Rotorcraft path following control for extended flight envelope coverage

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Abstract—This paper addresses the problem of steering a quadrotor vehicle along a predefined path. The problem is formulated so as to dynamically prescribe an adequate time evolution along the path and simultaneously bound the effect of position errors on the actuation. The proposed solution consists of a nonlinear state feedback controller for thrust and torque actuations combined with a path following timing law that i) guarantees global asymptotic convergence of the path following error to zero, for a large class of three-dimensional paths, ii) ensures that the actuation does not grow unbounded as function of the position errors, and iii) allows for zero thrust actuation to be applied when the vehicle is converging to the path. Simulation results are presented to assess the performance of the proposed controller.

I. INTRODUCTION

Motion control of underactuated vehicles is an active topic of research, which raises new and challenging problems when measured against motion control of its fully actuated counterpart. The quadrotor, in particular, is a typical example of an underactuated vehicle ideally suited for the development and test of new control strategies due to its simplicity and maneuverability. The simplified model commonly adopted for both quadrotors and helicopters is a 6-DoF rigid body actuated in force and torque with four inputs given by a scalar thrust and three torque inputs.

In recent years, several approaches to the problem of rotorcraft motion control have been proposed, ranging from PID control [1] to nonlinear methods such as feedback linearization [2] and backstepping control [3], [4]. While PID control has demonstrated good flying qualities for hovering and tracking of straight line trajectories in the low speed regime [1], more evolved control strategies are required to fully explore the vehicle’s flight envelope. Among these, we highlight the Lyapunov-based backstepping method.

Backstepping [5] is a well known technique extensively used for control of nonlinear systems although, in general, it is not applicable to underactuated systems. However, as shown in [2] the simplified rotorcraft model is feedback linearizable (by dynamic extension of the scalar thrust actuation) and hence stabilizable by means of backstepping, provided that the thrust does not cross the origin. This is in agreement with Brockett’s condition for smooth stabilizability of nonholonomic systems, since the trajectories that include zero-thrust conditions such as free fall trajectories belong to the category of equilibria that do not meet the referred condition.

Examples of trajectory tracking controllers that apply a backstepping procedure and build on the dynamic augmentation principle of appending two integrators to the thrust input can be found in [3], [4]. The resulting control laws are valid if non-zero thrust is applied at all times, meaning that zero thrust cannot be applied not only when tracking the desired trajectory but also when moving towards it. Since the thrust is commanded by its second derivative, it becomes difficult to determine the set of initial conditions that do not lead to zero thrust commands while the vehicle is converging to the path.

To lift the non-zero thrust restriction and thereby obtain a globally defined controller, we consider the problem of path following. As an alternative to trajectory tracking, path following solutions typically result in smoother convergence to the path and less demand on the control effort. A common approach to the path following problem is to parameterize the desired path using a scalar variable such as the arc-length and then select a timing law for that parameter [6], [7].

In this work, we explore the extra degree of freedom provided by the path following timing law so as to control the time evolution of the desired position along the path and thereby obtain convergence to the path even for extremely unfavorable initial conditions. The desired path is specified by a sufficiently smooth parameterized 3-D curve. The desired attitude of the vehicle is not prescribed since attitude convergence (up to a rotation about the body z axis) is naturally accomplished by solving the path following problem. For controller design, the attitude is handled in its natural space, SO(3), as a rotation matrix. This avoids the introduction of artifacts related only to the parameterization, as is the case of singularities with Euler angles and multiple coverings with the quaternion representation [8]. To ensure that the actuation does not grow unbounded as a function of the distance to the path, a saturation function is applied to the position errors used for feedback. We show that for a large class of sufficiently smooth curves, the resulting control law guarantees global convergence of the path following error to zero.

This paper is structured as follows: Section II introduces the quadrotor model. The proposed problem and control objective are stated in Section III. Controller design is described in Section IV. Simulation results illustrating the performance of the proposed control laws are presented in Section V and Section VI summarizes the contents of the paper.

