Non-isothermal, non-Newtonian Hele–Shaw flows within Cattaneo’s heat flux law

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Abstract

In this paper, we consider a non-isothermal, non-Newtonian injection process. This leads to the study of a novel elliptic–hyperbolic system. The hyperbolic nature of the system arises because we replace the infinite speed of propagation from classical, thermal elasticity by a finite propagation velocity. We present a formal derivation of the elliptic–hyperbolic system starting from conservations of mass, momentum, and energy in a three-dimensional domain, where the removal of the infinite propagation speed is achieved using Cattaneo’s law for heat conduction. The existence of weak solutions to certain elliptic–hyperbolic problems associated with the resulting equations is proved.

Keywords: Hyperbolic heat equation; Hele–Shaw approximation; Lubrication approximation

1. Introduction

Injection molding is a manufacturing process widely used for producing parts using a mold having a cavity in the shape of the part being manufactured. Nearly 20\% of goods manufactured nowadays use injection molding due to its versatility and low cost [1]. The analysis of injection molding is complicated because of its non-isothermal and non-Newtonian character. Usually the mold surface is cold and the polymer melt is hot. During the curing process initial stresses may occur in the mold; hence, it is essential that the injection process be well monitored with regard to controlling the pressure and temperature. We provide next a short review of the approach taken by Gilbert and Shi, 1992–1996, [20,21] in formulating the injection process (Problem I) for non-isothermal, non-Newtonian fluids and study the existence of its weak solutions.

Problem I. Find functions $\theta$ and $p$ defined in $\Omega_T$ such that

\begin{align}
\theta_t - \Delta \theta &= k(\theta)|\nabla p|^{r-1} + f \quad \text{in} \ \Omega_T, \\
-\nabla \cdot \{k(\theta)|\nabla p|^{r-2} \nabla p\} &= g \quad \text{in} \ \Omega_T.
\end{align}

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\[ \theta = \theta_0 \quad \text{on} \quad \partial \Omega \times (0, T), \]
\[ \theta = \varphi \quad \text{on} \quad \Omega \times \{0\}, \]
\[ p = p_0 \quad \text{on} \quad \Gamma_0 \times (0, T), \]
\[ -k(\theta) \nabla p + \frac{2}{\tau} \frac{\partial p}{\partial \tau} = l \quad \text{on} \quad \Gamma_1 \times (0, T). \]  

Here \( p \) is the pressure of the flow and \( \theta \) is the temperature.

Numerical solutions to this and similar problems appear in the engineering literature [25,24,9,10,27,51,53,52,1,50,34,61]. Some other related papers are [56,19,16,15].

In the present paper we replace the diffusion heat conduction equation used in Problem I with a hyperbolic heat conduction equation. We recall that the classical heat (diffusion) equation

\[
\begin{cases}
\Theta_t - \Delta \Theta = 0 & \text{in} \quad \mathbb{R}^n \times (0, \infty) \\
\Theta = g & \text{on} \quad \mathbb{R}^n \times \{t = 0\}
\end{cases}
\]

has the well-known fundamental solution

\[ \Theta(x, t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} g(y) \, dy. \]

If \( g \) is bounded, continuous, with \( g \geq 0 \), positive somewhere, then temperature \( \Theta(x, t) \) at any later time (no matter how soon) is everywhere positive. Therefore we say the heat equation forces \textit{infinite propagation speed} of the temperature.

From a macroscopic continuum sense, the heat conduction phenomenon is often described by Fourier’s law, relating the heat flux to the temperature:

\[ q = -K \nabla \theta. \]

Such an expression suffers from some pathological deficiencies, as has been mentioned on many occasions (Maxwell [39] 1867, Onsager [42,43] 1931). Firstly, it does not take into account memory effects which are present in some materials. Secondly, it predicts that all thermal disturbances at a given point in the body are felt immediately throughout the whole body [46].

Cattaneo [8] is credited as the first to eliminate this anomaly by proposing a damped version of Fourier’s law by introducing a heat flux relaxation term, namely,

\[ \tau \frac{\partial q}{\partial \tau} = -(q + K \nabla \theta) \]

where \( \tau \) is a relaxation time which represents the time required to establish a steady state of heat conduction in an element suddenly exposed to heat flux. He reasoned that heat flow does not start instantaneously, but grows gradually with a relaxation time \( \tau \), after the application of a temperature gradient. Similarly, heat flow does not cease immediately, but dies out gradually, after a temperature gradient is removed. As a consequence, numerous theories and postulates exist in the literature on the existence of a finite speed of heat propagation, termed the \textit{hyperbolic heat equation} [37,17,32,29,30,5,33,38,36,46,18,6,59,35]. The engineering literature on “hyperbolic conduction” is collected annually in the “Review of Heat Transfer Literature” in the \textit{International Journal of Heat and Mass Transfer}.

