Sequences with zero autocorrelation are of interest because of their use in constructing orthogonal matrices and because of applications in signal processing, range finding devices, and spectroscopy. Golay sequences, which are pairs of binary sequences (i.e., all entries are \(\pm 1\)) with zero autocorrelation, have been studied extensively, yet are known only in lengths \(2^n\) for \(n = 10, 26\). Ternary complementary pairs are pairs of \((0, \pm 1)\)-sequences with zero autocorrelation (thus, Golay pairs are ternary complementary pairs with no 0's). Other kinds of pairs of sequences with zero autocorrelation, such as those admitting complex units for nonzero entries, are studied in similar contexts. Work on ternary complementary pairs is scattered throughout the combinatorics and engineering literature where the majority approach has been to classify pairs first by length and then by deficiency (the number of 0's in a pair); however, we adopt a more natural classification, first by weight (the number of nonzero entries) and then by length. We use this perspective to redevelop the basic theory of ternary complementary pairs, showing how to construct all known pairs from a handful of initial pairs we call primitive. We display all primitive pairs up to length 14, more than doubling the number that could be inferred from the existing literature.

1. INTRODUCTION

A pair of (real-valued) sequences \(F = (f_1, \ldots, f_m); G = (g_1, \ldots, g_n)\) is complementary, or has zero autocorrelation, if

\[
(ff^* + gg^*)(x) = w \in \mathbb{R},
\]

where \(w\) is a constant for all \(x\) in the domain.
where $f(x) = \sum f_i x^i$ and $g(x) = \sum g_i x^i$ are (formal) Laurent polynomials, called the Hall polynomials of the sequences, and $f^*$ is the Laurent polynomial defined by $f^*(x) = f(x^{-1})$. Ternary complementary pairs are pairs of $(0, \pm 1)$-sequences (i.e., ternary sequences) with zero autocorrelation. The weight $w$, of a pair $F,G$, equals the total number of nonzero entries in the two sequences. We say that a sequence is reduced if its first and last entries are nonzero; a pair is reduced if it consists of two reduced sequences. Our first elementary lemma says that, in the discussion above, we may take $m = n$.

**Lemma 1.** If one of the sequences of a reduced ternary complementary pair has length greater than 1 then they have the same length; the first and last entries of one sequence are equal, while the other starts and ends with entries of opposite signs.

**Proof.** Let $F,G$ be reduced sequences of lengths $m$ and $n$. Then the highest-degree terms of $ff^*$ and $gg^*$ are $f_1 f_m x^{m-1} = \pm x^{m-1}$ and $g_1 g_n x^{n-1} = \pm x^{n-1}$, respectively. If max$(m, n) > 1$, and (1) is satisfied, then these terms must cancel, so $m = n$. Further, since they have opposite sign, the last statement follows.

We denote a ternary complementary pair of length $n$, with weight $w$, by $TCP(n,w)$. The deficiency of a $TCP(n,w)$ is $\delta = 2n - w$, the number of 0's in the two sequences.

Throughout this paper all sequences shall be assumed to be ternary unless otherwise specified. We shall use $a(x)$, $b(x)$, $c(x)$, ... for the Hall polynomials of $A = (a_1, ...,)$, $B = (b_1, ...,)$, $C = (c_1, ...,)$, ..., as above. We use $A^*$ for the sequence obtained by reversing the entries of the sequence $A$ (observe that $f_{A^*}(x) = x^{n+1}f_A(x)$). When displaying ternary sequences, $&$ shall be short for $&1$, a convention that eliminates the need for commas.

For example, $A = (11&); B = (101)$ is a $TCP(3,5)$, for

\[
\begin{align*}
a(x) a(x^{-1}) + b(x) b(x^{-1}) &= (x + x^2 - x^3)(x^{-1} + x^{-2} - x^{-3}) \\
&+ (x + x^2)(x^{-1} + x^{-3}) \\
&= (3 - x^2 - x^{-2}) + (2 + x^2 + x^{-2}) = 5.
\end{align*}
\]

This pair has deficiency 1 (which, as we shall see, is unusual).

Some examples of ternary complementary pairs may be found in Table I; see also [8, 16].

