$. Denote $u = [u_0, u_1, \ldots, u_{k-2}, \ldots, u_{k_2}]$, where each $u_k = (u_{k,1}, u_{k,2}) \in Mat(n, \mathbb{R})^2$. In order to simplify the exposition, both the matrices $u_{k,1}, u_{k,2}$ will be denoted as $u_k$. Indeed, what really matters is the bound $|u_{k,1}|, |u_{k,2}| \leq \|w_k\|^{-s}$ that finally will be applied to obtain the uniform estimates. The similar notation for $u_k$.

Let us introduce the linear operator $A^\dagger : \Omega^{s+1} \to \Omega^s$ defined as

$$(A^\dagger x)_k := \begin{cases} \left( Df^{(m)} \cdot x^m \right)_k & k = 0, \ldots, m-1 \\ A_k x_k, & k \geq m, \end{cases} \quad (24)$$
and consider the splitting
\[
DT(\bar{x} + ru) = [I - ADf(\bar{x} + ru)] rv = [I - AA^\dagger] rv - A[Df(\bar{x} + ru) - A^\dagger] rv.
\] (25)

The definition of \( Z(r) \) will follow as a result of different intermediate estimates: indeed we are going to introduce the vectors \( Z^0, Z^1, Z^2 \) such that
\[
\| [I - AA^\dagger] rv \| \leq_{cw} Z^0 r, \quad \forall v \in B(1),
\]
\[
\| [Df(\bar{x} + ru) - A^\dagger] rv \| \leq_{cw} Z^1 r + Z^2 r^2, \quad \forall u, v \in B(1).
\]

From (25) it follows that
\[
\| DT(\bar{x} + ru)rv \|_k \leq_{cw} \left( \| (I - AA^\dagger) rv \|_k + \| A(Df(\bar{x} + ru) - A^\dagger) rv \|_k \right),
\] (26)

Hence, the elements \( Z_k \), for \( k = 0, \ldots, M - 1 \) can be defined as
\[
Z_k(r) = \begin{cases} 
Z^0_k r + |A_m|(Z^1 r + Z^2 r^2)^m \| \ , & k = 0, \ldots, m - 1 \\
Z^0_k r + |\Lambda_k^{-1}|(Z^1_k r + Z^2 r^2), & k = m, \ldots, M - 1. 
\end{cases}
\]

Finally the element \( Z_M \) will be defined to satisfy (19).

The bound \( Z_0 \)

Since \( |v_k| \leq_{cw} \bar{w}^{-1}_k \), define \( Z^0 \) as
\[
(Z^0)_k = \begin{cases} 
\| I - AA^\dagger \| \{ w_j^{-1} 1 \}_{j=0}^{m-1} \| & k = 0, \ldots, m - 1 \\
0, & k \geq m
\end{cases}
\] (27)

so that
\[
\| I - AA^\dagger \| \leq_{cw} Z^0 r.
\]

Note that \( A^\dagger \) is an almost inverse of \( A \), indeed by definition \( A_m Df^{(m)} \approx I \). Then the size of \( Z^0 \) is small and depends on the accuracy of the numerical method that computes the inverse \( A_m \).

The bounds \( Z^1, Z^2 \)

Concerning the terms in \( Z^1, Z^2 \), consider the expansion as quadratic polynomial in \( r \)
\[
\| (Df(\bar{x} + ru) - A^\dagger) rv \|_k = \sum_{i=1,2} c_{k,i} r^i.
\] (28)

First note that
\[
(A \cdot Q)_{k,1} = \sum_{j+l=k} \left( Re(A_j)Q_{l,1} - Im(A_j)Q_{l,2} \right),
\]
\[
(A \cdot Q)_{k,2} = \sum_{j+l=k} \left( Im(A_j)Q_{l,1} + Re(A_j)Q_{l,2} \right),
\]

then, taking in mind that \( Q_{k,2} = -Q_{-k,2} \) and denoting with \( sg(l) = \text{sign}(l) \), one computes
\[
c_{0,1} = \left[ -\sum_{l+j=0 \atop |l| \geq m} \sum_{k \geq m} \left( Re(A_j) - sg(l)Im(A_j) \right) v_{j|l|} \right], \quad c_{0,2} = \left[ 0 \atop u_0 v_0 + v_0 u_0 \right]
\] (29)
for $k = 1, \ldots, m - 1$

$$c_{k,1} = -\sum_{l+j=k \atop |l| \geq m} \left[ \begin{array}{c} (\text{Re}(A_j) - \text{sgn}(l)\text{Im}(A_j)) v_{j|l|} \\ (\text{Im}(A_j) + \text{sgn}(l)\text{Re}(A_j)) v_{j|l|} \end{array} \right], \quad c_{k,2} = \left[ \begin{array}{c} u_k v_0 + v_k u_0 \\ u_k v_0 + v_k u_0 \end{array} \right].$$

(30)

and for $k \geq m$

$$c_{k,1} = -\sum_{l+j=k \atop |l| \neq k} \left[ \begin{array}{c} (\text{Re}(A_j) - \text{sgn}(l)\text{Im}(A_j)) v_{j|l|} \\ (\text{Im}(A_j) + \text{sgn}(l)\text{Re}(A_j)) v_{j|l|} \end{array} \right], \quad c_{k,2} = \left[ \begin{array}{c} u_k v_0 + v_k u_0 \\ u_k v_0 + v_k u_0 \end{array} \right].$$

(31)

Therefore $Z^1$, $Z^2$ need to be defined so that $Z^1_k \geq_{\text{cw}} |c_{k,1}|$ and $Z^2_k \geq_{\text{cw}} |c_{k,2}|$. To achieve this, it is enough to substitute in the above expression the bounds $|u_k|, |v_k| \leq_{\text{cw}} w^{-s}_k 1_n$ and $|\pm\text{Re}(A_j) \pm \text{Im}(A_j)| \leq_{\text{cw}} |\text{Re}(A_j)| + |\text{Im}(A_j)|$. Since $1_n 1_n = n 1_n$, one gets

$$|c_{0,2}| \leq_{\text{cw}} 2n \left[ \begin{array}{c} 0 \\ 1_n \end{array} \right] =: Z^2_0,$$

$$|c_{k,2}| \leq_{\text{cw}} 2n w^{-s}_k \left[ \begin{array}{c} 1_n \\ 1_n \end{array} \right] =: Z^2_k, \quad k \geq 1.$$

(32)

However this approach is not completely feasible for the computation of $|c_{k,1}|$, due to the presence of series. Therefore it is necessary to introduce further computational parameters

$$L_k > \max\{k, m\}$$

(33)

and matrices $H_k$ so that

$$|c_{0,1}| \leq_{\text{cw}} \sum_{L_0 \leq |l| \leq L_0} \left[ 2 \sum_{j=m}^{L_0} w^{-s}_j 1_n, \left( |\text{Re}(A_j)| + |\text{Im}(A_j)| \right) w^{-s}_l 1_n \right] + H_0 =: Z^1_0,$$

$$|c_{k,1}| \leq_{\text{cw}} \sum_{L_0 \leq |l| \leq L_0} \left[ \left( |\text{Re}(A_j)| + |\text{Im}(A_j)| \right) w^{-s}_l 1_n \right] + H_k =: Z^1_k, \quad k = 1, \ldots, m - 1$$

(34)

and similarly for $k \geq m$. It means that the bound $Z^1$ has been defined as sum of two factors: the first obtained by rigorous computation of a finite number of elements in the series, the second analytically defined to estimate the tail part of the series that have not been computed.

Define

$$\zeta(M, s) := \frac{1}{(M + 1)^s} + \frac{1}{(M + 2)^s} + \frac{1}{s - 1 (M + 2)^s - 1},$$

and

$$H_0 := \left[ \begin{array}{c} 2 \zeta(L_0, s) 1_n \\ h_0 1_n \end{array} \right], \quad H_k := h_k \left[ \begin{array}{c} 1_n \\ 1_n \end{array} \right],$$

(35)

where for $k \geq 0$

$$h_k = \frac{\sqrt{2n} \parallel A \parallel s^*}{(L_k + 1 - k)^{s^*}} \left( \zeta(L_k - k, 2s) + \zeta(L_k, 2s) \right).$$

Hence, one has the following result.
Lemma 2.7. Formula (34) holds for $H_0, H_k$ defined in (35).