The hyperbolic model of heat conduction is more physically realistic and more general than the parabolic model. For situations involving high heat flux devices such as lasers with short duration or high frequency and high temperature gradients, heat is found to propagate at a finite speed. With more emphasis on precision processing [31,41,2,26,45,4,54,23] it is likely that an increase in the application of the hyperbolic proposition will take place in the future. In this paper we use Cattaneo’s law for heat conduction to formulate a novel elliptic–hyperbolic system through an asymptotic argument. We also establish the existence of a weak solution for a resulting Newtonian non-isothermal case.

In Section 2 of the paper, we derive the two-dimensional non-isothermal, non-Newtonian systems of the significant terms by use of asymptotic expansions. In particular we take into account Cattaneo’s law (1.8) for heat conduction. In Section 3, we prove the existence of weak solutions to a resulting system of equations in the Newtonian case. Although the physical models are two dimensional, we shall carry out our proofs for \( N \) dimensions. Subsequent research will be
directed to the existence of weak solutions for the non-isothermal, non-Newtonian case and numerical investigation of such systems.

The statements of these problems are given below.

**Problem II.** Find functions θ and p defined in ΩT such that

\[
\frac{\partial \theta}{\partial t} + \tau \frac{\partial^2 \theta}{\partial t^2} - K \Delta \theta = \frac{\lambda k}{\rho c} |\nabla p - \vec{f}|^2 + q_0(x) \quad \text{in } \Omega_T, \tag{1.9}
\]

\[
\Delta p = \rho \nabla \cdot \vec{f} \quad \text{in } \Omega, \tag{1.10}
\]

\[
p = p_0 \quad \text{on } \partial \Omega \tag{1.11}
\]

\[
\theta = \theta_0 \quad \text{on } \partial \Omega \times (0, T), \tag{1.12}
\]

\[
\theta = \varphi \quad \text{and} \quad \partial_t = \psi \quad \text{on } \Omega \times \{0\}. \tag{1.13}
\]

We assume that Ω is a bounded domain in \(\mathbb{R}^n\) with a \(C^1\) boundary, where \(n \geq 2\). For a given time interval \((0, T)\), let \(\Omega_T := \Omega \times (0, T)\). We further assume that \(\theta_0, \varphi, \text{ and } \psi\) are given functions, while given functions \(\vec{f}\) and \(p_0\) are independent of time \(t\). We also assume that \(\tau, K, \lambda, k, \rho\) and \(c_p\) are positive constants related to physical quantities (such as relaxation time, thermal conductivity, density of the fluid, specific heat, etc.). In this paper, let \(H^{1,\alpha}(\Omega)\) and \(H_0^{1,\alpha}(\Omega)\) denote the usual Sobolev space equipped with the standard norm.

We assume that the boundary data \(\vec{f}, \theta_0, \text{ and } p_0\) for Problem II can be extended to functions defined on Ω such that

\[
\vec{f} \in [C^{1,\alpha}(\overline{\Omega})]^n, \quad \theta_0 \in C^{2,\alpha}(\overline{\Omega}) \quad \text{and} \quad p_0 \in C^{2,\alpha}(\overline{\Omega}), \tag{1.14}
\]

where \(0 < \alpha \leq 1\). We further assume that

\[
\varphi \in H_0^{1,\alpha}(\Omega), \quad \text{and} \quad \psi \in L^2(\Omega). \tag{1.15}
\]

**2. Derivation of the equations**

Let \(\Omega_\varepsilon = \Omega \times (-\varepsilon, \varepsilon)\) be a three-dimensional mold, where Ω is a bounded domain in \(\mathbb{R}^2\) with \(C^1\) boundary, and \(2\varepsilon\) represents the thickness of the mold. We begin with conservation of momentum and energy equations in \(\Omega_\varepsilon\):

\[
\rho^\varepsilon \frac{Dv^\varepsilon}{Dt} = -\nabla p^\varepsilon + \nabla \cdot s^\varepsilon + \rho^\varepsilon \vec{f}^\varepsilon \tag{2.1}
\]

\[
\rho^\varepsilon c^\varepsilon \frac{D\theta^\varepsilon}{Dt} = -\nabla \cdot q + s^\varepsilon d^\varepsilon_{ij} d^\varepsilon_{ij} \tag{2.2}
\]