Golay sequences are ternary complementary pairs with zero deficiency. These are known to exist in all lengths $2^a10^{26}$ [10]; we shall show how to construct these from pairs in Table I. Some suspect that there are no
<table>
<thead>
<tr>
<th>Primitive pair</th>
<th>Type</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>(1) ; (0)</td>
<td>TCP(1, 1)</td>
<td>1 Trivial</td>
</tr>
<tr>
<td>(1) ; (1)</td>
<td>TCP(1, 2)</td>
<td>0 Trivial</td>
</tr>
<tr>
<td>(11 - 1)(101)</td>
<td>TCP(3, 5)</td>
<td>1 See Section 3</td>
</tr>
<tr>
<td>(1010001)(111 - 1)</td>
<td>TCP(7, 10)</td>
<td>4 [8]</td>
</tr>
<tr>
<td>(10110 - 01)(11000 - 1)</td>
<td>TCP(8, 10)</td>
<td>6 New</td>
</tr>
<tr>
<td>(1100000 - 1)(10001010 - 1)</td>
<td>TCP(9, 8)</td>
<td>10 Can derive from [12]</td>
</tr>
<tr>
<td>(100 - 00 - 01)(10100011 - 1)</td>
<td>TCP(9, 10)</td>
<td>8 New</td>
</tr>
<tr>
<td>(11011 - 0 - 1)(10000010 - 1)</td>
<td>TCP(9, 10)</td>
<td>8 New</td>
</tr>
<tr>
<td>(10 - 1 - 0011)(100 - 0 - 11 - 1)</td>
<td>TCP(9, 13)</td>
<td>5 Can derive from [5]</td>
</tr>
<tr>
<td>(1000 - 01 - 001)(11100000001 - 1)</td>
<td>TCP(11, 10)</td>
<td>12 New</td>
</tr>
<tr>
<td>(1110 - 110 - 1 - 0)(1000 - 000101)</td>
<td>TCP(11, 13)</td>
<td>9 Can derive from [10]</td>
</tr>
<tr>
<td>(10000 - 10 - 001)(11100000001 - 1)</td>
<td>TCP(12, 10)</td>
<td>14 New</td>
</tr>
<tr>
<td>(100 - 0 - 11 - 01)(1101100 - 101)</td>
<td>TCP(12, 16)</td>
<td>8 New</td>
</tr>
<tr>
<td>(1000000000011)(1001 - 100010 - 1)</td>
<td>TCP(13, 10)</td>
<td>16 New</td>
</tr>
<tr>
<td>(10 - 0 - 010 - 101)(1110001 - 0101)</td>
<td>TCP(13, 16)</td>
<td>10 New</td>
</tr>
<tr>
<td>(100 - 001 - 1101)(101000 - 0 - 11 - 1)</td>
<td>TCP(13, 16)</td>
<td>10 New</td>
</tr>
<tr>
<td>(1 - 10 - 0001101)(1 - 0 - 010 - 1)</td>
<td>TCP(13, 17)</td>
<td>9 [17]</td>
</tr>
<tr>
<td>(1 - 00000 - 000011)(1000100001010 - 1)</td>
<td>TCP(14, 10)</td>
<td>18 New</td>
</tr>
<tr>
<td>(100 - 0 - 10010011)(110 - 100001000 - 1)</td>
<td>TCP(14, 13)</td>
<td>15 New</td>
</tr>
<tr>
<td>(1000000000011)(1010 - 0 - 10010 - 1)</td>
<td>TCP(14, 16)</td>
<td>12 New</td>
</tr>
<tr>
<td>(1010001000001 - 1)(1 - 0000011)(100000111 - 1)</td>
<td>TCP(14, 17)</td>
<td>11 [17]</td>
</tr>
<tr>
<td>(1 - 100100 - 0011)(100011 - 00 - 001)</td>
<td>TCP(14, 17)</td>
<td>11 New</td>
</tr>
<tr>
<td>(100010000000001)(1000000011 - 01 - 001)</td>
<td>TCP(14, 17)</td>
<td>11 New</td>
</tr>
<tr>
<td>(1 - 10 - 00 - 001 - 011)(101000 - 11 - 1110 - 1)</td>
<td>TCP(14, 20)</td>
<td>8 New</td>
</tr>
<tr>
<td>(10 - 11101 - 11 - 1)(100010 - 00 - 001 - 1)</td>
<td>TCP(14, 20)</td>
<td>8 New</td>
</tr>
</tbody>
</table>

Other cases; if there are, they are not of small length. It has long been known that the length of Golay sequences must be a sum of two squares. More recently, the following result of Eliahou et al. [5] provided a more general restriction.

**Theorem 2.** The length of Golay sequences is not divisible by any number congruent to 3 mod 4.

Theorem 2 is a consequence of the following more general result, given as Lemma 1.5 in [6]. Since the proof itself is of fundamental importance to the theory, we include it here.
Theorem 3. Suppose \( p \equiv 3 \mod 4 \) is prime and \( F, G \) is a pair of integer sequences satisfying (1), at least one of whose entries is not divisible by \( p \). Then \( w \) is not divisible by \( p \).

Proof. Let \( p \) be an odd prime divisor of \( w \), the weight of \( F, G \). Consider their Hall polynomials \( f, g \), and all polynomials in this proof, to be reduced modulo \( p \). Since \( \mathbb{Z}_p[x] \) is a principal ideal domain, greatest common divisors are well defined, and polynomials in this ring factor uniquely.

Since \( p \) does not divide all entries of \( F \) and \( G \), one of \( f, g \) is nonzero. Let \( h = \gcd(f, g) \), so that \( f(x) = h(x) k(x) \) and \( g(x) = h(x) r(x) \). Therefore,

\[
f(x) f(x^{-1}) + g(x) g(x^{-1}) = w = 0 = h(x) h(x^{-1}) (k(x) k(x^{-1}) + r(x) r(x^{-1})).
\]

Let \( d \) be the (common) degree of \( r \) and \( k \). Then,

\[
k(x) x^d k(x^{-1}) = -r(x) x^d r(x^{-1}).
\]

Now, \( k \) and \( r \) are relatively prime, so \( r(x) \mid x^d k(x^{-1}) \). Thus, \( r(x) = cx^d k(x^{-1}) \), \( c \in \mathbb{Z}_p \). So

\[
x^d [k(x) k(x^{-1}) + c^2 k(x^{-1}) k(x)] = (1 + c^2) x^d k(x) k(x^{-1}) = 0.
\]

It follows that \( c^2 = -1 \), which implies \([18]\) that \( p \equiv 1 \mod 4 \). \( \blacksquare \)

Observe that this result—and so also Theorem 2—is fundamentally about weight, rather than length, of sequences. Of course, we are particularly interested in its application to ternary complementary pairs, as follows.

Corollary 4. If \( w \neq 0 \) has a factor congruent to 3 mod 4, then TCP(\( n, w \)) does not exist.

The following result, handy in the search for complementary pairs, is easy to prove directly (i.e., let \( x = 1 \) in (1)); the restriction it implies on the value of \( w \) is, however, subsumed by Theorem 3.

Lemma 5. If \( a \) is the sum of the entries of \( F \) and \( b \) is the sum of the entries of \( G \), where \( F, G \) are integer sequences satisfying (1), then \( a^2 + b^2 = w \).