II. QUADROTOR MODEL

We model the quadrotor vehicle as a rigid body that is actuated in force and torque. Consider a fixed inertial frame
\{I\} and a frame \{B\} attached to the vehicle’s center of mass. The configuration of the body frame \{B\} with respect to \{I\} can be viewed as an element of the Special Euclidean group, \((R, p) = (\text{SE}(3), p) \in \text{SE}(3)\). The kinematic and dynamic equations of motion for the rigid body can be written as

\[
\begin{align*}
\dot{R} &= RS(\omega) \\
\dot{p} &= Rp \\
\dot{\omega} &= -J^{-1}S(\omega)\dot{p} + J^{-1}n \\
\dot{v} &= -S(\omega)v + \frac{1}{m}f,
\end{align*}
\]

where the position \(p\) is expressed in the inertial frame \{I\}, \(R\) is the rotation matrix from \{B\} to \{I\}, and the angular velocity \(\omega \in \mathbb{R}^3\) and the linear velocity \(v \in \mathbb{R}^3\) are expressed in the body frame \{B\}. The scalar \(n\) and the matrix \(J \in \mathbb{R}^{3 \times 3}\) represent the quadrotor’s mass and moment of inertia and \(f\) and \(n \in \mathbb{R}^3\) denote respectively the external force and torque expressed in the body frame. Aerodynamic drag forces due to the fuselage are neglected given the low speeds at which the quadrotor operates.

Given the geometry of the quadrotor and assuming that the forces and moments generated by each of the four rotors are approximately given by the thrust and torque components perpendicular to the rotor disk plane, we can consider a quadrotor model such that torques can be generated in any direction and the actuation force is always aligned with the body z axis. Figure 1 shows a sketch of the quadrotor setup, together with the force generated by each motor \(F_i\) and the direction of rotation for each propeller. The bijective correspondence between the motor forces and the total thrust \(T\) and torque \(n\) is given by (4)-(7), where \(k_n, k_t\) are constants intrinsic to the vehicle.

\[
\begin{align*}
T &= F_1 + F_2 + F_3 + F_4 \\
n_1 &= k_n(F_1 - F_3) \\
n_2 &= k_n(F_2 - F_4) \\
n_3 &= k_n(F_1 + F_3 - F_2 - F_4)
\end{align*}
\]

The external force in body coordinates is given by

\[
f = -Tu_3 + mgR^t u_3
\]

where \(u_3 = [0 \ 0 \ 1]^T\) and \(g\) is the gravitational acceleration. With full torque control, the Euler equations of motion (2) can be reduced to the integrator form \(\dot{\omega} = \tau\), using the input transformation

\[
n = J\tau + S(\omega)J\omega.
\]

The quadrotor is thus an underactuated vehicle, as evidenced by (3) and (8), making the control problem much more difficult to address than what it would be for a fully-actuated vehicle. In this particular case we only have one degree of freedom for the force actuation in body frame and we are required to control the linear position of the vehicle \(p \in \mathbb{R}^3\).

### III. Problem Statement

Let the desired path be described by a curve in three-dimensional space parameterized by \(\gamma \in \mathbb{R}\) such that \(p_d(\gamma) \in \mathbb{R}^3\). The desired path is required to be at least of class \(C^4\). The control objective consists of designing a control law for the quadrotor actuations \(T(t)\) and \(\tau(t)\) and a timing law for the path parameter \(\gamma(t)\) that ensures convergence of the vehicle’s position \(p(t)\) to the path \(p_d(\gamma(t))\) with the largest possible basin of attraction.

Throughout the remainder of the paper, time dependence is often omitted to lighten notation. In the sequel, the prime \(f'(x)\) denotes the partial derivative of the function \(f\) with respect to the parameter \(x\) and the upper dot \(f'(x(t))\) denotes the total time derivative of the function.