where \(\frac{Dv^\varepsilon}{Dt}\) is the material derivative of the velocity field \(v^\varepsilon = (v_1^\varepsilon, v_2^\varepsilon, v_3^\varepsilon)\), \(p^\varepsilon\) is the pressure, \(s^\varepsilon = (s_{ij}^\varepsilon)\) is the viscous stress tensor, \(\rho^\varepsilon\) is the density of the fluid, \(\vec{f}^\varepsilon\) is the volume force density, \(c^\varepsilon\) is the specific heat, \(K^\varepsilon\) is the thermal conductivity, and

\[
d^\varepsilon = (d^\varepsilon_{ij}) \quad \text{with} \quad d^\varepsilon_{ij} = \frac{1}{2} \left[ \frac{\partial v_i^\varepsilon}{\partial x_j} + \frac{\partial v_j^\varepsilon}{\partial x_i} \right] \tag{2.3}
\]

denotes the strain rate tensor. Repeated indices indicate that the summation convention is used. Note that the last term on the right hand side of (2.2) represents the heat generated by the deformation of the fluid under the action of the shear forces, the so called dissipation term. The physical meanings of the other terms appearing in (2.1) and (2.2) are self-evident. We further assume that the fluid is incompressible:

\[
\text{div } v^\varepsilon = 0. \tag{2.4}
\]

For more detailed information, we refer the reader to [1].
The stress tensor is given by the power law model (Eisele [13], Subbiah et al. [57], Advani et al. [1]):

\[ s^\varepsilon_{ij} = k^\varepsilon (\varepsilon^\alpha)^n \dot{e}^\varepsilon_{ij}, \tag{2.5} \]

where \( \dot{\varepsilon} \) is the strain rate given by \( \dot{\varepsilon} = 2\sqrt{(d^\varepsilon_{ij} d^\varepsilon_{ij})} \).

\[ 4d^\varepsilon_{ij} d^\varepsilon_{ij} = 2(d^\varepsilon_{11} d^\varepsilon_{11} + d^\varepsilon_{22} d^\varepsilon_{22} + d^\varepsilon_{33} d^\varepsilon_{33}) + 4(d^\varepsilon_{12} d^\varepsilon_{12} + d^\varepsilon_{13} d^\varepsilon_{13} + d^\varepsilon_{23} d^\varepsilon_{23}) \]

with \( n \) the power law index, and \( k^\varepsilon \) is a given positive function. The product \( \eta^\varepsilon = k^\varepsilon (\varepsilon^\alpha)^n / 2 \) is known as the viscosity of the fluid. We remark that when \( n = 1 \) the fluid is called Newtonian. When \( n \neq 1 \), the constitutive equation (2.5) represents shear thinning \((n < 1)\) and shear thickening \((n > 1)\) fluids.

In this section, indices with Greek letters range from 1 to 2 while indices with Roman letters range from 1 to 3. For example, we use \((x_\alpha) := (x_1, x_2)\) to designate two coordinates and \((x_i) := (x_1, x_2, x_3)\) to designate three coordinates.

2.1. Lubrication approximation

We shall want to compare the magnitudes of the various terms in (2.1) and (2.2). However in the \( x_i \) coordinate system the terms have a misleading appearance, for the velocity components vary much more rapidly in the \( x_3 \)-direction than the lateral directions. To normalize the system, we scale variables in a manner analogous with that of Reynolds’ theory of lubrication. (For more details, see, for example, [60] or [58, Section 2.2].) We stretch the \( x_3 \) coordinate according to the rule

\[ x_3 = \varepsilon x_3. \tag{2.6} \]

At the same time we assume that the physical parameters \( K^\varepsilon, k^\varepsilon, c^\varepsilon, \rho^\varepsilon \) and the time \( t \) remain unchanged:

\[ K^\varepsilon = K, \quad k^\varepsilon = k, \quad c^\varepsilon = c, \quad \rho^\varepsilon = \rho, \quad \text{and} \quad t = t. \tag{2.7} \]

From the equation of continuity (2.4), we obtain that the magnitude of \( v_3^\varepsilon \) will be \( \varepsilon \) times the magnitude of \( v_3^0 \). Using the dimensionless variable \( y_3 \) introduced above, we compute \( 4d^\varepsilon_{lm} d^\varepsilon_{lm} \): \n
\[ 4d^\varepsilon_{lm} d^\varepsilon_{lm} = \frac{\partial v_\beta^\varepsilon}{\partial x_\alpha} \frac{\partial v_\beta^\varepsilon}{\partial x_\alpha} + \frac{\partial v_\alpha^\varepsilon}{\partial x_\alpha} + \frac{\partial v_\beta^\varepsilon}{\partial x_\alpha} \frac{\partial v_\alpha^\varepsilon}{\partial x_\alpha} + \frac{\partial v_\beta^\varepsilon}{\partial y_3} \frac{\partial v_\alpha^\varepsilon}{\partial y_3} + \frac{\partial v_\beta^\varepsilon}{\partial y_3} \frac{\partial v_\alpha^\varepsilon}{\partial y_3} + \frac{\partial v_\beta^\varepsilon}{\partial x_\alpha} + \frac{\partial v_\alpha^\varepsilon}{\partial x_\alpha}. \tag{2.8} \]