Some Golay sequences of small lengths—namely \( n = 34, 50, 58, 68 \) \([1, 6, 14]\), whose existence is not ruled out by Theorem 2—have been eliminated by other methods. Our theoretical knowledge of Golay sequences appears, for the present, to have reached a dead end, so it is natural to look at
generalizations such as ternary complementary pairs, which are useful in their own right. Complex [2, 4, 13] and dihedral [15] Golay sequences generalize in a different direction, allowing entries \(\pm 1, \pm i\) or elements of the dihedral signed group, respectively; polyphase complementary pairs work over the complex units [7]. Results analogous to Theorem 3 have proved useful in these contexts as well.

2. RESTRICTIONS WHEN \(\delta\) IS SMALL

It is well known that \(TCP(n, w)\) implies a weighing matrix \(W(2n, w)\). This is a Hadamard matrix if \(\delta = 0\) \((w = 2n)\), in which case it is well known that \(n = 1\) or \(n\) is even.

**Lemma 6.** If there is a TCP\((n, w)\), then \(w \leq 2n\); equality cannot hold if \(n\) is an odd number greater than 1.

That is, \(\delta \neq 0\) for odd \(n > 1\). The following result, from [8], says that \(\delta = 1\) only in the case of \(TCP(1, 1)\) or \(TCP(3, 5)\). We defer the proof of this result until Section 7.

**Lemma 7.** If there is a TCP\((n, 2n - 1)\), then \(n = 1\) or 3.

Only under certain conditions can \(\delta = 2\). The following result is also given in [8].

**Lemma 8.** If \(A; B\) is a reduced TCP\((n, 2n - 2)\) and \(a_i = 0\), then either \(b_i = 0\) or \(b_{n-i+1} = 0\).

**Proof.** Without loss of generality, let us assume that the zero entry of lowest index in any of \(A, B, A^*, B^*\) is \(a_i = 0\). If the \(i\)th entries of \(B, A^*, \) and \(B^*\) are all nonzero, then the coefficient of \(x^{n-i}\) in (1) is odd—a contradiction. Thus, one of \(b_i, a_{n-i+1}\) and \(b_{n-i+1}\) is zero. If \(a_{n-i+1} = 0\), then the coefficient of \(x^{(n-i+1)-i}\) is odd, unless \(i = \frac{n-1}{2}\), in which case only one zero is accounted for in \(A\). So, in any case, either \(b_i = 0\) or \(b_{n-i+1} = 0\), as required.

The following result, which addresses the case \(\delta = 3\), is from [8] which, unfortunately, does not include a proof. Because it is shown more naturally in the setting of Boolean sequences, we refer the reader to [3] for a proof.

**Lemma 9.** If \(A; B\) is a reduced TCP\((n, 2n - 3)\), then \(n = 4m + 2\) and \(A; B\) can be interchanged and/or reversed (these terms are introduced just before Theorem 10) as necessary so that \(a_{2m+1} = b_{m+1} = b_{3m+2} = 0\).
3. NEW SEQUENCES FROM OLD

The next result gives a number of transformations under which the set of ternary complementary pairs of a given weight is closed. We shall consider two pairs equivalent if a series of these transformations converts one into the other. The first six operations we shall refer to as interchanging, shifting, reversing, negating, alternating, and expanding the pair, respectively; the reverse of shifting is reducing and the reverse of expanding is contracting.

**Theorem 10.** The set of ternary complementary pairs of weight \( w \) is closed under each of the following operations:

1. exchanging the two sequences for each other;
2. appending any number of 0's to either or both ends of either or both sequences;
3. reversing one or both sequences;
4. negating one or both sequences;
5. negating every second entry of both sequences;
6. inserting a fixed number of 0's between all pairs of consecutive entries of both sequences;
7. reversing any of the above operations, when possible.

**Proof.** Let \( F \) and \( G \) be real sequences of length \( m \). Let \( H, K \) be the pair of sequences obtained by one of the following operations:

1. letting \( H = G \) and \( K = F \). Then \( h = g \) and \( k = f \);
2. adding \( t \) 0's to the end of \( F \) to obtain \( H \), while \( K = G \). Then \( h = f \) and \( k = g \);
3. letting \( H = F^* \) and \( K = G \). Then \( h(x) = x^{m+1}f(x^{-1}) \) and \( k = g \);
4. letting \( H = -F \) and \( K = G \). Then \( h = -f \) and \( k = g \);
5. negating every second entry of \( F \) and \( G \) to obtain \( H \) and \( K \), respectively. Then \( h(x) = -f(-x) \), \( k(x) = -g(-x) \);
6. inserting \( t \) 0's between consecutive entries of \( F, G \). Then \( h(x) = f(x^{t+1}) \) and \( k(x) = g(x^{t+1}) \).

In each case, simple algebra shows that (1) holds if and only if \( hh^* + kk^* = w \). The operations described in parts 1 to 6 of the theorem are all combinations of the above set of six, which clearly preserve ternary sequences.
Reversing these operations when possible thus also preserves the set of \(\text{TCP}(r, w)'s\).

An obvious motivation for defining equivalence this way is economy: there is no need to record pairs equivalent to previously recorded pairs. Thus, among other considerations, we may restrict our attention to reduced ternary complementary pairs that cannot be contracted, in which the first sequence starts and ends with 1 and the second starts with 1 and ends with \(-1\).

Define the support of a sequence as the set of positions in which it is nonzero; a pair of sequences is disjoint if the two sequences have disjoint support and is conjoint if the two sequences have the same support (which is possible only when \(w\) is even).

Observe that every TCP is equivalent to a disjoint pair (shifting alone is sufficient, though combinations of the equivalence operations could achieve this in various ways). On the other hand, a given TCP may or may not be equivalent to a conjoint pair.