Due to the underactuated nature of the vehicle, the desired attitude cannot be arbitrarily selected. From (3) and (8), it is easy to observe that the equilibrium for path following satisfies

\[
R_d T_d u_3 = mgu_3 - m\dot{p}_d.
\]

Consequently, the desired rotation matrix \(R_d\) is automatically prescribed up to a rotation about the body z axis \((R_d R_2(\psi)T_d u_3 = mgu_3 - m\dot{p}_d)\). The symmetry exhibited by the quadrotor vehicle dictates that there is an additional degree of freedom. Rotations about the body z axis do not influence the control as it is possible to generate any torque and thrust, within the vehicle’s limits of operation, regardless of its heading angle.

Though the vehicle’s velocity is not required to converge to a given value in the path following problem, its progression along the path can be controlled through an adequate choice of the timing law \(\gamma(t)\).

As sensor measurements we consider the full state to be available for feedback. In an experimental setup, the state readings are typically produced by inertial measurement units, which combine triads of rate-gyros, accelerometers, magnetometers and GPS.

### IV. Controller Design

To devise a control law that steers the aerial vehicle along the path \(p_d(\gamma)\), we start by introducing the position error vector

\[
e_1 = p - p_d(\gamma).
\]

For path following to be attained the controller must drive the position error to zero. Towards that goal, we define the
first tentative Lyapunov function as the squared norm of the error $e_1$

$$V_1 = \frac{1}{2} e_1^T e_1.$$  

Computing the time derivatives of $e_1$ and $V_1$, we can write

$$\dot{e}_1 = R v - \dot{p}_d,$$
$$\dot{V}_1 = -W_1(e_1) + k_1 e_1^T (\sigma(e_1) + \frac{1}{k_1} \dot{e}_1),$$  \hspace{1cm} (11)

where $W_1(e_1) = k_1 e_1^T \sigma(e_1)$, which is positive definite for $k_1 > 0$. Similarly to $W_1$, $W_i(\cdot)$ $i \in \{2,3,4\}$ is used in the sequel to denote a quadratic function of its arguments that can be made positive definite by appropriately selecting a set of scalar coefficients. In (11), $\sigma(\cdot)$ is a sigmoidal saturation function given by

$$\sigma(x) = p_{max} \frac{x}{1 + ||x||}$$  \hspace{1cm} (12)

that limits the effect of the position error on $e_2$ and subsequent backstepping errors. The parameter $p_{max}$ controls the influence of the position error on the subsequent errors.

Continuing to apply the standard backstepping procedure, we define a second error and Lyapunov function and, as previously, compute its derivative to obtain

$$e_2 = \sigma(e_1) + \frac{1}{k_1} \dot{e}_1,$$
$$V_2 = \frac{1}{2} \sum_{i=1}^{2} e_i^T e_i,$$
$$\dot{V}_2 = -W_2(e_1, e_2) + e_2^T \left( k_1 k_2 e_2 + \left( \sigma(e_1) + \frac{1}{k_1} \dot{e}_1 \right) \right),$$  \hspace{1cm} (13)

where

$$\dot{e}_1 = \frac{1}{m} R f - \dot{p}_d = -\frac{1}{m} R u_3 + g u_3 - \dot{p}_d.$$  

Defining the desired thrust $T_d$ and desired thrust direction $r_{3d}$ as

$$T_d = m \| k_1 k_2 e_2 + k_1 \sigma(e_1) + g u_3 - \dot{p}_d \|$$  \hspace{1cm} (14)
$$r_{3d} = \frac{k_1 k_2 e_2 + k_1 \sigma(e_1) + g u_3 - \dot{p}_d}{\| k_1 k_2 e_2 + k_1 \sigma(e_1) + g u_3 - \dot{p}_d \|}$$  \hspace{1cm} (15)

we can rewrite (13) as an explicit function of the desired thrust given by

$$\dot{V}_2 = -W_2(e_1, e_2) + \frac{1}{k_1} e_2^T \left( \frac{1}{m} T_d r_{3d} - \frac{1}{m} T_r \right).$$  \hspace{1cm} (16)

The desired thrust direction is only well-defined if $T_d$ is non-zero. At this point we proceed with the computations assuming that $T_d$ is always non-zero and in the next section we present a timing law for $\gamma(t)$ that enforces that condition.