Proof. First note that for any $M \geq 1$ and $s \geq 2$

$$
\sum_{k=M+1}^{\infty} \frac{1}{k^s} < \zeta(M, s). \quad (36)
$$

That can be seen from the fact that $\sum_{k=M+1}^{\infty} \frac{1}{k} = \frac{1}{(M+1)^s} + \frac{1}{(M+2)^s} + \sum_{k=M+4}^{\infty} \frac{1}{k} < \frac{1}{(M+1)^s} + \frac{1}{(M+2)^s} + \int_{M+2}^{\infty} k^{-s} dk$. Hence, one has that

$$
\left\| 2 \sum_{j=L_0+1}^{\infty} w_j^{-s} \mathbb{I}_n \right\| \leq \text{cw} \left\| \sum_{j=L_0+1}^{\infty} w_j^{-s} \mathbb{I}_n \right\|.
$$

For the remaining terms note that $(|Re(A_j)| + |Im(A_j)|) \leq \text{cw} \sqrt{2} |A_j| \leq \text{cw} \sqrt{2} \|A\| \mathbb{I}_n$, then, for any $k \geq 0$, the tail part of the series can be bounded by

$$
\left\| \sum_{l=|t|=L_{k+1}}^{\infty} \left( |Re(A_j)| + |Im(A_j)| \right) w_l^{-s} \mathbb{I}_n \right\| \leq \text{cw} \sqrt{2} \|A\| \mathbb{I}_n \sum_{l=L_{k+1}}^{\infty} \left( \frac{1}{w_l^{-s}} + \frac{1}{w_{l+1}^{-s}} \right) \mathbb{I}_n w_l^{-s} \mathbb{I}_n
$$

In the last passage we have used the fact that $L_k > k$, $s^* \geq s$ and the relation $\mathbb{I}_n \mathbb{I}_n = n \mathbb{I}_n$. The result follows by applying (36) once the last series has be rewritten as

$$
\sum_{l=L_{k+1}}^{\infty} \left( \frac{1}{w_l^{-s}} + \frac{1}{w_{l+1}^{-s}} \right) w_l^{-s} = \sum_{l=L_{k+1}}^{\infty} \left( \frac{1}{w_l^{-s}} \right) w_l^{-s} + \sum_{l=L_{k+1}+2}^{\infty} \left( \frac{1}{w_l^{-s}} \right) w_l^{-s} \leq \sum_{l=L_{k+1}}^{\infty} w_l^{-2s} + \sum_{l=L_{k+1}+2}^{\infty} w_l^{-2s}.
$$

The bound $Z_M$

From [23], one has that

$$
\sum_{k_1+k_2=k, k_1 \neq k} \frac{1}{w_{k_1} w_{k_2}} \leq \frac{1}{w_k^2} \left[ 2 + \sum_{l=1}^{M} \frac{1}{l^s} + \frac{2}{M^{s-1}(s-1)} + \eta_M - 1 - \frac{1}{w_{2k}^2} \right],
$$

where

$$
\eta_k = 2 \left[ \frac{k}{k-1} \right]^s + \left[ \frac{4 \log(k-2)}{k} + \frac{\pi^2 - 6}{3} \right] \left[ \frac{2}{k} + \frac{1}{2} \right]^{s-2}.
$$
Recall the definition of $c_{k,1}$ and $c_{k,2}$ given in (28), with a more explicit form in (31) for the case $k \geq m$. Then for $k \geq M$ one has that
\[
|c_{k,1}|_\infty \leq \sqrt{2}\|A\|_s^r \sum_{l+j=k, l \neq k} w_j^r w_l^r \leq \sqrt{2}\|A\|_s^r \sum_{l=1}^M \frac{1}{l^s} + \frac{2}{M^{s-1}(s-1)} + \eta_M - 1 =: \sqrt{2}\|A\|_s^r C_1,
\]
\[
|c_{k,2}|_\infty \leq \frac{2n}{w_k^r}.
\]
Since for $k \geq M$ the first term on the right hand side of (26) is zero, the following estimate holds
\[
\left|DT(\bar{x} + ru)rv\right|_k \leq c_w \left|A \left(Df(\bar{x} + ru) - A^1\right)rv\right|_k \leq \|\Lambda_k^{-1}\|_\infty (|c_{k,1}|_\infty r + |c_{k,2}|_\infty r^2).
\]
Finally, combining (21) and that $k \geq M$, one gets that $\|\Lambda_k^{-1}\|_\infty \leq \frac{C_2}{M}$, and we can define $Z_M$ as
\[
Z_M = \frac{CA}{M^2}(\sqrt{2}\|A\|_s^r C_1 r + 2nr^2).
\]

3 Computing stable and unstable tangent bundles of periodic orbits using Floquet normal forms

Consider an autonomous differential equation
\[
\dot{y} = g(y), \quad g \in C^1(\mathbb{R}^n)
\]
and suppose that $\gamma(t)$ is a $\tau$-periodic solution with $\gamma(0) = \gamma_0$. Denote by $\Gamma = \{\gamma(t), t \in [0, \tau]\}$ the support of $\gamma$ and for any $\theta \in [0, \tau]$, define $\gamma_\theta(t) = \gamma(t + \theta)$ the phase-shift re-parametrization of $\Gamma$. Being autonomous, system (38) has the property that any of the curves $\gamma_\theta(t)$ is a $\tau$-periodic solution satisfying $\gamma_\theta(0) = \gamma(\theta)$. We refer to $\Gamma$ as the periodic orbit and $\gamma_\theta$ as the periodic solutions.

**Definition 3.1** (Monodromy matrix). Let $\gamma : \mathbb{R} \rightarrow \mathbb{R}^n$ be a $\tau$-periodic solution (38) and let $\Phi_\theta(t)$ be the unique solution of the non-autonomous linear problem
\[
\begin{cases}
\dot{\Phi}_\theta = \nabla g(\gamma_\theta(t))\Phi_	heta \\
\Phi_\theta(0) = I.
\end{cases}
\]

The matrix $\Phi_\theta(\tau)$ is called the *monodromy matrix* of $\gamma_\theta(t)$.

Having chosen $\gamma(t) = \gamma_0(t)$, in the following we identify $\Phi(\tau) = \Phi_\theta(\tau)$. The next two Lemmas are classical results and are direct consequence of $\Phi_\theta(t)$ being a fundamental matrix solution. For sake of completeness, we present their proofs.

**Lemma 3.2.** For any $\theta \in [0, \tau]$, the solution $\Phi_\theta(t)$ of (38) satisfies
\[
\Phi_\theta nt + t = \Phi_\theta(t)\Phi_\theta(\tau)^n, \quad \forall t \in \mathbb{R}, \quad \forall n \in \mathbb{N}.
\]
Proof. Without loss of generality let us consider \( \theta = 0 \). By induction on \( n \geq 0 \). For \( n = 0 \) the result is obvious. Suppose it holds for \( n - 1 \). Then
\[
\Phi(n\tau) = \Phi((n - 1)\tau + \tau) = \Phi(\tau)\Phi(\tau)^{n-1} = \Phi(\tau)^n.
\]
Define
\[
\Psi(t) = \Phi(t + n\tau)\Phi(n\tau)^{-1}.
\]
It follows that \( \Psi(0) = I \) and that
\[
\dot{\Psi}(t) = \dot{\Phi}(n\tau + t)\Phi(n\tau)^{-1} = A(n\tau + t)\Phi(n\tau + t)\Phi(n\tau)^{-1} = A(t)\Psi(t)
\]
For the uniqueness of solutions of the initial value problem, \( \Psi(t) = \Phi(t) \) thus
\[
\Phi(t + n\tau) = \Phi(t)\Phi(n\tau) = \Phi(t)\Phi(\tau)^n, \ \forall t \in \mathbb{R}.
\]
\[\square\]