We now show how to construct various ternary complementary pairs by elementary means.

Trivially, there are disjoint \(\text{TCP}(1, w)'s\) with \(w = 0, 1\), namely \((0);(0)\) and \((1);(0)\) (the latter appears in Table I). The simple weight-doubling trick given in the following lemma gives \(\text{TCP}(1, 2)\) (which may also be considered trivial). Addition and subtraction of sequences is as with vectors.

**Lemma 11.** If \(F;G\) is a disjoint \(\text{TCP}(n, w)\), then \(F + G;F - G\) is a conjoint \(\text{TCP}(n, 2w)\).

**Proof.**\((f + g)(f + g)^* + (f - g)(f - g)^*)(x) = 2(f^* + g^*)(x) = 2w.  \)

By essentially the same argument, the following converse is also true.

**Lemma 12.** If \(F;G\) is a conjoint \(\text{TCP}(n, 2w)\), then \((F + G)/2,(F - G)/2\) is a disjoint \(\text{TCP}(n, w)\).

Lemma 11 doubles the weight of a pair; Lemma 12 halves it. Although neither changes the actual length of sequences, observe that pairs of length greater than 1 cannot be both reduced and disjoint. Thus, a pair whose weight is halved by Lemma 12 can be shortened by reducing; conversely, the weight of reduced pairs cannot be doubled by Lemma 11 until they are first lengthened by shifting.

We shift the \(\text{TCP}(1, 2)\) obtained above to obtain \((1, 0);(0, 1)\). Lemma 11 yields Golay sequences \((11);(1-)\) of length 2. From these we similarly
obtain sequences $(111 \ldots 1)(11 \ldots 1)$ of length 4; following this procedure iteratively, we obtain Golay sequences of all lengths $2^n$.

Let us say that $A$ is **symmetric** if $A^* = A$ and **skew** if $A^* = -A$.

**Lemma 13.** If $n$ is odd, $B$ is skew, and $C$ is the sequence obtained by replacing the middle entry of $B$ with 1, then $A;B$ is a TCP($n, w$) if and only if $A;C$ is a TCP($n, w + 1$).

**Proof.** The middle entry of $B$ must be 0, so $C$ is a ternary sequence if and only if $B$ is. Writing $n = 2k - 1$, we have $(aa^* + cc^*)(x) = aa^*(x) + (b(x) + x^k)(b(x^{-1}) + x^{-k}) = (aa^* + bb^*)(x) + x^{-k}(b(x) - b(x)) + 1 = (aa^* + bb^*)(x) + 1$. The result follows.

Shifting TCP(1, 2) to obtain (100);(001) and applying Lemma 11, we obtain (101);(10\ldots) (alternately, these could be obtained by expanding the Golay sequences of length 2 above). Since the second sequence is skew, Lemma 13 tells us that (101);(11\ldots) is a TCP(3, 5). Shifting and using Lemma 11 again, we obtain the pair (1101);(11\ldots) of weight 10. Reversing the second sequence, shifting and using Lemma 11 once more, we obtain Golay sequences of length 10, (1101\ldots11);(11111\ldots). We can now double repeatedly to obtain Golay sequences of lengths $2^k \cdot 5$, $k > 0$; by slight modifications of this construction we obtain a variety of other ternary complementary pairs of these weights as well.

To obtain Golay sequences from the TCP(11, 13) in Table I, shift the second sequence by three positions and apply Lemma 11 to obtain the following TCP(14, 26):

$$(111 \ldots 11 \ldots 1 \ldots 101);(111 \ldots 11 \ldots 1 \ldots 0 \ldots).$$

TCP(26, 52), and therefore Golay sequences of length 26, are now obtained by reversing one of these sequences, shifting, and applying Lemma 11. As above, we may continue the process, obtaining Golay sequences of lengths $2^t \cdot 13$, $t > 0$, and various other TCP$(*, 2^t \cdot 13)$'s.

For sequences $A = (a_1, \ldots, a_n)$, $B$, let us write $A \otimes B = (a_1B, \ldots, a_nB)$, where $aB$ denotes scalar multiplication. Thus, $A \otimes B$ has Hall polynomial $x^{-m}a(x^n) b(x)$, where $m$ is the length of $B$. This is the **Kronecker product** of sequences, which we use to define the following product of pairs, which has been given in various forms (e.g., [11, Theorem 1; 8, Theorem 4]); it appears to have originated with Golay [9] in the context of Golay pairs). We shall refer to this product simply as **multiplication** of TCP's.

---

1 All three parts of this theorem are specific instances.
Theorem 14. Suppose \( A;B \) is a TCP \((m, w)\), \( C;D \) is a TCP \((n, z)\), and one of the pairs is disjoint. Then

\[
U = A \otimes C + B \otimes D;
\]
\[
V = A \otimes D^* - B \otimes C^*
\]
is a TCP \((mn, wz)\).

Proof. The Hall polynomials of \( U \) and \( V \) are, respectively,

\[
u(x) = x^{-n}(a(x^n) c(x) + b(x^n) d(x)) \quad \text{and} \quad v(x) = x^{-n}(a(x^n) d(x^{-1}) - b(x^n) c(x^{-1})).
\]

Clearly, \( U \) and \( V \) are ternary sequences of length \( mn \), and

\[
(uu^* + vv^*)(x) = aa^*(x^n) cc^*(x) + ab^*(x^n) cd^*(x) + ba^*(x^n) dc^*(x) + bb^*(x^n) dd^*(x) + aa^*(x^n) dd^*(x) - ab^*(x^n) cd^*(x) - ba^*(x^n) dc^*(x) + bb^*(x^n) cc^*(x) = (aa^* + bb^*)(x^n)(cc^* + dd^*)(x) = wz.
\]

The result follows.