For the power law structure (2.5) to be meaningful, the velocity has to become

\[ v_\beta^\varepsilon = v_\beta^0 + v_\beta^1 \varepsilon + v_\beta^2 \varepsilon^2 + \cdots, \quad \alpha = 1, 2 \]

\[ v_3^\varepsilon = v_3^0 + v_3^1 \varepsilon + v_3^2 \varepsilon^2 + \cdots. \tag{2.9} \]

Then

\[ 4d^\varepsilon_{lm} d^\varepsilon_{lm} = A + O(\varepsilon) \tag{2.10} \]

where

\[ A = \left( \frac{\partial v_\beta^0}{\partial y_3} \frac{\partial v_\alpha^0}{\partial y_3} \right). \tag{2.11} \]

From conservation of momentum (2.1) and energy (2.2), the pressure \( p^\varepsilon \), body force \( f_i^\varepsilon \), and the temperature \( \theta^\varepsilon \) transfer according to the rules

\[ p^\varepsilon = \frac{1}{\varepsilon} p^0 + p^1 + p^2 \varepsilon + \cdots, \]

\[ \theta^\varepsilon = \theta^0 + \theta^1 \varepsilon + \theta^2 \varepsilon^2 + \cdots, \tag{2.12} \]

\[ f_i^\varepsilon = \frac{1}{\varepsilon} f_i^0 + f_i^1 + f_i^2 \varepsilon + \cdots, \quad i = 1, 2, 3. \]

Similar expansions can be found in \([11, 21]\).
We shall not notationally distinguish a function using different variables. For instance, we employ the same notation:

\[ v^\varepsilon = v^\varepsilon(x_1, x_2, x_3) = v^\varepsilon(x_1, x_2, \varepsilon y_3). \]

Therefore from (2.5), we approximate the stress:

\[ s^\varepsilon_{ij} = k(\theta^0)A^{(n-1)/2}d^\varepsilon_{ij} + O(\varepsilon). \]

(2.13)

In particular,

\[ s^\varepsilon_{3\alpha} = k(\theta^0)A^{(n-1)/2}d^\varepsilon_{3\alpha} + O(\varepsilon) \]

\[ = \frac{1}{2}k(\theta^0)A^{(n-1)/2}\left( \frac{\partial v^\varepsilon_\alpha}{\partial x_3} + \varepsilon \frac{\partial v^\varepsilon_3}{\partial x_\alpha} \right) + O(\varepsilon) \]

\[ = \frac{1}{2}k(\theta^0)A^{(n-1)/2}\left[ \frac{\partial v^0_\alpha}{\partial y_3} + \varepsilon \frac{\partial v^0_3}{\partial y_\alpha} + \cdots \right] + O(\varepsilon) \]

\[ = \frac{1}{2}k(\theta^0)A^{(n-1)/2}\frac{\partial v^0_\alpha}{\partial y_3} + O(\varepsilon). \]

From the computations of (2.8) and (2.9) in [21], we have

\[ \frac{\partial}{\partial x_j} s^\varepsilon_{\beta j} = \frac{1}{2\varepsilon} \frac{\partial}{\partial y_3} \left\{ k(\theta^0)A^{(n-1)/2}\frac{\partial v^0_\beta}{\partial y_3} \right\} + O(1) \]

(2.14)

\[ \frac{\partial}{\partial x_j} s^\varepsilon_{3 j} = \frac{\partial}{\partial y_3} \left\{ k(\theta^0)A^{(n-1)/2}\frac{\partial v^0_3}{\partial y_3} \right\} + \frac{1}{2} \frac{\partial}{\partial x_\mu} \left\{ k(\theta^0)A^{(n-1)/2}\frac{\partial v^0_\mu}{\partial y_3} \right\} + O(1). \]

(2.15)

2.2. Hele–Shaw approximation

We next assume that viscosity effects predominate. In particular, the inertial term is dropped from the balance of linear momentum. This assumption has been widely used in the literature (e.g. [13,57,48,3,1]). Therefore, (2.1) and (2.5) become

\[ \frac{\partial}{\partial x_i} \left\{ k^\varepsilon (\theta^\varepsilon) \gamma^\varepsilon_{n-1} a^\varepsilon_{ij} \right\} = \frac{\partial p^\varepsilon}{\partial x_j} - \rho^\varepsilon f^\varepsilon_{j} \]

(2.16)

which is the quasistatic form of the conservation of momentum.