**Lemma 3.3.** The matrices \( \Phi_\theta(\tau) \) are equivalent under conjugation. In particular
\[
\Phi_\theta(\tau) = \Phi(\theta)\Phi(\tau)\Phi(\theta)^{-1}.
\] (40)

**Proof.** The matrix \( \tilde{\Phi}(t) := \Phi(t + \theta) \) is solution of the equation \( \dot{y} = \nabla g(\gamma(t+\theta))y = \nabla g(\gamma_\theta(t)) \), with \( \tilde{\Phi}(0) = \Phi(\theta) \). Since \( \Phi_\theta(t) \) is the principal fundamental solution of the previous system,
\[
\tilde{\Phi}(t) = \Phi_\theta(t)\Phi(\theta).
\]
It follows
\[
\Phi_\theta(t) = \tilde{\Phi}(t)\Phi(\theta)^{-1} = \Phi(t + \theta)\Phi(\theta)^{-1}, \ \forall t.
\] (41)
Thus
\[
\Phi_\theta(\tau) = \Phi(\tau + \theta)\Phi(\theta)^{-1} = \Phi(\theta)\Phi(\tau)\Phi(\theta)^{-1}
\]
where, in the last passage, Lemma 3.2 has been used. \[\square\]

The previous result implies that all monodromy matrices \( \Phi_\theta(\tau) \) have the same eigenvalues. That motivates the following definition.

**Definition 3.4.** The eigenvalues \( \sigma_j \) of the monodromy matrix \( \Phi(\tau) \) are called the *Floquet multipliers* of the periodic orbit \( \Gamma \).

As already mentioned in Section 1, in the theory of dynamical systems the monodromy matrix \( \Phi(\tau) \) associated to a periodic solution \( \gamma(t) \) plays a fundamental role since it encompasses the information about the stability character of \( \gamma \). Indeed, as shown in Proposition 2.122 in [2], the Floquet multipliers of \( \gamma(t) \) are in fact the eigenvalues of \( DP(\gamma(0)) \), where \( P(x) \) denotes the Poincaré map of \( \gamma(t) \) on a \((n-1)\)-dimensional hypersurface transversal to \( \gamma \) at \( \gamma(0) \). Moreover, it can be proved that at least one of the Floquet multipliers \( \sigma_j \) of \( \Phi(\tau) \) is equal to one, corresponding to the eigenvector \( \dot{\gamma}(0) \). Hence, we will denote by \( \sigma_n = 1 \) the Floquet multiplier corresponding to \( \dot{\gamma}(0) \) and denote by \( \{\sigma_j\}_{j=1,...,n-1} \) the set of non trivial Floquet multipliers. We refer to Section 2.4 in [2] for a more extensive analysis of the links between Poincaré sections and Floquet theory. Based on the above discussion, we are now ready to introduce the definition of stability of a periodic orbit.

**Definition 3.5.** Let \( \Gamma = \{\gamma(t), t \in [0, \tau]\} \) be a \( \tau \)-periodic orbit of the system (38) and let \( \{\sigma_j\}_{j=1,...,n-1} \) be the corresponding set of non trivial Floquet multipliers. We say that
• $\Gamma$ is stable if $\forall j \in \{1, \ldots, n-1\}, |\sigma_j| < 1$;

• $\Gamma$ is unstable if $\exists j \in \{1, \ldots, n-1\}$ such that $|\sigma_j| > 1$.

Moreover, if $p < n-1$ Floquet multipliers have modulus less than one, and $q < n-p$ Floquet multipliers have modulus greater than one, $\Gamma$ is said to have $p$ stable directions and $q$ unstable directions.

Let us mention that there is a variant to the Floquet normal form introduce in Theorem 1.1, namely there exist a constant (possibly complex) matrix $B$ such that $\Phi(\tau) = e^{B\tau}$. We refer to Theorem 2.83 in [2] for more details and for the proof. Therefore, there exists a (possibly complex) matrix $B$ such that $\Phi(\tau) = e^{B\tau}$. Denoting by $\lambda_j$ the eigenvalues of $B$, it follows that $\sigma_j = e^{\tau\lambda_j}$ is a Floquet multiplier. Note that for a given $\sigma_j$, the solution $\lambda_j$ of $\sigma_j = e^{\tau\lambda_j}$ is not uniquely defined. Indeed for any $k \in \mathbb{Z}$, $e^{(\lambda_j + 2\pi k)\tau} = \sigma_j$. This reflects the fact that in the complex Floquet normal form $\Phi(t) = P(t)e^{Bt}$, the matrix $B$ is also not uniquely defined. In the literature it is common to call a Floquet exponent associated to $\sigma_j$ any complex number $\lambda_j$ so that $\sigma_j = e^{\tau\lambda_j}$. On the converse, for any $\sigma_j$ there is a unique real number $l_j$ so that $|\sigma_j| = e^{l_j\tau}$. That motivates the following definition.

**Definition 3.6.** A Lyapunov exponent associated to a Floquet multiplier $\sigma_j$ is the unique real number $l_j$ so that $|\sigma_j| = e^{l_j\tau}$.

Note that using the notion Lyapunov exponents, a definition of stability of a periodic orbit similar to the one of Definition 3.5 can be introduced. Indeed, given a $\tau$-periodic orbit $\Gamma = \{\gamma(t), t \in [0, \tau]\}$ of (38) and considering $\{l_j\}_{j=1,\ldots,n-1}$ to be the corresponding set of non trivial Lyapunov exponents, we say that $\Gamma$ is stable if $l_j < 0$, $\forall j = 1, \ldots, n-1$ and that $\Gamma$ is unstable if there exists $j \in \{1, \ldots, n-1\}$ such that $l_j > 0$.

Given a real $n \times n$ diagonalizable matrix $A$, let us introduce the notation $\Sigma(A) = \{\alpha_k, v_k\}_{k=1,\ldots,n}$ to denote the eigendecomposition of the square matrix $A$, i.e. $Av_k = \alpha_k v_k$, for all $k = 1, \ldots, n$.

The following result shows how the information from the couple $(R, Q(t))$ coming from the Floquet normal form $\Phi(t) = Q(t)e^{Rt}$ can directly be used to study the dynamical properties of the periodic orbit $\Gamma$. More explicitly, it demonstrates that the stability of $\Gamma$ can be determined by the eigenvalues of $R$ while the stable and unstable tangent bundles of $\Gamma$ can be retrieved from the action of $Q(t)$ (with $t \in [0, \tau]$) on the eigenvectors of $R$.

**Theorem 3.7.** Assume that $\Gamma = \{\gamma(t), t \in [0, \tau]\}$ is a $\tau$-periodic orbit of (38) and consider $\Phi(t)$ the fundamental matrix solution of the non-autonomous linear equation $\dot{y} = \nabla g(\gamma(t))y$ such that $\Phi(0) = I$. Suppose that a Floquet normal form decomposition of Theorem 1.1) $\Phi(t) = Q(t)e^{Re^{Rt}}$ is known. Assume that the real $n \times n$ matrix $R$ is diagonalizable and let $\Sigma(R) = \{\mu_j, v_j\}_{j=1,\ldots,n}$ the eigendecomposition of $R$. Then the Lyapunov exponents $l_j$ of $\Gamma$ are given by

$$l_j = Re(\mu_j).$$

Furthermore, for any $\theta \in [0, \tau]$, if one defines

$$w^\theta_j := Q(\theta)v_j,$$

then $w^\theta_j$ is an eigenvector of $\Phi_\theta(\tau)$ associated to the Lyapunov exponent $l_j$. Note that $w^\theta_j$ is a smooth $2\tau$-periodic function of $\theta$. 