We now introduce a control law for the thrust, based on the desired thrust and thrust direction, and rewrite $\dot{V}_2$ by substituting the thrust in (16), so that

$$T = r_{3d}^T r_3 T_d$$  \hspace{1cm} (17)
$$\dot{V}_2 = -W_2(e_1, e_2) - \frac{T_d}{m k_1} e_2^T S(r_3) r_{3d}$$

In addition to incorporating the thrust law (17) we define a new error and augment the tentative Lyapunov with a term designed to drive the angle between the desired and the actual thrust direction to zero.

$$e_3 = r_3 - r_{3d}$$
$$V_3 = \frac{1}{2} e_3^T e_3.$$

The Lyapunov function $V_3$ time derivative can be computed as

$$\dot{V}_3 = -W_3(e_1, e_2, r_3, r_{3d}) + r_{3d}^T R \left( -k_3 S(u_3)^2 R^T r_{3d} + S(u_3) (\omega - R^T R_d \omega_d) - \frac{T_d}{m k_1} S(u_3)^2 R^T e_2 \right)$$

where $R_d$ is any time dependent rotation matrix such that $R_d u_3 = r_{3d}$. Notice that with the defined $e_3$ error we only require convergence of $R u_3$ to $R_d u_3$ and not of the whole $R$ matrix to $R_d$.

For the last backstepping procedure we define a fourth error and the corresponding fourth Lyapunov function and derivative as

$$e_4 = -k_3 S(u_3)^2 R^T r_{3d} + S(u_3) (\omega - R^T R_d \omega_d) - \frac{T_d}{m k_1} S(u_3)^2 R^T e_2$$  \hspace{1cm} (18)
$$\dot{V}_4 = V_3 + \frac{1}{2} e_4^T e_4$$
$$\dot{V}_4 = \dot{V}_3 + e_4^T \left( R^T r_{3d} - k_3 S(u_3)^2 (R^T r_{3d} + R^T \dot{r}_{3d} + S(u_3) (\tau - R^T R_d \omega_d + R^T R_d \tau_d) - \frac{1}{m k_1} S(u_3)^2 \left( \dot{T}_d R^T e_2 + T_d R^T \dot{e}_2 + T_d R^T \dot{e}_2 \right) \right).$$

Finally, as a control law for the torque actuation we consider

$$\tau = -S(u_3) \left( -k_4 e_4 - R^T r_{3d} + k_3 S(u_3)^2 (R^T r_{3d} + R^T \dot{r}_{3d}) + \frac{1}{m k_1} S(u_3)^2 (T_d R^T e_2 + T_d \dot{R}^T e_2 + T_d R^T \dot{e}_2) \right) + \dot{R}^T R_d \tau_d + R^T R_d \tau_d + \frac{1}{2} \tau_3(t)^T$$  \hspace{1cm} (19)

where $\tau_3(t)$ is an arbitrary function. This rotational degree of freedom is due to the axial symmetry exhibited by the quadrotor and is used in the sequel to control the heading of the vehicle.

Notice that neither $T$ nor $\tau$ depend directly on $e_1$. This is a desired property since the position error can be arbitrarily large, depending on the initial position. The other errors depend only on a bounded function of the position error, $\sigma(e_1)$, leading to an actuation that does not grow unbounded with the position errors.

We can now state the main result, assuming the control law is well-defined. In the following Section we show how the timing law can be chosen so that the control law is always well-defined, meaning that the expression for $T_d$ given in (14) never goes to zero.