Consider now the right hand side of (2.16). From (2.9), (2.12), (2.14) and the equation

\[ -\rho^\varepsilon f^\varepsilon_{\alpha} + \frac{\partial p^\varepsilon}{\partial x_\alpha} = \frac{\partial}{\partial x_j} s^\varepsilon_{\alpha j} \]

we have

\[ -\rho f^0_{\alpha} + \frac{\partial p^0}{\partial x_\alpha} = \frac{1}{2} \frac{\partial}{\partial y_3} \left\{ k(\theta^0)A^{(n-1)/2}\frac{\partial v^0_\alpha}{\partial y_3} \right\}. \]

(2.17)

On the other hand, from (2.9), (2.12), (2.15) and the equation

\[ -\rho^\varepsilon f^\varepsilon_{3} + \frac{\partial p^\varepsilon}{\partial x_3} = \frac{\partial}{\partial x_j} s^\varepsilon_{3 j} \]

we have

\[ -\rho f^0_{3} + \frac{1}{\varepsilon} \frac{\partial p^0}{\partial y_3} = -\frac{1}{\varepsilon} \rho f^0_{3} - \rho f^1_{3} - \cdots + \frac{1}{\varepsilon^2} \frac{\partial p^0}{\partial y_3} + \frac{1}{\varepsilon} \frac{\partial p^1}{\partial y_3} + \frac{\partial p^2}{\partial y_3} + \cdots \]

(2.18)
which implies
\[ \frac{\partial p^0}{\partial y_3} = 0. \]

This shows that the first term of the pressure expansion does not depend on the mold thickness.

We now assume that the velocity field of the flow is symmetric about the center of the mold, and that the fluid does not slip on the surfaces of the mold. These notions translate to
\[ \frac{\partial v^e}{\partial x_3} = 0 \text{ at } x_3 = 0, \quad \text{and} \quad v^e = 0 \text{ at } x_3 = \pm \varepsilon. \] (2.19)

Both of these assumptions have been well accepted in engineering literature (e.g. see Subbiah et al. [57] and Advani et al. [1]). We also assume that the \( f^{0}_\alpha \) are independent of the mold thickness. A similar Hele–Shaw analogy has been used by other authors (e.g. see [47,44,12,49,55,7,40]). Using (2.19), Eq. (2.17) may be integrated over \((0, y_3)\) to obtain
\[ -2y_3\rho f^0_\alpha + 2y_3 \frac{\partial p^0}{\partial x_\alpha} = k(\theta^0) A^{(n-1)/2} \frac{\partial v^0_\alpha}{\partial y_3}. \] (2.20)

Solving for \( \frac{\partial v^0_\alpha}{\partial y_3} \) to give the form
\[ \frac{\partial v^0_\alpha}{\partial y_3} = \frac{2y_3}{k(\theta^0)} A^{(n-1)/2} \left( \frac{\partial p^0}{\partial x_\alpha} - \rho f^0_\alpha \right) \]

another integration yields
\[ v^0_\alpha = - \left( \frac{\partial p^0}{\partial x_\alpha} - \rho f^0_\alpha \right) \int_{y_3}^1 \frac{2\xi}{k(\theta^0)} A^{(1-n)/2} d\xi. \]

Now averaging this \( v^0_\alpha(\varepsilon y_i) \) over \([0, 1]\) and denoting this average as \( \bar{v}^0_\alpha \), we have
\[ \bar{v}^0_\alpha = \int_0^1 v^0_\alpha \, dy_3 = -s^0(x_\mu) \left( \frac{\partial p^0}{\partial x_\alpha} - \rho f^0_\alpha \right) \] (2.21)

where
\[ s^0 = 2 \int_0^1 \int_{y_3}^1 \frac{\xi}{k(\theta^0)} A^{(1-n)/2} d\xi \, dy_3 \] (2.22)

and we have replaced \( \theta^0 \) by its average \( \bar{\theta}^0 \) over the interval \((0, \varepsilon)\).

Using the second part of the assumption (2.19), the incompressibility condition (2.4) averages to
\[ \frac{\partial \bar{v}^0_\alpha}{\partial x_\alpha} = 0 \]
which with (2.21) yields a pressure equation:
\[ \frac{\partial}{\partial x_\alpha} \left( s^0 \left( \frac{\partial p^0}{\partial x_\alpha} - \rho f^0_\alpha \right) \right) = 0. \] (2.23)

This can be improved upon summing the squares of both sides of (2.20) to obtain
\[ |\nabla p^0 - \tilde{\rho} f^0| = \frac{k(\bar{\theta})}{2y_3} A^{n/2}. \] (2.24)