18
Proof. Consider the eigendecomposition $\Sigma(R) = \{\mu_j, v_j\}_{j=1,...,n}$ of the diagonalizable matrix $R$, meaning that the set $\{v_1, \ldots, v_n\}$ consists of $n$ linearly independent eigenvectors of $R$. By Lemma 3.2, one has that $\Phi(\tau)^2 = \Phi(2\tau)$. Since $Q(t)$ is $2\tau$-periodic and $Q(0) = I$, it follows that $\Phi(2\tau) = e^{R2\tau}$. Since $R$ is diagonalizable, $\Phi(2\tau) = e^{R2\tau}$ is also diagonalizable. Since $\Phi(2\tau) = \Phi(\tau)^2$ and since the matrix $\Phi(\tau)$ is invertible and defined over the field of complex number (which has zero characteristic), then it can then be showed that $\Phi(\tau)$ is also diagonalizable. Now, since $\Phi(2\tau)$ is invertible and defined over the field of complex number (which has zero characteristic), then it can then be showed that $\Phi(\tau)$ is also diagonalizable. Now, since $\Phi(2\tau) = \Phi(\tau)^2$ one has that if $(\sigma, w) \in \Sigma(\Phi(\tau))$, then $(\sigma^2, w) \in \Sigma(\Phi(2\tau))$. Combining this last point with $\Phi(\tau)$, $\Phi(2\tau)$ being diagonalizable implies that the eigenspaces of $\Phi(\tau)$ and $\Phi(2\tau)$ are in one-to-one correspondence. That implies the existence of a set $\{\sigma_j\}_{j=1,...,n}$ such that $\Sigma(\Phi(\tau)) = \{\sigma_j, v_j\}_{j=1,...,n}$. From the property of the exponential matrix operator, $\Sigma(\Phi(2\tau)) = \{e^{\mu_j 2\tau}, v_j\}_{j=1,...,n} = \Sigma(\Phi(\tau)^2) = \{\sigma_j^2, v_j\}_{j=1,...,n}$.

This implies that $\sigma_j^2 = e^{\mu_j 2\tau}$ for any $j = 1, \ldots, n$. Note that $l_j = Re(\mu_j)$ is the unique real number so that $|\sigma_j| = e^{l_j \tau}$. Hence, $l_j$ is a Lyapunov exponent associated to the Floquet multipliers $\sigma_j$.

Now, from (41), one has that

$$\Phi_\theta(2\tau) = \Phi(2\tau + \theta)\Phi(\theta)^{-1} = Q(\theta)e^{(2\tau+\theta)R}e^{-R\theta}Q(\theta)^{-1}, \quad \forall \theta \in [0, \tau]$$

thus

$$\Phi_\theta(2\tau)Q(\theta)v_j = Q(\theta)e^{2\tau R}v_j = e^{2\tau \mu_j}Q(\theta)v_j$$

showing that $\Sigma(\Phi_\theta(2\tau)) = \{e^{2\tau \mu_j}, Q(\theta)v_j\}$. Applying the same argument than above, one can conclude that $\Sigma(\Phi_\theta(\tau)) = \{\sigma_j, Q(\theta)v_j\}_{j=1,...,n}$ forms an eigendecomposition of the matrix $\Phi_\theta(\tau)$. Hence, $w_j^\theta = Q(\theta)v_j$ is an eigenvector of $\Phi_\theta(\tau)$. By the smoothness and the $2\tau$-periodicity of the matrix function $Q(\theta)$, one can conclude that $w_j^\theta = Q(\theta)v_j$ is also a smooth $2\tau$-periodic function of $\theta$.

Recall (43) and consider $w_j^\theta = a_j^\theta + ib_j^\theta$. We define the stable and unstable subspaces $E_s^\theta, E_u^\theta \subset \mathbb{T}_{\gamma(\theta)}\mathbb{R}^n$ of the periodic orbit $\Gamma$ at the point $\gamma(\theta)$ as

$$E_s^\theta = \text{Span}\{a_j^\theta, b_j^\theta : |\sigma_j| < 0\}$$

$$E_u^\theta = \text{Span}\{a_j^\theta, b_j^\theta : |\sigma_j| > 0\}.$$

That allows us to define the following

**Definition 3.8.** We define the stable and unstable tangent bundles of $\Gamma$ respectively by

$$E_s, E_u \subset \mathbb{T}_\Gamma\mathbb{R}^n$$

$$E_s = \bigcup_{\theta \in [0, \tau]} \{\gamma(\theta)\} \times E_s^\theta, \quad E_u = \bigcup_{\theta \in [0, \tau]} \{\gamma(\theta)\} \times E_u^\theta.$$

It is important to remark that from the conclusion of Theorem 3.7, the complete structure of the stable and unstable bundles can be recovered by the action of the matrix function $Q(t)$ on the eigenvectors of $R$, which themselves correspond to the stable and unstable directions at the point $\gamma(0)$ on $\Gamma$. Also, the proof of Theorem 3.7 is constructive in the sense that combined with the rigorous computational method of Section 2, it provides a computationally efficient direct way to obtain the eigenvectors $w_j^\theta$ of $\Phi_\theta(\tau)$, which are the ingredients defining the bundles of Definition 3.8. Note that one could be tempted to use the fact that $\Phi(\tau) = Q(\tau)e^{R\tau}$ and then attempt to compute the eigendecomposition of $\Phi(\tau)$ directly. However, that would imply having to compute the exponential of an
interval valued matrix, which turns out to be a difficult task (e.g., see [32], [33]). This being said, the rigorous computation of the eigendecomposition of the interval matrix $R$ is not completely straightforward. We addressed this problem by adapting the computational method based on the radii polynomials in order to enclose all the solution $\{\mu_k, v_k\}$ of the nonlinear problem $(R - \mu I)v = 0$ with constrain $|v|^2 = 1$. Further details on the enclosure of the eigendecomposition of interval matrices are postponed on a future work of the same authors [34].

4 Applications

In this section, we present some applications, where we construct rigorously tangent stable and unstable bundles of some periodic orbits of the Lorenz equations in Section 4.1 and of the $\zeta^3$-model in Section 4.2. Note that all rigorous computations were performed in Matlab with the interval arithmetic package Intlab [30].

4.1 Bundles of periodic orbits in the Lorenz equations

Consider the following three dimensional system of ODEs, known as the Lorenz equations

$$\begin{cases}
\dot{u}_1 = \sigma (u_2 - u_1) \\
\dot{u}_2 = \rho u_1 - u_2 - u_1 u_3 \\
\dot{u}_3 = u_1 u_2 - \beta u_3
\end{cases} \quad (44)$$

with the classical choice of parameters $\beta = 8/3, \sigma = 10$ and $\rho$ left as a bifurcation parameter. Suppose to have rigorously proved the existence of a real $\tau_\gamma$-periodic solution $\gamma(t) = [\gamma^1, \gamma^2, \gamma^3](t)$ of (44) in the form

$$\gamma^j(t) = \sum_{k \in \mathbb{Z}} \xi_k^j e^{ik \frac{2\pi}{\tau_\gamma} t}, \quad j = 1, 2, 3 \quad (45)$$

in a ball of radius $r_\gamma$ and centered at $[\bar{\tau}_\gamma, \bar{\xi}_k]$, $|k| \leq M_\gamma$, with respect to the $\Omega^{s*}$ norm, meaning that

$$|\tau_\gamma - \bar{\tau}_\gamma| \leq r_\gamma$$

$$|\Re(\xi_k) - \Re(\bar{\xi}_k)|_\infty \leq r_\gamma w_k^{-s*}, \quad |\Im(\xi_k) - \Im(\bar{\xi}_k)|_\infty \leq r_\gamma w_k^{-s*}, \quad |k| = 0, \ldots, M_\gamma \quad (46)$$

for a decay rate $s^* \geq 2$. Note that $\xi_k \in \mathbb{Z}^3$ and $\xi_{-k} = \mathcal{C}(\xi_k)$. The existence of such solution could be achieved by applying a modified version of the method discussed in the previous section. Even with some technical differences, the philosophy is the same. Rewrite the system of ODEs as a infinite dimensional algebraic system where $\tau_\gamma$ and the Fourier coefficients $\xi_k$ are the unknowns, then consider a finite dimensional projection and compute a numerical approximate solution $\bar{\tau}_\gamma, (\bar{\xi}_k)_k$. Then, by means of the radii polynomials, prove the existence, in a suitable Banach space, of a genuine solution $\tau_\gamma, (\xi_k)_k$ of the infinite dimensional problem in a small ball containing the approximate solution. Note that this is not the first time that the radii polynomials are used to prove existence of periodic solutions of differential equations (e.g., see [16], [24], [26], [27], [25]).