**Theorem 1**: Let the quadrotor kinematics and dynamics be described by (1)-(3) and (8)-(9). Consider the control laws (17) and (19) and assume that these are well-defined for all time. For an appropriate choice of coefficients $k_i$ and independently of the timing law for $\gamma(t)$, the resulting closed-loop error system has an asymptotically stable equilibrium point at the origin and the quadrotor system achieves global convergence to the path.
Proof: Considering (19) together with (17) as actuation laws, the time derivative of the Lyapunov function (18) along the system’s trajectories is given by
\[
\dot{V}_4 = -k_1 e_1^T \sigma(e_1) + k_1 e_1^T e_2 - k_1 k_2 e_2^T e_2 - k_2 r_{33}^T S(r_3) S(r_3)^T r_{33} - k_1 e_1^T e_4,
\]
and can be made negative semi-definite with an appropriate choice of the scalar coefficients $k_1$ as long as either $\|e_1\| < p_{max}^1 - 1$ or $\|e_2\| < \|\sigma(e_1)\|$ are observed. We can apply LaSalle’s invariance principle and conclude that the error trajectories converge to the largest invariant set of the Lyapunov function $\dot{V}_4$, which consists of the points $e_1 = 0, e_2 = 0, e_3 = 0, r_3 = \pm r_{33}$. For sufficiently small initial errors, including the orientation error $e_3$, $\dot{V}_4$ is negative definite, implying that the origin of the error system is asymptotically stable. For the case where none of the referred conditions is satisfied it can be shown that the closed-loop trajectories of the error system will reach in finite time an invariant set where they are verified. We define a new Lyapunov function and express its time derivative when using the control laws (17) and (19).
\[
\dot{V}_5 = \frac{1}{2} (e_2^T e_2 + e_3^T e_3 + e_4^T e_4)
\]
\[
\dot{V}_5 = -k_1 k_2 e_2^T e_2 - k_3 r_{33}^T S(r_3) S(r_3)^T r_{33} - k_4 e_4^T e_4
\]
The time derivative (21) is negative semi-definite for all $k_i > 0$. Once again, we can apply LaSalle’s invariance principle to show that $e_2$ and $e_4$ converge to zero and $r_3$ to $\pm r_{33}$. Thus, we can conclude that $\|e_2\|$ is ultimately bounded by $2 + \epsilon$, for an arbitrarily small $\epsilon$. Choosing $p_{max}^2 > 3$, we can guarantee that there is a time $0 < t_1 < \infty$ such that $\|e_2(t+t_1)\| < p_{max}^1 - 1$ and (20) is negative semi-definite for all $t \geq t_1$. As such $e_1, e_2$, and $e_4$ converge to zero for any initial conditions, meaning that path following is achieved globally.

A. Path following timing law

The proposed controller achieves path following as long as the control law is well-defined. In this section we explore the use of the degree of freedom provided by the path parameter $\gamma(t)$ to guarantee that the path following controller is well-defined. That is to say, the timing law for $\gamma(t)$ ensures that
\[
k_1 k_2 e_2 + k_3 \sigma(e_1) + gu_3 - \dot{p}_d = 0
\]
is never satisfied. As a secondary control objective we drive the path speed $\dot{\gamma}$ to a desired speed $\dot{\gamma}_c$.

In the sequel we present two control laws for $\dot{\gamma}$ that solve the speed assignment objective while keeping the path following control law well-defined. We first present a law for desired paths that verify
\[
\frac{1}{\|S(p_d(\gamma))p_d(\gamma)\|} > \alpha, \quad \text{if } \|p_d(\gamma)\| \neq 0,
\]
and then another one for paths that verify
\[
\|p_d(\gamma)^T u_1\| > \alpha
\]
for all $\gamma$. Without loss of generality we consider that the desired path has a non-zero parameter derivative, $\|p_d'(\gamma)\| \neq 0$. When $p_d''(\gamma) \neq 0$ we can define a frame $\{T\}$ tangent to the path such that its $x$ and $y$ axis are aligned with the tangent and normal vectors to the desired path, respectively. To fully describe $\{T\}$, we define the rotation matrix from the tangent frame to the inertial frame at points where $p_d''(\gamma) \neq 0$.

\[
\frac{1}{\|p_d''(\gamma)\|} \xi(\gamma) S(p_d'((\gamma))p_d'(\gamma)) \|p_d''(\gamma)\| S(p_d'(\gamma)p_d'(\gamma))
\]
where $\xi(\gamma)$ denotes a sign function that can change sign at the path’s inflexion points, characterized by $p_d''(\gamma) = 0$, so as to keep the rotation matrix $\frac{1}{\|p_d''(\gamma)\|} R(\gamma)$ a function of at least class $C^2$. For straight line segments and inflexion points, where $p_d''(\gamma) = 0$, we are at liberty to define any rotation matrix such that

\[
\frac{1}{\|p_d''(\gamma)\|} R(\gamma) \frac{1}{\|p_d''(\gamma)\|} p_d'(\gamma) = u_1
\]
and that makes $\frac{1}{\|p_d''(\gamma)\|} R(\gamma)$ a function of class $C^2$.