Hence
\[ \frac{\partial}{\partial x_\alpha} \left( m(\bar{\theta}) |\nabla p^0 - \tilde{\rho} f^0|^{1/n-1} \left( \frac{\partial p^0}{\partial x_\alpha} - \rho f^0_\alpha \right) \right) = 0 \] (2.25)
where
\[ m(\bar{\theta}^0) = |k(\bar{\theta}^0)|^{-1/n} \int_0^1 \int_{y_3}^1 (2\zeta)^{1/n} \, d\zeta \, dy_3. \]

2.3. Hyperbolic heat equations

We now turn to the energy equation (2.2) and Cattaneo’s equation (1.8). Combining (1.8) and (2.2) and eliminating \( q \) gives rise to
\[
\rho^e \epsilon^0 \frac{D\theta^e}{Dt} - s_{ij} \frac{d^e_d}{dt} + \tau \frac{\partial}{\partial t} \left[ \rho^e \epsilon^0 \frac{D\theta^e}{Dt} - s_{ij} \frac{d^e_d}{dt} \right] = \nabla \cdot (K \nabla \theta). \quad (2.26)
\]

From (2.13), we obtain
\[
s_{ij} \frac{d^e_d}{dt} = \frac{k(\theta^0)}{4} A^{(n+1)/2} + O(\epsilon)
\]
so that (2.26) becomes
\[
\rho c \left\{ \left( \frac{\partial \theta^0}{\partial t} + \epsilon \frac{\partial \theta^1}{\partial t} + \cdots \right) + \epsilon (v^0 + \epsilon v^1 + \cdots)(\frac{\partial \theta^0}{\partial x_\alpha} + \epsilon \frac{\partial \theta^1}{\partial x_\alpha} + \cdots) \right. \\
+ \frac{1}{\epsilon} (v^0_{\epsilon^2} + v^1_{\epsilon^3} + \cdots)(\frac{\partial \theta^0}{\partial y_3} + \epsilon \frac{\partial \theta^1}{\partial y_3} + \cdots)
\]
\[
+ \epsilon (v^0 + \epsilon v^1 + \cdots)(\frac{\partial \theta^0}{\partial x_\alpha} + \epsilon \frac{\partial \theta^1}{\partial x_\alpha} + \cdots) \right) - \frac{k(\theta^0)}{4} A^{(n+1)/2} + O(\epsilon)
\]
\[
\left. + \tau \rho c \left( \left( \frac{\partial \theta^0}{\partial t} + \epsilon \frac{\partial \theta^1}{\partial t} + \cdots \right) + \epsilon (v^0 + \epsilon v^1 + \cdots)(\frac{\partial \theta^0}{\partial x_\alpha} + \epsilon \frac{\partial \theta^1}{\partial x_\alpha} + \cdots) \right) \right. \\
+ \frac{1}{\epsilon} (v^0_{\epsilon^2} + v^1_{\epsilon^3} + \cdots)(\frac{\partial \theta^0}{\partial y_3} + \epsilon \frac{\partial \theta^1}{\partial y_3} + \cdots)
\]
\[
+ \epsilon (v^0 + \epsilon v^1 + \cdots)(\frac{\partial \theta^0}{\partial x_\alpha} + \epsilon \frac{\partial \theta^1}{\partial x_\alpha} + \cdots) \right) - \frac{1}{\rho c} \frac{k(\theta^0)}{4} A^{(n+1)/2} + O(\epsilon) \right) \\
= \frac{\partial}{\partial x_\alpha} \left\{ K \left( \frac{\partial \theta^0}{\partial y_3} + \epsilon \frac{\partial \theta^1}{\partial y_3} + \cdots \right) \right. \\
+ \frac{1}{\epsilon^2} \frac{\partial}{\partial y_3} \left\{ K \left( \frac{\partial \theta^0}{\partial y_3} + \epsilon \frac{\partial \theta^1}{\partial y_3} + \cdots \right) \right. \right\}. \quad (2.27)
\]

By considering the lower order terms, we obtain
\[
\frac{\partial}{\partial y_3} \left( K \frac{\partial \theta^i}{\partial y_3} \right) = 0, \quad i = 0, 1 \\
\rho c \frac{\partial \theta^0}{\partial t} - \frac{k(\theta^0)}{4} A^{(n+1)/2} + \tau \frac{\partial}{\partial t} \left\{ \rho c \frac{\partial \theta^0}{\partial t} - \frac{k(\theta^0)}{4} A^{(n+1)/2} \right\} = \frac{\partial}{\partial x_\alpha} \left( K \frac{\partial \theta^0}{\partial y_3} \right) + \frac{\partial}{\partial y_3} \left( K \frac{\partial \theta^2}{\partial y_3} \right).
\]