In the following we aim to combine the rigorous computational method of Section 2 together with Theorem 3.7 to rigorously compute the stable and unstable tangent bundles of
the periodic orbit \( \gamma(t) \) given by (45). This first requires the computation of the fundamental matrix solution of the linearized system along \( \gamma(t) \), that is the solution for \( t \in [0, \tau_\gamma] \) of the non-autonomous system

\[
\begin{align*}
\dot{\Phi} &= \nabla g(\gamma(t))\Phi \\
\Phi(0) &= I
\end{align*}
\]  

(47)

where \( g \) is the right hand side of (44), \( \nabla g \) denotes the Jacobian of the right hand side of system (44) and \( I \) is the \( 3 \times 3 \) identity matrix. The former system is nothing more than a particular case of (1), where \( A(t) = \nabla g(\gamma(t)) \) and \( n = 3 \). We now apply the computational method presented in Section 2 to compute the principal fundamental matrix solution of the non-autonomous linear system \( \dot{y} = \nabla g(\gamma(t))y \). In particular a constant matrix \( R \) and the Fourier coefficients \( Q_k \) of a \( 2\tau_\gamma \)-periodic function \( Q(t) \) will be computed, so that

\[
\Phi(t) = Q(t)e^{Rt}
\]

is the unique solution of (47). Once the computation of \( R \) and the \( Q_k \) is done, following the conclusion of Theorem 3.7, we will compute \( \Sigma(\gamma) = \{ (\mu_j, v_j) \mid j = 1, \ldots, n \} \), derive from the Lyapunov exponents \( l_j := \text{Re}(\mu_j) \) the stability of the periodic orbit \( \Gamma \) and from the eigenvectors \( \{ v_1, \ldots, v_n \} \) of \( R \) we will construct the tangent bundles as defined in Definition 3.8 and given by the formula (43).

**Computation of \( R \) and \( Q_k \)**

To begin with, let us explicitly write the Jacobian

\[
\nabla g(u) = \begin{bmatrix}
-\sigma & \sigma & 0 \\
\rho - u^3 & -1 & -u^1 \\
u^2 & u^1 & -\beta
\end{bmatrix}
\]

and, as consequence, the coefficients \( A_k \)

\[
A_0 = \begin{bmatrix}
-\sigma & 0 \\
\rho - \xi_0^3 & -1 \\
\xi_0^2 & \xi_0^1 & -\beta
\end{bmatrix}, \quad A_k = \begin{bmatrix}
0 & 0 & 0 \\
0 & -\xi_k^3 & -\xi_k^1 \\
\xi_k^2 & \xi_k^1 & 0
\end{bmatrix}, \quad k \geq 1.
\]

The hypothesis (46) for \( \xi_k \) to lie in a ball centered at \( \xi_k \) implies that \( \| A_k \|_* < \infty \). Although this bound is sufficient to proceed with the computational process, we want to stress out that precise informations are known about the \( |A_k|_\infty \) of the tail elements of the sequence \( \{ A_k \} \). Indeed it can be easily seen that

\[
|A_k|_\infty \leq \sqrt{2\tau_\gamma} \frac{1}{w_k}, \quad \forall k > M_\gamma.
\]

(48)

The computation of the approximate solution \( \bar{R}, \bar{Q}_{k,1}, \bar{Q}_{k,2} \) has been addressed as follow: consider the approximation \( \bar{\gamma}(t) = \sum_{|k| \leq M_\gamma} \xi_k e^{ik2\pi t/\tau_\gamma} \) of the periodic orbit \( \gamma(t) \) and numerically solve system (47) up to time \( 2\tau_\gamma \). Denote by \( \bar{y}(2\tau_\gamma) \) the obtained result and numerically compute

\[
\mathcal{R} = \log(\bar{y}(2\tau_\gamma)).
\]

Neglect the imaginary part and consider only the real part. Then numerically integrate the system (3) up to time \( 2\tau_\gamma \) with \( \mathcal{R} \) in place of \( R \) yielding the solution \( Q(t_j) \). Fix the positive finite dimensional parameter \( m \) and compute from \( Q(t_j) \) the matrices \( Q_{k,1}, Q_{k,2} \), respectively the real and imaginary part of the Fourier coefficients with \( |k| < m \). Finally the vector \( (\mathcal{R}, Q_k) \) is considered as starting point for a Newton iteration scheme applied on the
finite dimensional reduction defined generally in (11). Denote the output of the iterative process by \( \tilde{x} = (\tilde{R}, \tilde{Q}_k) \), that is an approximated solution \( f^{(m)}(\tilde{x}) \approx 0 \) up to a desired accuracy, where \( f^{(m)} \) is defined in (11).

Consider \( \Lambda_k \) given by (13). Note that in the case of the three-dimensional vector field (44), \( \Lambda_k \) is a 6 × 6 matrix and one could compute its inverse analytically using the mathematical software Maple. After having computed \( \Lambda_k^{-1} \) one needs to check that the chosen \( m \) satisfies \( m > K \) where \( K \) is the same as in Lemma 2.2, otherwise increase \( m \).

Then for a choice of \( M > m \) and \( 2 \leq s \leq s^* \) one can compute the coefficients \( Y_k \) and \( Z_k \), \( k = 0, \ldots, M \) and \( Y_M, Z_M \) as shown in Section 2.5. It only remains to define the computational parameters \( L_k \) introduce in (33). In the computation presented here \( L_k \) has been chosen as

\[
L_k = \max\{M + M_\gamma\} + k.
\]

This choice assures that the tail elements \( H_0, H_k \) in (34) only contain the terms \( A_j \) satisfying \( |A_j|_\infty \leq \sqrt{2} r_\gamma w_j^{-s^*} \). Therefore the subsequent estimate for \( h_k \) can be improved by replacing \( \|A\|_{s^*} \) with \( r_\gamma \).

Again, the knowledge of the particular behavior of the coefficients \( A_k \) allows to provide a better estimate for \( Z_M \). Indeed note that \( |Re(A_k)| \leq_{cw} |Re(\tilde{A}_j)| + w_j^{-s^*} 1_n \), where \( \tilde{A}_k \) denotes the matrix \( A_k \) with the entries \( \tilde{c} \) in place of \( \xi \) and the same holds for \( |Im(A_k)| \). Therefore \( |Re(A_k)|_\infty + |Im(A_k)|_\infty < \sqrt{2} |\xi|_\infty + 2 r_\gamma w_j^{-s^*} \) for \( 1 \leq |k| \leq M_\gamma \) and (48) for \( |k| > M_\gamma \).