Having completely defined the rotation matrix from the inertial to the tangent frame we left multiply (22) with its transpose and use

\[
\dot{p}_d = \dot{\gamma}^2 p_d'(\gamma) + \dot{\gamma} p_d''(\gamma)
\]
to write the following equivalent set of equations
\[
v_1 + 4 r_{33} - \dot{\gamma} \|p_d'\| = 0
\]
\[
v_2 + 2 r_{33} = 0
\]
\[
v_3 + 3 r_{33} = 0
\]
where $v_i = u_1^T R (k_1 k_2 e_2 + k_1 \sigma(e_1) - \dot{p}_d')$, $r_{33}$ are the components of the vector $r_3 = \dot{R} u_3$, and dependency on $\gamma$ is omitted to lighten notation. At this point, we design a control law for $\dot{\gamma}$ that drives $\dot{\gamma}$ to $\dot{\gamma}_c$ but perturb it when necessary so that (26)-(27) is never verified. To perturb $\dot{\gamma}$ only locally we start by defining two bump-like functions of class $C^\infty$ that are used in the design of the timing law,

\[
\Psi(x) = \begin{cases} \exp(-x^2/(x^2 - x^2)), & |x| < \epsilon \\ 0, & \text{otherwise} \end{cases}
\]
\[
\Phi(x) = 1 - \Psi(x),
\]
where $\epsilon > 0$.

Let us first concentrate on desired paths verifying (23) and define the parameter’s second order time derivative as follows

\[
\ddot{\gamma} = -\Phi(v_3 + 4 r_{33}) k_3 \sigma(\dot{\gamma} - \dot{\gamma}_d)
\]
\[
+ \frac{1}{\|p_d'\|} \Psi(v_3 + 4 r_{33}) (v_1 + v_3 + 4 r_{33})
\]

Lemma 2: Let $p_d'(\gamma) \in \mathbb{R}^3$ be a function of at least class $C^4$ such that (23) is verified for all $\gamma$. Controlling the parameter $\gamma$ through (28) with $\epsilon < g \alpha$ guarantees that the path following control laws (17) and (19) are always well-defined and $\dot{\gamma}$ converges to $\dot{\gamma}_d$.

Proof: If (27) holds, (26) becomes $v_3 = -g r_{33} = 0$, which contradicts assumption (23) since the latter implies

\[
|\dot{r}_{33}(\gamma)| > \alpha
\]
for all $\gamma$ if the rotation matrix (25) is chosen so as to have $|\dot{r}_{33}\alpha$ when $p_d''(\gamma) = 0$. The control law (28) for the
evolution of $\dot{\gamma}$ guarantees then that (22) is never verified. Additionally, if $\epsilon < g_\alpha$, in a zero path following error situation we have $v_3 = 0$ and $\Psi_e(s_{33}) = 1$, $\Psi_e(s_{33}) = 0$, leading to convergence of $\dot{\gamma}$ to $\dot{\gamma}_d$.

We now discuss desired paths observing (24) for all $\gamma$ and consider, without loss of generality, that the path derivative with respect to the parameter has unit norm, that is, $\|p_d'(\gamma)\| = 1$. Departing from (26)-(27) we design the following control law for the parameter's second order time derivative
\[ \ddot{\gamma} = -k_\gamma \sigma(\dot{\gamma} - \dot{\gamma}_d) + v_1 \]  
(29)

**Lemma 3:** Let $p_d(\gamma) \in \mathbb{R}^3$ be a function of at least class $C^4$ such that $|p_d'(\gamma)\times u_3| > \alpha$ and $|p_d'(\gamma)| = 1$ for all $\gamma$. Controlling the timing law through (29) with $k_{\gamma} < g_\alpha$ guarantees that the path following control laws (17) and (19) are always well-defined and $\dot{\gamma}$ converges to $\dot{\gamma}_d$.