Observe that Lemma 11 is subsumed by this result (i.e., take \( C = D = (1) \)).

Now, since Golay pairs are conjoint, they give appropriate sequences \( F, G \) for Lemma 12. Using the resulting disjoint ternary complementary pair for \( A;B \) and another Golay pair for \( C;D \) in Theorem 3, we obtain Golay sequences whose length is the product of the lengths of the original pairs; consequently, we have the following well-known result, which we mentioned earlier.

Theorem 15. The set of lengths of Golay sequences is closed with respect to multiplication. In particular, there are Golay sequences of all lengths \( 2^{a}10^{b}26^{c} \).

4. PRIMITIVE PAIRS

When should we say that a certain TCP is “known”? Surely a pair is known if an equivalent pair is known. Further, we must regard pairs obtained by multiplying known pairs as known. Pairs not obtainable from previously known pairs by equivalence or multiplication are new. Observe that every new pair essentially doubles the variety of sequences that can be constructed.

This motivates the following definition: If a reduced ternary complementary pair cannot be obtained by the multiplication of pairs of smaller
weight or shortened by contraction, we say that it is *primitive*. Apparently
the term *seed*, used in [8], is intended to convey much the same idea—that
is, it denotes pairs not obtained from simpler pairs by standard construc-
tions.

The problem of constructing ternary complementary pairs is thus
reduced to the problem of constructing primitive pairs, up to equivalence.

Table I contains all primitive pairs up to length 14 (for which only
weights 1, 5, 8, 10, 13, 16, 17, and 20 are possible), obtained by an
exhaustive computer search. In spite of our definition, we have (arbitrarily)
decided to count \((1;0)\) as primitive, but not \((0;0)\).

We claim that all ternary complementary pairs previously appearing in
the literature may be constructed from the 9 pairs numbered 1, 2, 3, 4, 6,
9, 11, 17, and 21 in Table I, using the methods in Section 3—that is, equi-
valence operations and multiplication of *TCP*’s. The seventeen new pairs
thus significantly broaden the scope of the existing theory.

5. “RANDOM” PRODUCTS?

Let us now measure the claim of the previous section against the
“randomly” occurring products mentioned in [11]—probably the most
ambitious work of its type to date. The authors of that paper display a
number of product constructions for *TCP*s, indicating that that they were
unable to associate some of them with known standard constructions such
as we have given. They offer the product

\[
X = (-Q, -B, A, B, P, -Q, P); \\
Y = (-Q, -P, -Q, -A, B, A, -P),
\]
as evidence of such constructions, where \(A;B\) is a conjoint *TCP* \((n, 2w)\) and
\(P;Q\) is the disjoint *TCP* \((n, w)\) obtained from \(A;B\) by Lemma 12. Then \(X;Y\)
is a *TCP* \((7n, 10w)\). They ask whether there is an (unknown) standard result
that leads to this and other “random” products. Observe how we obtain
this product as a combination of the standard constructions of Section 3.

By Lemmas 11 and 12, \(A;B\) is a conjoint *TCP* \((n, 2w)\) if and only if \(P;Q\)
is a disjoint *TCP* \((n, w)\). Thus we may regard the construction as a product
involving \(P;Q\), rather than \(A;B\), as follows.

\[
X = (-Q, Q - P, P + Q, P - Q, P, -Q, P) \\
= (0 - 11101) \oplus P + (-11 - 0 - 0) \oplus Q = S \oplus P + T \oplus Q,
\]

\[
Y = (-Q, -P, -Q, -P - Q, P - Q, P + Q, -P) \\
= (0 - 0 - 11 - ) \oplus P - (0 - - - - - 10) \oplus Q = T^* \oplus P - S^* \oplus Q,
\]
where \( S; T = (0 - 11101); (-11 - 0 - 0) \), a (non-reduced) \( TCP(7, 10) \), easy to construct from known pairs. The equivalent pair, \( X; -Y^* = S \otimes P + T \otimes Q; S \otimes Q^* - T \otimes P^* \), is the result of multiplying \( S; T \) by \( P; Q \).

All the products in Table I of [11] can be similarly decomposed into the constructions given in Section 3; among those products, four involve reversal of sequences in a way that requires the following less obvious step. Suppose \( U; V \) is a ternary complementary pair. So, then, is \( U; V^* \). Thus, if \( X; Y \) is another ternary complementary pair, and one of these pairs is disjoint then, by Theorem 14, so is \( X \otimes U + Y \otimes V^*; X \otimes V - Y \otimes U^* \).

For example, the first such product listed in Table I of [11] is given as

\[
(-Q^*, -A^*, P^*, P, B, Q); (-Q^*, -B^*, -P^*, P, -A, -Q),
\]

which we represent solely in terms of \( P; Q \) as

\[
(-Q^*, -P^* - Q^*, P^*, P, P - Q, Q) = (0, 1) \otimes (P, P - Q, Q) + (1, 0) \otimes (P, P - Q, -Q^*);
\]

\[
(-Q^*, Q^* - P^*, -P^*, P, -P - Q, -Q) = (0, 1) \otimes (P, P - Q, -Q) + (1, 0) \otimes (P, P - Q, Q^*).
\]

This is the step described above, with \( X; Y = (0, 1); (1, 0) \) and \( U; V = (P, P - Q, Q); (P, -P - Q, -Q), \) which is equivalent, by reversing the second sequence, to \( (P, P - Q, Q); (Q^*, P^* + Q^*, -P^*) \), which can be obtained by multiplying \((110); (0 - 1)\) by \( P; Q \).

The fractional multiplication factors given in [11] are accounted for by the fact that the products are stated in terms of \( A; B \) instead of \( P; Q \).