Therefore, the computation of the bound for \( |c_{k,1}|_\infty \) when \( k \geq M \), necessary for the definition of \( Z_M \), has been slightly modified as follows.

\[
|c_{k,1}| = \left| \sum_{l+j=k \atop |l| \neq k} (|Re(A_j)| + |Im(A_j)|) w_l^{-s} 1_n \right| \leq_{cw} \sum_{l+j=k \atop j \neq 0} (|Re(A_j)| + |Im(A_j)|) w_l^{-s} 1_n + 2 r_\gamma \sum_{l+j=k \atop |l| \neq k} w_j^{-s} w_l^{-s} 1_n 1_n.
\]

Then, passing to the infinity absolute value, for any \( k \geq M \)

\[
|c_{k,1}|_\infty \leq_{cw} 3 \sqrt{2} \sum_{j=1}^{M_\gamma} |\bar{x}|_\infty (w_k^{-s} + w_k^{-s}) + 2 n r_\gamma \sum_{l+j=k \atop |l| \neq k} w_j^{-s} w_l^{-s}
\]

\[
\leq_{cw} \frac{3}{k^s} \left[ \sqrt{2} \sum_{j=1}^{M_\gamma} |\bar{x}|_\infty (w_k^{-s} + w_k^{-s}) + 2 r_\gamma \sum_{l=1}^{M} \frac{1}{l} + 2 \frac{2}{M^{s-1} (s-1)} + \eta_M \right]
\]

\[
\leq_{cw} \frac{3}{k^s} \left[ \sqrt{2} \sum_{j=1}^{M_\gamma} |\bar{x}|_\infty \left( \frac{1}{(1 - \frac{j}{M})^s} + 1 \right) + 2 r_\gamma \sum_{l=1}^{M} \frac{1}{l} + 2 \frac{2}{M^{s-1} (s-1)} + \eta_M \right].
\]

**Computational results**

For the choice \( \sigma = 10, \beta = 8/3 \) it is known that there exists a branch of periodic solutions parametrized by \( \rho \) joining a Hopf bifurcation at \( \rho = \sqrt{79} / 19 \approx 24.736 \) and a homoclinic point at \( \rho \approx 13.9265 \). Figure 1(a-b) shows the bifurcation graph and some of the periodic orbit of the continuous family.

22
The computation of the rigorous enclosure of the periodic orbits and successively of their tangent bundles have been performed for a set of different periodic orbits of the Lorenz system lying on the mentioned bifurcation branch and corresponding to values of $\rho$

$$\rho_1 = 18.0815, \rho_2 = 18.6815$$
$$\rho_3 = 20.8815, \rho_4 = 23.8815, \rho_5 = 24.1816$$

Figure 1(c) reports the graphics of the numerical approximation $\bar{\gamma}_i$ of the orbits corresponding to the choice $\rho_i$, while the Table 1 contains the computational parameter $M_\gamma$ that have been chosen for the rigorous enclosure of the orbit, the period $\bar{\tau}_\gamma$ and the radius $r_\gamma$ resulting from the computations. The growth rate $s^*$ has been fixed $s^* = 2$ for all the cases.

<table>
<thead>
<tr>
<th>#</th>
<th>sol</th>
<th>$M_\gamma$</th>
<th>$\bar{\tau}_\gamma$</th>
<th>$r_\gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>32</td>
<td>1.027854</td>
<td>6.844864508150837e−09</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>30</td>
<td>0.978271</td>
<td>7.151582969846857e−09</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>26</td>
<td>0.822883</td>
<td>4.260379031142465e−09</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>20</td>
<td>0.683813</td>
<td>5.368959115576269e−09</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>30</td>
<td>0.672595</td>
<td>2.360935240171144e−08</td>
<td></td>
</tr>
</tbody>
</table>

Table 1: The $i$-th row concerns the computation of the periodic orbit for the Lorenz system corresponding to $\rho = \rho_i$. $M_\gamma$ is the finite dimensional reduction parameter chosen in the computation, $\bar{\tau}_\gamma$ the approximated period of the solution and $r_\gamma$ the resulting enclosing radius.

In the Appendix the first 15 Fourier coefficients of $\bar{\gamma}_1$ and $\bar{\gamma}_4$ are listed. As shown in Figure 2, one can notice that the Fourier coefficients of the five orbits under consideration are decaying to zero with a different speed. This is due to the fact that the closer we are to the
homoclinic orbit, the flatter the periodic solution is, meaning that a larger number of Fourier coefficients contributes to the Fourier expansion, hence leading to a slower decay. Table 2 contains information about the computation of the fundamental matrix solution associated to each of the previous periodic orbits. More precisely, it contains the finite dimensional reduction parameter $m$, the computational parameter $M$ and the resulting radius $r$.

<table>
<thead>
<tr>
<th># sol</th>
<th>$m$</th>
<th>$M$</th>
<th>$r$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>100</td>
<td>180</td>
<td>$1.98645943e - 05$</td>
</tr>
<tr>
<td>2</td>
<td>90</td>
<td>140</td>
<td>$7.52145121e - 06$</td>
</tr>
<tr>
<td>3</td>
<td>80</td>
<td>80</td>
<td>$9.66152623e - 07$</td>
</tr>
<tr>
<td>4</td>
<td>60</td>
<td>66</td>
<td>$9.91268997e - 07$</td>
</tr>
<tr>
<td>5</td>
<td>60</td>
<td>70</td>
<td>$3.77687574e - 06$</td>
</tr>
</tbody>
</table>

Table 2: Computing the fundamental matrix solution for each of the periodic orbit $\gamma_i$. $m$ and $M$ are respectively the finite dimensional reduction parameter and the computational parameter that have been chosen. $r$ is the radius of the ball centered approximate solution in $\Omega^s$ within which a genuine solution of (6) exists.

Some of the radii polynomials $p_k(r)$ built during the computation of solution #4 have been plotted in Figure 3. The bold line on the $x$-axis remarks the interval

$$\text{INT} = [9.91268997 \cdot 10^{-7}, \ 1.4574858482 \cdot 10^{-3}]$$

where all the radii polynomials are negative.

From the computations we noticed that the odd Fourier coefficients of $Q(t)$ are almost vanishing, suggesting that $Q(t)$ is a $\tau_\gamma$ periodic function, rather than $2\tau_\gamma$ periodic. This is not in contradiction with Floquet Theorem. Again in the Appendix we report the numerical approximation $\bar{R}$ and the first even Fourier coefficients $\bar{Q}_k$ for the solution #1 and solution #4. As in the previous case, the Fourier coefficients $\bar{Q}_k$ corresponding to periodic orbits closer to homoclinic decrease slower. This justifies the fact that larger values of $m$ and $M$ were necessary to obtain successful computations.
Figure 3: Plot of some of the radii polynomials $p_k(r)$ constructed in the computation of the fundamental matrix solution associated to $\gamma_4$. On the right: magnification close to $r = 0$. The red line denotes the interval $\text{INT}$ where all the $p_k(r)$ are negative.

We now have all the ingredients necessary to construct the tangent bundles: first we compute the intervals containing the spectrum and the eigenvectors of each the interval value matrix $R$, then, in light of Theorem 3.7, the multiplication of the stable and unstable directions with the function $Q(\theta)$ yields the tube enclosing the complete stable and unstable bundles.

Table 3 lists the Lyapunov exponents of the periodic orbits, as defined in Definition 3.6, and it also contains the radius of the intervals enclosing the stable and unstable eigen-couple of $R$ while in Figure 4 the tangent bundles are depicted. In Appendix the complete list of the eigen-decomposition of the interval matrices $R$ is also provided.

<table>
<thead>
<tr>
<th>Sol #</th>
<th>Center</th>
<th>Radius</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-14.2953855130260</td>
<td>6.801248614 \cdot 10^{-5}</td>
</tr>
<tr>
<td></td>
<td>0.6287188463595</td>
<td>9.510853040 \cdot 10^{-4}</td>
</tr>
<tr>
<td>2</td>
<td>-14.2174898849454</td>
<td>2.814282939 \cdot 10^{-5}</td>
</tr>
<tr>
<td></td>
<td>0.550823182790</td>
<td>3.733964559 \cdot 10^{-4}</td>
</tr>
<tr>
<td>3</td>
<td>-13.9620493680589</td>
<td>2.774785811 \cdot 10^{-5}</td>
</tr>
<tr>
<td></td>
<td>0.2953827013923</td>
<td>3.653168667 \cdot 10^{-5}</td>
</tr>
<tr>
<td>4</td>
<td>-13.7210150091049</td>
<td>2.544262339 \cdot 10^{-6}</td>
</tr>
<tr>
<td></td>
<td>0.0543483424385</td>
<td>5.248456341 \cdot 10^{-5}</td>
</tr>
<tr>
<td>5</td>
<td>-13.7013292393391</td>
<td>9.729262854 \cdot 10^{-6}</td>
</tr>
<tr>
<td></td>
<td>0.0346625726730</td>
<td>3.336819199 \cdot 10^{-4}</td>
</tr>
</tbody>
</table>

Table 3: Lyapunov exponents for each of the periodic orbit $\gamma_i$. For each solution we report the center and the radius of the interval vectors enclosing the exponents. Note that we could prove the existence of the eigenvectors $v_j$ associated to $\mu_j$ within the same accuracy given by $r$.