If we assume that the temperature profile is symmetric over the mold thickness and that on the surface of the mold the conductive heat flux, \(-q_0(x)\), is known, this implies that
\[
\frac{\partial \theta^e}{\partial x_3} = 0 \quad \text{at} \quad x_3 = 0, \quad \text{and} \quad K \frac{\partial \theta^e}{\partial x_3} = q_0(x) \quad \text{at} \quad \pm \epsilon. \quad (2.28)
\]

Averaging both sides of (2.27) and employing (2.24) and (2.28), we obtain a two-dimensional model for the energy equation:
\[
\rho c \frac{\bar{\theta}^0}{\bar{\theta}^0} - \lambda m(\bar{\theta}^0) |\nabla p^0 - \rho f^0(\alpha + 1/n) + \tau \frac{\partial}{\partial t} \left( \rho c \frac{\bar{\theta}^0}{\bar{\theta}^0} - \lambda m(\bar{\theta}^0) |\nabla p^0 - \rho f^0(\alpha + 1/n) \right) \\
= \frac{\partial}{\partial x_\alpha} \left( K \frac{\partial \bar{\theta}^0}{\partial x_\alpha} \right) + q_0(x) \quad (2.29)
\]
where \( \lambda \) is the constant that equates
\[
[k(\bar{\theta}^0)]^{-1/n} \int_0^1 (2\xi)^{(n+1)/n} \, d\xi = \lambda m(\bar{\theta}^0).
\]

Eqs. (2.25) and (2.29) give the desired two-dimensional model for the injection molding and these are independent of mold thickness \( 2\epsilon \). For \( \tau = 0 \), (2.29) reduces to the parabolic diffusion equation.

It is not our aim in this article to provide an analytical solution to the system of Eqs. (2.25) and (2.29). In the next section, we will consider a simpler case with constant coefficients \( \rho, c, \tau, m(\bar{\theta}^0), K \).

3. Weak solution of Problem II

In this section, we consider the initial–boundary value Problem II for non-isothermal, Newtonian Hele–Shaw flows; moreover, we prove that Problem II has a unique weak solution for \( n = 1 \).

By setting \( n = 1 \), it is easy to see that Eq. (1.10) is a non-dimensional form of Eq. (2.25) upon assuming that \( m(\bar{\theta}^0) \) is a constant coefficient.

Assuming that the boundary condition \( p_0 \) in Eq. (1.12) and body forces \( \vec{f} \) in Eq. (1.10) are time independent, the Dirichlet problem
\[
\Delta p = \rho \nabla \cdot \vec{f} \quad \text{in } \Omega, \quad p = p_0 \quad \text{on } \partial \Omega,
\]
is also time independent. Therefore Eq. (2.29) reduces to (1.9).

**Definition 1.** We say that \( \{ \theta, p \} \) is a weak solution of Problem II, namely (1.9)–(1.13), if
\[
\begin{align*}
\theta &\in L^2(0, T; H_0^{1,2}(\Omega)) \quad (3.2) \\
\frac{\partial \theta}{\partial t} &\in L^2(0, T; L^2(\Omega)) \\
\frac{\partial^2 \theta}{\partial t^2} &\in L^2(0, T; H^{-1,2}(\Omega)) \\
p - p_0 &\in H_0^{1,2}(\Omega)
\end{align*}
\]
and for all \( v \in C_0^\infty(\Omega) \) and for almost all \( t \in (0, T) \),
\[
\int_\Omega \left( \frac{\partial \theta}{\partial t} v + \tau \frac{\partial^2 \theta}{\partial t^2} v + K \nabla \theta \cdot \nabla v \right) \, dx \, dt = \int_\Omega \left( \frac{\lambda k}{\rho c} |\nabla p - \vec{f}|^2 + q(x) \right) v \, dx;
\]
\[
\theta = \varphi \quad \text{and} \quad \theta_t = \psi \quad \text{on } \Omega \times \{0\}
\]
and for all \( \xi \in L^r(0, T; H_0^{1-r}(\Omega)) \),
\[
\int_\Omega (\nabla p - \vec{f}) \nabla \xi \, dx = 0.
\]

**Theorem 1.** Assume that (1.14) and (1.15) hold. Then there exists a unique weak solution to Problem II, that is, (1.9)–(1.13) in the sense of Definition 1.