What about “interleaving” of complementary sequences, as used in that paper and elsewhere? This doesn’t, by itself, exceed the scope of the results in Section 3, for interleaving may be accomplished by expanding a pair, shifting one sequence by one position to make the pair disjoint, and applying Lemma 11.

Another common construction “doubles” pair \( A; B \) to \((A, B); (A, -B)\), different from Lemma 11 in that \( A; B \) need not be disjoint. However, this can be accomplished by applying Lemma 11 to the equivalent (but disjoint) pair \( F; G = (A, 0_n); (0_n, B) \).

6. MINIMUM LENGTH FOR A GIVEN WEIGHT

The question of existence of ternary complementary pairs has been resolved for weights up to 29, which is now the first unresolved case. Table II gives the state of the art up to weight 100. A unique feature of this
The table is that it gives the minimum length for a given weight, whereas most previous work considers minimum deficiency, and therefore the largest weight, for a given length.

If the actual shortest length for a weight is not known, the table gives a lower bound, reflecting an exhaustive computer search and/or direct analysis; if an upper bound is given, it is the smallest known length, though not proven minimum. If no upper bound is given, no pairs with this weight

<table>
<thead>
<tr>
<th>$w$</th>
<th>Min. length</th>
<th>Source of upper bound</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>Trivial</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>Golay sequences obtained from $TCP(1, 1)$ by Lemma 13</td>
</tr>
<tr>
<td>4</td>
<td>2</td>
<td>Golay sequences obtained from $TCP(1, 2)$ by Lemma 11</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>Obtained from $TCP(3, 4)$ by Lemma 13</td>
</tr>
<tr>
<td>8</td>
<td>4</td>
<td>Golay sequences obtained from $TCP(2, 4)$ by Lemma 11</td>
</tr>
<tr>
<td>10</td>
<td>6</td>
<td>Obtained from $TCP(3, 5)$ by Lemma 11</td>
</tr>
<tr>
<td>13</td>
<td>9</td>
<td>Primitive pair</td>
</tr>
<tr>
<td>16</td>
<td>8</td>
<td>Golay sequences obtained from $TCP(4, 8)$ by Lemma 11</td>
</tr>
<tr>
<td>17</td>
<td>13</td>
<td>Primitive pair</td>
</tr>
<tr>
<td>20</td>
<td>10</td>
<td>Golay sequences obtained from $TCP(6, 10)$ by Lemma 11</td>
</tr>
<tr>
<td>25</td>
<td>18</td>
<td>Shift $TCP(3, 5)$ by 3 places and multiply by $TCP(3, 5)$</td>
</tr>
<tr>
<td>26</td>
<td>14</td>
<td>Obtained from $TCP(9, 13)$ by Lemma 11</td>
</tr>
<tr>
<td>32</td>
<td>16</td>
<td>Golay sequences obtained from $TCP(8, 16)$ by Lemma 11</td>
</tr>
<tr>
<td>34</td>
<td>$19 \leq n \leq 24$</td>
<td>Shift $TCP(13, 17)$ by 11 places and apply Lemma 11 ($n=18$ eliminated in [16])</td>
</tr>
<tr>
<td>37</td>
<td>$n \geq 20$</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>$n \geq 22$</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>$26 \leq n \leq 30$</td>
<td>Shift $TCP(3, 5)$ by 2 places and multiply by $TCP(3, 5)$ to obtain $TCP(21, 25)$; shift this by 9 places and apply Lemma 11</td>
</tr>
<tr>
<td>52</td>
<td>26</td>
<td>Golay sequences obtained from $TCP(14, 26)$ by Lemma 11</td>
</tr>
<tr>
<td>53</td>
<td>$n \geq 29$</td>
<td></td>
</tr>
<tr>
<td>58</td>
<td>$n \geq 30$</td>
<td></td>
</tr>
<tr>
<td>61</td>
<td>$n \geq 33$</td>
<td></td>
</tr>
<tr>
<td>64</td>
<td>32</td>
<td>Golay sequences obtained from $TCP(16, 32)$ by Lemma 11</td>
</tr>
<tr>
<td>65</td>
<td>$34 \leq n \leq 42$</td>
<td>Multiply disjoint $TCP(14, 13)$, obtained from $TCP(11, 13)$, by $TCP(3, 5)$</td>
</tr>
<tr>
<td>68</td>
<td>$35 \leq n \leq 40$</td>
<td>Apply Lemma 11 to disjoint $TCP(40, 34)$ obtained from $TCP(24, 34)$</td>
</tr>
<tr>
<td>73</td>
<td>$n \geq 38$</td>
<td></td>
</tr>
<tr>
<td>74</td>
<td>$n \geq 38$</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>40</td>
<td>Golay sequences obtained from $TCP(20, 40)$ by Lemma 11</td>
</tr>
<tr>
<td>82</td>
<td>$n \geq 42$</td>
<td></td>
</tr>
<tr>
<td>85</td>
<td>$45 \leq n \leq 72$</td>
<td>Multiply disjoint $TCP(24, 17)$ obtained from $TCP(14, 17)$ by $TCP(3, 5)$</td>
</tr>
<tr>
<td>89</td>
<td>$n \geq 46$</td>
<td></td>
</tr>
<tr>
<td>97</td>
<td>$n \geq 50$</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>$51 \leq n \leq 58$</td>
<td>Shift $TCP(30, 50)$ by 28 places and apply Lemma 11</td>
</tr>
</tbody>
</table>
are known. Weights not listed are eliminated by Corollary 4—the only way we currently know how to eliminate weights.

Gysin and Seberry [12] have recently independently examined minimum length for given weight; they provide better lower bounds for $w = 29, 34, 37$.