4.2 $\zeta^3$-model: non orientable tangent bundles

It is known that if a Floquet multipliers of a periodic orbit is negative, then the corresponding tangent bundle is not orientable. Moreover, in the case of a saddle periodic orbit of a three-dimensional system, the two non-trivial Floquet multipliers are real and their product is positive. Therefore both the tangent bundles are either orientable or not orientable and, in
Figure 4: Plot of the tangent stable (turquoise) and unstable (red) bundles of each of the periodic orbits $\gamma_i$. The central figure concerns Sol# 2, with a magnification, while figures (a)-(b)-(d)-(e) concern respectively $\gamma_1, \gamma_3, \gamma_4, \gamma_5$. 
the latter case, they are topologically equivalent to a M"obius strip, see [35].

An example of a dynamical system with periodic orbits that exhibit this behavior is the so called $\zeta^3$-model considered in [36]

\[
\begin{align*}
\dot{x} &= y \\
\dot{y} &= z \\
\dot{z} &= \alpha x - x^2 - \beta y - z.
\end{align*}
\]

For $\beta = 2$, as $\alpha$ varies, the periodic orbits of system (52) produce an interesting bifurcation diagram. We refer to [35] and [37] for a detailed analysis of the bifurcation diagram and on the genesis of periodic orbits, called twisted periodic orbit, with non orientable invariant manifolds. We focus on a particular twisted periodic orbit corresponding to $\alpha = 3.372$ lying on the branch emanating from a period-doubling bifurcation that occurs at $\alpha \approx 3.125$.

Following the same procedure as before, we rigorously compute the enclosure of the periodic orbit $\gamma(t)$ and subsequently the enclosure of the matrix $R$ and of the matrix function $Q(t)$, hence producing an explicit Floquet normal form as in (2). Then, we extract the necessary stability parameters and we recover the stable and unstable tangent bundles using (43). Figure 5 shows the resulting bundles.

Having computed the intervals enclosing the period $\tau$ of the orbit and the eigenvalues of $R$, we realize that the absolute values of the two nontrivial Floquet multipliers satisfy

\[
|\sigma_1| \in [7.037325782193 \cdot 10^{-3} \quad 7.037944324307 \cdot 10^{-3}] = \Delta_{st}
\]

\[
|\sigma_2| \in [1.526609276443494 \quad 1.528421395487018] = \Delta_{unst}
\]

To conclude we emphasize the role played by the continuous function $Q(\theta)$ in the construction of the tangent bundles. As proved in Theorem 3.7, as $\theta$ changes, the eigenvector $w^\theta_j$ of $\Phi_\theta(T)$ associated to the Floquet multiplier $\sigma_j$ is given by $w^\theta_j = Q(\theta)v_j$, where $v_j$ is the eigenvector of $R$ relative to the eigenvalue $\mu_j$. The function $Q(\theta)$ is continuous and $2\tau$-periodic, but the tangent bundles are smooth manifolds. That implies that $w^\tau_j = Q(\tau)v_j$ has to be an eigenvector of $\Phi(\tau)$ associated to the Floquet multiplier $\sigma_j$, i.e. $\text{span}\{v_j\} = \text{span}\{w^\tau_j\}$. In the case of the Lorenz system $Q(\tau)$ turns to be the identity matrix, therefore the last relation is simply verified. But in case of the $\zeta^3$-model and in general when the bundles are not orientable $Q(\tau)$ need not be the identity matrix. Indeed, in the considered example, $Q(\tau)$ results to stay in a small interval around

\[
\bar{Q} = \begin{bmatrix}
-1.663148705259992 & -1.018593776943882 & -0.446703151142258 \\
1.323227706784005 & 1.032472501143805 & 0.891338521225762 \\
0.936264065136750 & 1.43809784773345 & -0.369323795883880
\end{bmatrix}.
\]

Denoting by $\bar{R}$, $\bar{\tau}$, $\bar{v}_1$ the centers of the intervals the genuine $R$, $\tau$ and $v_1$ belong to and defining $\bar{\Phi} = \bar{Q}(\tau)e^{\bar{R}\bar{\tau}}$ the numerical approximation of $\Phi(T)$, we compute

\[
\bar{\Phi}\bar{Q}\bar{v} = \begin{bmatrix}
-0.002642211417990 \\
-0.003294408641461 \\
0.011434535907588
\end{bmatrix}, \quad \bar{Q}\bar{v} = \begin{bmatrix}
0.37544264619642 \\
0.468116019931565 \\
-1.624780050494352
\end{bmatrix}.
\]

The component-wise ratio between the two computed vectors is $\bar{\sigma}_1 = -7.03759044326 \cdot 10^{-3} \pm 10^{-13}$, whose absolute value is indeed in the interior of $\Delta_{st}$. If the unstable eigenvector $v_2$ is considered, the same operations produce $\bar{\sigma}_2 = -1.527515067244305 \pm 10^{-13}$. Although not rigorous, these computations confirm the above theoretical discussion and moreover provide a method to recover the sign of the Floquet multipliers, information that is not possible to achieve following the presented computational technique.
Figure 5: Stable (turquoise) and unstable (red) tangent bundles of a periodic orbit for the $\zeta^3$ model with negative Floquet multipliers
5 Acknowledgments

We would like to thank Marcio Gameiro and Jason D. Mireles James for helpful discussions.

6 Appendix

Period and Fourier coefficients of $\tilde{\gamma}_1$ and $\tilde{\gamma}_4$.

**Solution # 1**

\[
\tau_\gamma \equiv 1.0276544075218, \quad \xi_0 = \begin{bmatrix}
-6.40635466884038 \\
-6.40635466884038 \\
13.533127936581090
\end{bmatrix}
\]

\[
\xi_1 = \begin{bmatrix}
-2.45759444025310 \\
-0.66251105495624 \\
-0.0012154683778 \\
-0.00002121714006 \\
-0.00000344784845 \\
-1.06949120821600 \\
-5.273294172610276 \\
-1.910007347872115 \\
-0.00000000000006 \\
-0.00000000000005
\end{bmatrix}, \quad \xi_2 = \begin{bmatrix}
-0.840331595816763 \\
-0.17558636707219 \\
-0.003176204341056 \\
-0.000005425303769 \\
-0.00000000000005 \\
-0.11548781924051 \\
-0.00044052745140 \\
-0.000000135871818 \\
-0.00000000000000 \\
-0.00000000000000
\end{bmatrix}, \quad \xi_3 = \begin{bmatrix}
-2.087540965775885 \\
-0.08975904313526 \\
-0.01622039074351 \\
-0.00000000000000 \\
-0.00000000000000 \\
-0.00000000000000 \\
-0.00000000000000 \\
-0.00000000000000 \\
-0.00000000000000 \\
-0.00000000000000
\end{bmatrix}, \quad \xi_4 = \begin{bmatrix}
-2.3653481981223071 \\
-0.058924136908184 \\
-0.058924177198641 \\
-0.00000000000000 \\
-0.00000000000000 \\
-0.00000000000000 \\
-0.00000000000000 \\
-0.00000000000000 \\
-0.00000000000000 \\
-0.00000000000000
\end{bmatrix}
\]

**Solution # 4**

\[
\tau_\gamma \equiv 0.683613500457546, \quad \xi_0 = \begin{bmatrix}
-2.52125210993276 \\
-7.52125210993276 \\
-7.52125210993276 \\
-7.52125210993276 \\
-7.52125210993276
\end{bmatrix}
\]

\[
\xi_1 = \begin{bmatrix}
-1.246453590201949 \\
-1.107986038743539 \\
-1.138858736824905 \\
-0.000483713119288 \\
-0.00000000000000
\end{bmatrix}, \quad \xi_2 = \begin{bmatrix}
-0.00000000000000 \\
-0.00000000000000 \\
-0.00000000000000 \\
-0.00000000000000 \\
-0.00000000000000
\end{bmatrix}, \quad \xi_3 = \begin{bmatrix}
-0.00000000000000 \\
-0.00000000000000 \\
-0.00000000000000 \\
-0.00000000000000 \\
-0.00000000000000
\end{bmatrix}, \quad \xi_4 = \begin{bmatrix}
-0.00000000000000 \\
-0.00000000000000 \\
-0.00000000000000 \\
-0.00000000000000 \\
-0.00000000000000
\end{bmatrix}
\]

Numerical approximation $R$ and even Fourier coefficients $Q_k$.