**Proof.** Under the hypotheses of Theorem 1, the Dirichlet problem
\[
\Delta p = \rho \nabla \cdot \vec{f} \quad \text{in } \Omega, \quad p = p_0 \quad \text{on } \partial \Omega,
\]
has a unique solution \( p \) lying in \( C^{2,\alpha}(\Omega) \) [22, Theorem 6.14]. This implies that the first term in the right hand side of Eq. (1.9)
\[
\frac{\lambda k}{\rho c} |\nabla p - \vec{f}|^2 \in C^{1,\alpha}(\Omega) \in L^2(\Omega).
\]
Letting \( \tilde{t} = \tau - \tau e^{-t/\tau} \) and \( 0 \leq t \leq T \), then Eq. (1.9) becomes
\[
\frac{\partial^2 \theta}{\partial t^2} - \left( \frac{\tau}{\tau - \tilde{t}} \right)^2 K \Delta \theta = \frac{\lambda k}{\rho c} \left( \frac{\tau}{\tau - \tilde{t}} \right)^2 |\nabla p - \tilde{f}|^2 + \frac{\tau^2}{(\tau - \tilde{t})^2} q(x), \quad \text{in } \Omega_T. \tag{3.7}
\]
For expository reasons, we assume that \( \theta_0 = 0 \). The existence and uniqueness of the weak solutions to the initial–boundary value problem
\[
\frac{\partial^2 \theta}{\partial t^2} - \left( \frac{\tau}{\tau - \tilde{t}} \right)^2 K \Delta \theta = \frac{\lambda k}{\rho c} \left( \frac{\tau}{\tau - \tilde{t}} \right)^2 |\nabla p - \tilde{f}|^2 + \frac{\tau^2}{(\tau - \tilde{t})^2} q(x), \quad \text{in } \Omega_T \tag{3.8}
\]
\[
\theta = \theta_0 \quad \text{on } \partial \Omega \times (0, \tau (1 - e^{-T/\tau})),
\]
\[
\theta = \varphi \quad \text{and } \theta_t = \psi \quad \text{on } \Omega \times \{0\}
\]
is an immediate application of Galerkin approximations [14, Chapter 7]. It can be easily extended to the case where \( \theta_0 \neq 0 \) because Eq. (1.9) is linear for temperature \( \theta \).

The existence and uniqueness of a solution of the coupled system, Problem II, follow immediately from the existence and uniqueness of the elliptic problem (3.6) and hyperbolic problem (3.8). \( \square \)

Using the regularity results of elliptic equations (see, for example, [22, Theorems 6.14 and 6.19]) and hyperbolic equations (see, for example, [14, Theorem 7, p. 393]), we can establish the regularity theorem for the non-isothermal Newtonian Problem II.

**Theorem 2.** Assume that \( \Omega \) is a bounded domain with a \( C^\infty \) boundary and
\[
p_0, \varphi, \psi, q(x) \in C^{\infty, \alpha}(\overline{\Omega}), \quad \text{and } \tilde{f} \in [C^{\infty, \alpha}(\overline{\Omega})]^n. \tag{3.9}
\]
Suppose also the following \( m \)th-order compatibility conditions hold for \( m = 0, 1, 2, \ldots : \)
\[
\begin{align*}
\varphi_0 & := \varphi \in H^1_0(\Omega), \psi_2 := \phi \in H^1_0(\Omega), \ldots, \\
\varphi_{2l} & := \frac{d^{2l-2}g}{dt^{2l-2}}(\cdot, 0) + \left( \frac{\tau}{\tau - \tilde{t}} \right)^2 K \Delta \psi_{2l-2} \in H^1_0(\Omega) \quad \text{if } m = 2l \\
\psi_{2l+1} & := \frac{d^{2l-1}g}{dt^{2l-1}}(\cdot, 0) + \left( \frac{\tau}{\tau - \tilde{t}} \right)^2 K \Delta \varphi_{2l-1} \in H^1_0(\Omega) \quad \text{if } m = 2l,
\end{align*} \tag{3.10}
\]
where \( g := \frac{\lambda k}{\rho c} \left( \frac{\tau}{\tau - \tilde{t}} \right)^2 |\nabla p - \tilde{f}|^2 + \frac{\tau^2}{(\tau - \tilde{t})^2} q(x) \). This implies that the elliptic–hyperbolic Problem II has a unique solution \( \{p, \theta\} \in C^{\infty}(\overline{\Omega}) \) and \( \theta \in C^{\infty}(\overline{\Omega_T}) \).

**Proof.** Applying Theorem 6.19 in [22] for \( m = 0, 1, 2, \ldots, \) we have the infinite differentiability result for pressure \( p \).

In order to establish the infinite differentiability result for temperature \( \theta \), it is enough to show that the \( m \)th-order compatibility requirement can be satisfied for the solution \( p \) of elliptic problem (3.6). We notice that \( \tilde{t} = \tau (1 - e^{-t/\tau}) < \tau \) and \( |\cdot|^2 \) is infinitely differentiable and continuous. The infinite differentiability of \( \theta \) is an immediate result of Theorem 7 in [14, p. 393]. \( \square \)

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**References**