7. SOME ADDITIONAL THOUGHTS

One may wonder—Why fuss about classifying sequences first by weight rather than by length? It is this perspective that drives our definition of equivalence. Since it is not always the case that reduced pairs of maximum weight for a given length have minimum length for their weight, or vice-versa, the two approaches are definitely at odds. Note that the minimum length of sequences is not a monotone function of weight.

Nevertheless it is clear that weight is the principal consideration. Multiplication (as in Theorem 14) and factorization (as in the proof of Theorem 3) are arguably the central theoretical results about ternary complementary pairs; both reflect the importance of weight, while length plays a minor role.

It is not only for theoretical reasons that weight is the principal issue. In combinatorics one uses sequences with zero autocorrelation to construct orthogonal designs having difficult weights. Once a weight is established this way for some length, this length can be increased arbitrarily by the operation we have called shifting. Thus, weight is fundamental, and length is arbitrarily large.

In signal processing, one is likely to ask first what strength of received signal (corresponding to weight) is desired before one considers its duration (corresponding to length); deficiency is a measure of inefficiency resulting from one’s choice of weight. Minimum deficiency with respect to weight appears more useful in this setting than with respect to length.

Since one can apparently work with non-reduced pairs with impunity in all current applied contexts, it is surprising that so much effort has gone into a notion of optimality that requires consideration only of the reduced case.

For example we argue that, since there are Golay sequences of length 10, the minimum deficiency for length 11 ought to be $\delta = 2$, rather than $\delta = 6$, as reported elsewhere (see, for example, [8])—even though the former cannot be attained by reduced sequences.

Finally, the theory we have outlined here—in which all pairs are derived from a fundamental set of primitive pairs, demands that principal emphasis be placed on weight. From this perspective the current state of the art for ternary complementary pairs is seen to be wonderfully compact: all known pairs arise from a handful of primitive cases.
We now make a number of rather obvious conjectures, motivated by observable patterns among known sequences and, of course, wishful thinking.

Our first conjecture says that the weights of primitive pairs are precisely those numbers not eliminated by Corollary 4.

**Conjecture 1.** If \( w \) has no factor congruent to 3 mod 4, then a \( TCP(n, w) \) exists, for some \( n \).

As shown in Table II, this conjecture is established for all \( w < 29 \). The weights less than 100 for which this conjecture remains unresolved are \( w = 29, 37, 41, 53, 58, 61, 73, 74, 82, 89, \) and 97. Because of Theorem 14, the weights of ternary complementary pairs form a multiplicative monoid; so to prove Conjecture 1, it would suffice to show that every prime number congruent to 1 mod 4 is attainable as the weight of TCPs. The following is weaker than even this, but useful because it focuses attention on prime weights. It may also be considered as a weak partial converse to Theorem 14.

**Conjecture 2.** If \( TCP(n, w) \) exists and \( p \mid w \), then \( TCP(m, p) \) exists, for some \( m \).

In other words, the monoid of valid weights of TCP’s is freely generated by a set of prime numbers. An obvious way to attempt to prove this conjecture would involve showing that, if \( w \) is composite, then a \( TCP(n, w) \) must factor over multiplication of pairs. However, since we have displayed primitive pairs with composite weights, it is unlikely that any such approach would work, unless it involves a radically different kind of product.

If there is one ternary complementary pair of weight \( w \), then there are infinitely many distinct such pairs. This can be established by expansion alone (of course pairs so produced are all equivalent, and all but the original are contractible). But how many inequivalent primitive pairs of weight \( w \) can exist?

**Conjecture 3.** For any \( w \), there are only finitely many primitive ternary complementary pairs of weight \( w \).

In other words, primitive \( TCP(n, w) \)'s exist for only finitely many \( n \); there do not exist infinitely many inequivalent \( TCP(*, w) \)'s. This conjecture can easily be verified for weights smaller than 8.

For a given weight, how large is the smallest \( n \) such that \( TCP(n, w) \) exists? Here is another setting in which it is useful to classify ternary complementary pairs by weight, for the evidence on hand suggests that this number behaves differently for different weight classes.

Ternary complementary pairs with odd weight seem to behave differently from pairs of even weight, so this is one natural division. Among those odd
weights for which the minimum length is known, the relationship between these two numbers is remarkably linear—consider the data in Table III. The regression line is $n = 0.732w - 0.136$; the coefficient of correlation is 0.997, and each of the known minimum lengths is predicted by this line to within 1. But 5 points of data, especially for small values having to satisfy combinatorial constraints, cannot be used reliably to make inferences about larger values. Further, this line predicts a minimum length of 47.5 for $w = 65$, whereas we can construct a pair of length 42 (the actual minimum, which is unknown, must then be at least 5.5 units off this line). The predicted value for weight 85, on the other hand, is about 62, comfortably between the bounds we have given.

Perhaps, one might suggest, primality is the key to understanding the relationship between odd $w$ and $n$. One will immediately find, for example, that the values of $n$ for the first four weights in Table III exactly fit the quadratic function $n = (w^2 + 18w + 29)/48$, but it would be raw speculation to form a hypothesis from this, particularly since $n$ increases much faster according to this curve than one would infer from experience. For example, this parabola predicts a minimum length of 29 for weight 29; in comparison, the regression line gives a value close to 21, which experience would suggest is a more reasonable guess.

There is at least one concrete benefit of recording the above regression line: It provides a rough measure of what to regard as a “small” deficiency for a ternary complementary pair of odd length. If we can plot a new point under the line we may consider it a candidate for optimum length, whereas a point very much above the line is probably not of minimum length, and it is worth looking for shorter pairs of that weight. Further, when searching for pairs of larger weights, one can expect failure among sequences significantly shorter than this line predicts. Thus, one can reasonably begin a search for $TCP(n, 29)$ at $n = 20$, since lengths 16, 17, 18, and 19 are unlikely to bear fruit.