**Proof:** Applying (29) to $\dot{\gamma}$ reduces (26) to
\[ k_\gamma \sigma(\dot{\gamma} - \dot{\gamma}_d) - gr_3 = 0, \]
which is impossible if $k_\gamma < g|r_3(\gamma)|$. This condition is verified for all $\gamma$ if $k_\gamma < g_\alpha$ as
\[ |r_3| = |p_d'(\gamma)^T u_3| > \alpha. \]

The timing law defined by (29) ensures that (22) is never verified. Convergence of $\dot{\gamma}$ to $\dot{\gamma}_d$ is achieved when the path following errors $e_1$ and $e_2$ converge to zero as $v_1 = 0$ if $e_1 = 0$, $e_2 = 0$, and $|p_d'| = 1$.

B. Yaw degree of freedom

As specified in (10), if path following is attained then the rotation matrix from the body to inertial frame is automatically prescribed up to a rotation about the body $z$ axis. This is corroborated by the control law (19) obtained in the previous section, where the torque $\tau_3 = \omega_3$ is arbitrary. We can thus use this extra degree of freedom to achieve an additional control objective, which should be carefully chosen so that it does not conflict with the original path following objective. One possibility is to simply guarantee convergence of $\omega_3$ to zero, applying the control law $\tau_3 = -k_3 \omega_3$. A more involved goal that is still attainable would be to ensure that the vehicle follows the desired path with zero velocity component along the body $y$ axis. Combining (10), which describes the path following equilibrium, with the additional constraint
\[ [0 \ 1 \ 0]^T v_d = v_{2d} = 0 \]
we can arrive at the following expression for the desired rotation matrix
\[ R_d = \begin{bmatrix} -\frac{S(r_{2d})p_{d}'}{\|S(r_{2d})p_{d}'\|} & \frac{S(r_{3d})p_{d}'}{\|S(r_{3d})p_{d}'\|} & r_{3d} \end{bmatrix} \quad (30) \]
where the third column of $R_d$ is given by (15).

Clearly, (30) can only be used if $S(r_{3d}) \neq 0$. When this is the case, any pair $(r_{1d}, r_{2d})$ such that $R_d = [r_{1d} \ r_{2d} \ r_{3d}] \in SO(3)$ yields $v_{2d} = 0$. Since the control law for path following already ensures that $r_3$ converges to $r_{3d}$, to guarantee that $v_{2d}$ approaches zero, we need only have convergence of $r_2$ to $r_{2d}$ or equivalently of $r_{2d}^T r_2$ to one. Once again, we stress that working directly with the rotation matrix as opposed to adopting a parameterization has the advantage of avoiding singularities and unwinding behavior of the system’s trajectories [8].