**Solution # 1**

\[
R = \begin{bmatrix}
1.37770562612841 \\
6.244108010218366 \\
-7.445538072862637 \\
-9.08675622671404 \\
6.862010295473324
\end{bmatrix}, \quad Q_0 = \begin{bmatrix}
4.11735582468368 \\
-9.09232886309471 \\
1.30072865437973 \\
6.862010295473324 \\
0.843422174867108
\end{bmatrix}, \quad Q_2 = \begin{bmatrix}
-0.73040946201914 \\
-0.991279291038973 \\
-0.00135658535843 \\
-0.00000000000000 \\
-0.00000000000000
\end{bmatrix}
\]

29
\[
Q_4 = 0.00323244126082 + 0.07219153526703 + 0.08040749773392 + 0.08598096774482 - 0.13901636811394 + 0.08074733121947 - 0.03875624142790 + 0.22701958561727 = 0.2771120560736 - 0.13857298306892.
\]

\[
Q_6 = 0.03222464783581 + 0.04335507463808 + 0.00947598772244 + 0.08785281969847 + 0.00890496774482 + 0.01357185159069 - 0.086779888674130 - 0.118920918017351 = 0.18999209299670 - 0.03526099963182.
\]

\[
Q_6 = 0.01487677547204 + 0.01679777242045 + 0.00342106751010 + 0.040627402366845 + 0.0063567805971 - 0.03954081630743 - 0.041468737604576 + 0.07799141046786 = 0.0337235078127 - 0.042860237785526.
\]

\[
Q_6 = 0.01642111764788 + 0.00111352816472 + 0.0068163885290 + 0.00545043426706 - 0.09850956051616 + 0.06619388137483 + 0.048237550260254 - 0.00673784786048 = 0.03017805068046 - 0.00673784786048.
\]

\[
Q_10 = \begin{pmatrix}
2.293328952532320 & 0.003607621081908 & 0.014369458507765 & 0.003095806603627 & 0.000167141543813 & 0.006228163197581 + 0.001899418469947
\end{pmatrix}
\]

Solution \# 4
\[
\beta_1 = \begin{pmatrix}
-2.108771563265242 - 0.62930195418498 + 0.864619706987080 &
-6.29527840812851 - 3.46886782670139 - 2.93807740489710 &
0.865350589016370 - 0.54207282704893 - 0.41391813643666 + 0.049692461586980 - 0.02562381143980
\end{pmatrix}
\]

\[
\theta_1 = -10.10382700749006 - 5.011512125068070 - 4.18159233282406 + 1.37659272203086 - 2.108771563265242 - 0.62930195418498 + 0.864619706987080 &
-6.29527840812851 - 3.46886782670139 - 2.93807740489710 - 0.10756150664544 + 0.33122459010300 + 0.46903810585610 - 0.38333667504131 - 0.051976791264511
\]

\[
Q_4 = 0.18441507850539 + 0.000302584783012 + 0.00323830325558 + 0.0011374276095116 - 0.000341594785350 - 0.00002398349382 + 0.0012782759910915 + 0.00172727258882
\]

\[
Q_4 = 0.00257164032931 + 0.00187573220509 + 0.00135047432202 + 0.0011073090390066 + 0.000760730537533 + 0.0002999015051655 + 0.0000257164032931 + 0.00187573220509 + 0.00135047432202 + 0.0011073090390066 + 0.000760730537533 + 0.0002999015051655 + 0.00002398349382 + 0.0012782759910915 + 0.00172727258882
\]

\[
Q_4 = -0.00280553568511 - 0.00023994443313 - 0.000167141543813 - 0.000063749030420i + 0.0001253096573098 - 0.00022759976102 + 0.00024818870303i - 0.00059911668090 - 0.00003299205205 + 0.00016486816545i
\]

\[
\beta_{10} = 1.1 - 0.93 - 0.42 + 1.02i
\]

\[
\theta_{10} = 0.02779827570345 + 0.005964383585053 + 0.0076673507992 + 0.006378869029096 - 0.04974858722010 + 0.00258862458980 + 0.000613936052197 + 0.11457660525219 + 0.29493056057831 - 0.27354789078776 + 0.45190975032486 + 0.22853295233220 + 0.11501688035055 + 0.05599195616628 + 0.2808666052373x + 0.29483393191544 - 0.22958271626269 - 0.049143411715591
\]

\[
q_{12} = 1.1 + 0.94 - 0.42 + 1.02i
\]

\[
\theta_{14} = 0.02184467611758 - 0.00885662195616 - 0.0093733147877201 + 0.12298641569192 + 0.33405719249540 + 0.10093631671518 + 0.16094807523844 - 0.25162634533053 + 0.12332423393918 + 0.092568425206174 + 0.005787490826114 + 0.334771573278523 + 0.252425144830831 - 0.01484326217575
\]
Solution # 2

<table>
<thead>
<tr>
<th>E. values</th>
<th>Stable</th>
<th>Unstable</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.427488849454</td>
<td>-0.000000008000372</td>
<td>0.55623218257069</td>
</tr>
<tr>
<td>-1.31141283044274</td>
<td>0.31301870843905</td>
<td>0.362984001361858</td>
</tr>
</tbody>
</table>

E. vectors

Solution # 3

<table>
<thead>
<tr>
<th>E. values</th>
<th>Stable</th>
<th>Unstable</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.962049368058929</td>
<td>-0.000000000000126</td>
<td>0.295382701392358</td>
</tr>
<tr>
<td>1.347327907101522</td>
<td>0.210153254038267</td>
<td>0.27128549606597</td>
</tr>
<tr>
<td>1.071215739018561</td>
<td>0.999348750677595</td>
<td>0.3730644559021</td>
</tr>
</tbody>
</table>

E. vectors

Solution # 4

<table>
<thead>
<tr>
<th>E. values</th>
<th>Stable</th>
<th>Unstable</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.721015009104903</td>
<td>-0.000000000000309</td>
<td>0.5434834238550</td>
</tr>
<tr>
<td>1.349284428490609</td>
<td>0.051023279023768</td>
<td>0.128543563503133</td>
</tr>
<tr>
<td>0.266150110668596</td>
<td>1.19867456309931</td>
<td>1.25276532277167</td>
</tr>
<tr>
<td>0.926058357486949</td>
<td>1.27737843119246</td>
<td>1.19813836972578</td>
</tr>
</tbody>
</table>

E. vectors

Solution # 5

<table>
<thead>
<tr>
<th>E. values</th>
<th>Stable</th>
<th>Unstable</th>
</tr>
</thead>
<tbody>
<tr>
<td>-1.701329239339196</td>
<td>-0.000000000000487</td>
<td>0.3466257273008</td>
</tr>
<tr>
<td>1.451175793211715</td>
<td>0.02013320407690</td>
<td>-0.100604795866982</td>
</tr>
<tr>
<td>-0.333782791906608</td>
<td>1.11578057644323</td>
<td>-1.200723375868350</td>
</tr>
<tr>
<td>0.384950830190627</td>
<td>1.324623855708436</td>
<td>1.138422530506485</td>
</tr>
</tbody>
</table>

E. vectors

Rad 10^{-3}

References


[34] Roberto Castelli and Jean-Philippe Lessard. On the rigorous enclosure of the eigendecomposition of interval matrices. In preparation.


33