<table>
<thead>
<tr>
<th>$w$</th>
<th>1</th>
<th>5</th>
<th>13</th>
<th>17</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n$</td>
<td>1</td>
<td>3</td>
<td>9</td>
<td>13</td>
<td>18</td>
</tr>
</tbody>
</table>

TABLE III
Known Minimum Lengths $n$ for Odd Weight $w$
Lemma 13 looks promising initially as a way of capitalizing on special internal structure of TCPs, but the only known examples of ternary complementary pairs in which one of the sequences is skew are all obvious cases with weights 0, 1, and 4.

Even so, it is tempting to specialize the hypotheses of Lemma 13 even further, as follows: if \( A; B \) is a TCP\((n, w) \) and \( C; D \) is a TCP\((n, z) \), where \( A \) and \( D \) are symmetric and \( B \) and \( C \) are skew, and \( A, C \) are disjoint and \( B, D \) are disjoint, then \( A + C; B + D \) is a TCP\((n, w + z) \). For, among known cases, if one sequence is skew, the other is symmetric. However, the following result implies that there are no other examples of this type (since all sequences of weight less than 8 are known).

**Theorem 16.** If \( A; B \) is a TCP\((n, w) \), where \( A \) is symmetric and \( B \) is skew, then \( w \in \{0, 1, 4\} \).

**Proof.** If \( n \) is even, then \( A = (X, X^*) \), \( B = (Y, -Y^*) \). So \( A + B = (X + Y, X^* - Y^*) \); \( A - B = (X - Y, X^* + Y^*) \) = \((X + Y, X^* - Y^*)^* \) is a complementary pair of integer sequences. But then \((X + Y, X^* - Y^*) \) is a (single!) sequence with zero autocorrelation, which therefore has at most one nonzero entry. It follows that \( X \) and \( Y \) each have at most one nonzero entry, so \( w = 0 \) or 4. If \( n \) is odd then, without loss of generality, \( A = (X, e, X^*) \), \( B = (Y, 0, -Y^*) \), where \( e = 0, \pm 1 \). As above, we obtain that \((X + Y, e, X^* - Y^*) \) has zero autocorrelation, so \( w = 0, 1 \) or 4. □

This result also provides us with the key point for establishing Lemma 7.

**Proof of Lemma 7.** Let \( A; B \) be a TCP\((n, w) \), \( w = 2a - 1 \), the only 0 occurring in the \( k \)th position of \( A \). If \( k \neq \frac{a - 1}{2} \), we may assume without loss of generality that \( k < \frac{a - 1}{2} \). Then the coefficient of \( x^{a - k} \) in the polynomial on the left hand side of (1) is odd, and hence nonzero—a contradiction. Thus the 0 must occur in the center position of \( A \).

So we can assume that \( n = 2m + 1 \), \( A = (a_1, ..., a_m, 0, c_m, ..., c_1) \) and, without loss of generality, \( B = (b_1, ..., b_m, 1, d_m, ..., d_1) \). The coefficient of \( x^{a - 1} \) in (1) is thus \( a_1 c_1 + b_1 d_1 = 0 \). Since there are two terms, both equal to \( \pm 1 \), it follows that they have opposite signs, so \( a_1 b_1 c_1 d_1 = -1 \). Similarly the coefficient of \( x^{a - 2} \) is \( a_1 c_2 + a_2 c_1 + b_1 d_2 + b_2 d_1 = 0 \). Since there are four terms, two are 1 and two are \(-1 \), so \( a_1 c_1 a_2 c_1 b_1 d_2 + b_2 d_1 = (a_1 b_1 c_1 d_1)(a_2 b_2 c_2 d_2) = 1 \). Thus also, \( a_2 b_2 c_2 d_2 = -1 \). Continuing this procedure we obtain \( a_i b_i c_i d_i = -1 \) for \( i = 1, ..., m \).

Now the product of the terms making up the coefficient of \( x^m \) is

\[
a_2 \cdots a_m c_2 \cdots c_m b_1 \cdots b_m d_1 \cdots d_m = (-1)^{m-1} b_1 d_1.
\]


Since there are $2m$ terms in this coefficient, their product is $(-1)^m$, so $b_1d_1 = -1$. By similarly considering the coefficients of $x^{m-1}$, ..., $x^1$, it follows that $b_id_i = -1$, $a_ic_i = 1$, $i = 1, ..., m$.

Thus, $A = (S, 0, S^*)$, $B = (T, 1, -T^*)$, where $S = (a_1, ..., a_m)$ and $T = (b_1, ..., b_m)$ are binary sequences. By Lemma 13, $(S, 0, S^*); (T, 0, -T^*)$ is a TCP$(2m + 1, 4m)$. By Theorem 16, $m = 0$ or 1. It follows that $n = 1$ or 3.

What else can we say about candidate sequences for use in Lemma 13, seeing that the necessary condition is weaker than in Theorem 16? Since a skew sequence sums to 0, Lemma 5 gives us the following result.

**Theorem 17.** If $A;B$ is a TCP$(n, w)$ and $B$ is skew, then $w$ is square. Further, the sum of the entries of $A$ is $\pm \sqrt{w}$.

The next conjecture is supported by Theorems 16 and 17.

**Conjecture 4.** If $A;B$ is a TCP$(n, w)$ and $B$ is skew, then $w \in \{0, 1, 4\}$.

Our final conjecture seems likely in light of the evidence in Table I.

**Conjecture 5.** If there exists a TCP$(n, w)$, $w \neq 4$, then there exists a primitive TCP$(m, w)$, for some $m$.

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REFERENCES