Having defined $R_d$ completely, straightforward but rather lengthy computations provide expressions for both $\omega_{3d}$ and its time derivative $\dot{\omega}_{3d}$. Gathering all these elements, we define the following PD-like control law for $\tau_3$
\[ \tau_3 = -l_2(\omega_3 - \omega_{3d} + l_3 r_{2d}^T r_1) + \dot{\omega}_{3d} - l_1 \frac{d}{dt}(r_{3d}^T r_1) \quad (31) \]
with $l_1 > 0$ and $l_2 > 0$. Computing the time derivative of the Lyapunov function $W = (\omega_3 - \omega_{3d} + l_3 r_{2d}^T r_1)^2$ is negative definitive, we can immediately conclude $\omega_{3d}$ converges to $\omega_{3d} - l_1 r_{2d}^T r_1$. Clearly this is not enough to guarantee that $r_2$ approaches $r_{2d}$, however some insight that this convergence is happening can be gained by noting that the term $-l_1 r_{2d}^T r_1$ opposes growth in the angular distance between $r_2$ and $r_{2d}$. More formally, we can define the states
\[ \xi = \omega_3 - \omega_{3d} + l_1 r_{2d}^T r_1 \in \mathbb{R} \]
\[ \eta = 1 - r_{2d}^T r_2 \in [0, 2] \]
and verify that, at the path following equilibrium given by $e = 0$ and $\xi = 0$, the dynamic system for $\eta$ are described by
\[ \dot{\eta} = -l_1(2 - \eta) \eta \quad (32) \]
and the origin of this system is asymptotically stable. If we consider the closed-loop quadrotor system that results from applying the control laws (17) and (19) with (31), the system (32) can be thought of as the quadrotor zero dynamics. We can therefore conclude that the overall closed-loop system has an asymptotically stable equilibrium point at the origin, which follows from the asymptotic stability of $e = 0$ and $\xi = 0$. Determining the respective region of attraction requires further investigation and is left for future work.

V. SIMULATION RESULTS

In this section we present the results from a simulation run of the controller. The distance to the target position is such that the saturation (12) is active. The reference path to follow is a class $C^\infty$ curve defined by
\[ p_d(\gamma) = \begin{bmatrix} 20 \cos(0.1\gamma) \\ 10 \sin(0.2\gamma) \\ 0.1\gamma \end{bmatrix} \]
and the desired parameter derivative is defined as follows
\[ \dot{\gamma}_d = \frac{1}{2} \Psi_{p_{\text{max}}}(|e_1|). \quad (33) \]
For large position errors the desired parameter derivative is zero to ensure that the desired position waits for the vehicle to reach it. When the vehicle is on the path, $\dot{\gamma}_d$ takes a constant value to enforce a constant progression along the path.

At the initial time instant the vehicle state has $p = [-2, 10, -10]^T \text{m}$, $v = [2, 0, 0]^T \text{ms}$, $r_3 = [0, 0, -1]^T$ and $\omega = [0, 0, 0]^T \text{rad/s}$. This extremely unfavorable orientation is chosen to illustrate a negative thrust actuation condition. The vehicle mass is $m = 4.76 \text{kg}$ and the controller law coefficients are $k_1 = 1$, $k_2 = 1$, $k_3 = 10$, $k_4 = 1$, and $p_{\text{max}} = 5$. For the timing law we use $k_\gamma = 1$ and $\gamma(0) = 1$. 

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Figure 2 depicts all the path following and backstepping errors and shows the time evolution of their components. We verify that all the errors converge to zero. The thrust direction is the fastest to converge whereas the position is the slowest. Notice that due to the initial unfavorable orientation, the direction of \( r_3 \) suffers an inversion, as can be noticed by the evolution of the error component \( e_{33} \).

As shown in Fig. 3, apart from the initial transient, the actuations do not vary significantly from their nominal values. The large initial difference between the thrust actuation \( T \) and desired one \( T_d \) is due to the initial orientation of the vehicle, chosen to generate almost symmetrical thrust \( r_3 \) and desired thrust \( r_{3d} \) directions, and thus lead to a negative initial thrust.

The time evolution of the path parameter \( \gamma \) and its derivative is depicted in Fig. 4. The evolution of the parameter derivative is dictated by the timing law and tries to follow (33). The large initial position error leads to an initial convergence of the parameter’s derivative to zero and only when the path following error is small does the parameter’s derivative converge to the final value of 0.5.

VI. CONCLUSIONS

This paper presented a state feedback solution to the problem of steering a quadrotor vehicle along a predefined path. The proposed solution guarantees global convergence of the path following error to zero, for a large class of three-dimensional paths. The controller was designed with backstepping and ensures that the actuation does not grow unbounded as function of the position errors and allows for zero thrust actuation to be applied when the vehicle is converging to the path. The vehicle’s time evolution along the path is controlled to follow a predefined specification and simultaneously maintain the path following control law well-defined. Simulation results were presented to assess the performance of the proposed controller.

REFERENCES