Routing Vehicles to Minimize Fuel Consumption

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Abstract

We consider a generalization of the capacitated vehicle routing problem known as the cumulative vehicle routing problem in the literature. Cumulative VRPs are known to be a simple model for fuel consumption in VRPs. We examine four variants of the problem, and give constant factor approximation algorithms. Our results are based on a well-known heuristic of partitioning the traveling salesman tours and the use of the averaging argument.

Keywords: Approximation Algorithms, Fuel Consumption, Cumulative Vehicle Routing Problem

1. Introduction

Fuel consumption in transportation and logistics is an important area of study. Fuel consumption can account for as much as 60% of the operating cost of a vehicle according to one study [17]. Therefore, a direct reduction in the operating costs can be obtained by minimizing the fuel consumption of the vehicle. Fuel consumption by a vehicle is affected by various factors, important ones being the distance traveled, the weight of the vehicle, vehicle speed, road inclination, aerodynamic drag, traffic congestion etc. [8]. An important factor that determines the fuel consumption is the total weight of the vehicle i.e., the weight of the empty vehicle plus the weight of the cargo being carried by the vehicle. In a simplified model of fuel consumption, the fuel consumed by the vehicle per unit distance is assumed to be proportional to the total weight of the vehicle [16, 11, 12, 19].

In the vehicle routing problem (VRP) introduced by Dantzig and Ramser [7], a vehicle with finite capacity is located at a depot in a graph G(V, E) with positive edge lengths. Each vertex of G except the depot has an object of positive weight located at it. The objective is to devise a travel schedule for the vehicle such that all the objects are brought to the depot and the total distance traveled by the vehicle is minimized.

Several authors have studied generalizations of VRPs where the objective is to minimize fuel consumed by the vehicle under the simplified model discussed above. For background and description of the vehicle routing problem please refer to the excellent books by Golden and Assad [9] and by Toth and Vigo [18].

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1.1. Cumulative VRPs

Kara et al. [11, 12] define cumulative VRPs (CumVRPs) where the objective is to minimize fuel consumption of the vehicle given that fuel consumed per unit distance is proportional to the total weight of the vehicle. Further, they note that cumulative VRPs generalize several previously studied problems in literature such as the minimum latency problem and the k-traveling repairman problem. A linear model of fuel consumption and heuristic algorithms for the capacitated vehicle routing problem with the objective of minimizing fuel consumption are given in [19]. We now describe the cumulative vehicle routing problem as defined by Kara et al. [11, 12].

CumVRP-Input.

1. We are given a complete, undirected graph G(V, E) with positive lengths on each edge e ∈ E. The edge lengths satisfy triangle inequality and form a metric space. If the graph is not complete or the edges do not satisfy triangle inequality, we take the metric closure of graph G(V, E). The vertices of graph G(V, E) are numbered from 0 to n. Vertex 0 is called the depot.
2. For each i ∈ V \ {0}, there is an object of positive weight w_i which is located at vertex i.
3. An empty vehicle is located at the depot, which at any point of time can carry objects of total weight not exceeding Q.

The goal is to devise a travel schedule for the vehicle so that all the objects are brought to the depot and the fuel consumed is minimized. We allow the vehicle to offload cargo at the depot an arbitrary number of times.

Fuel Consumption Model. We assume that the fuel consumed per unit distance is proportional to the total weight of the vehicle. Let the weight of the empty vehicle be W_0 and suppose the cargo being carried has weight w. Then, fuel consumed by the vehicle to travel distance l is given by µ(W_0 + w)l = µW_0l + µwl, where µ is a constant.
In the following we take \( a = \mu W_0 \) and \( b = \mu \), so that the fuel consumed per unit distance by a vehicle with cargo weight \( w \) is \( a + bw \).

**Feasible Solution to CumVRP.** A feasible solution \( S \) to CumVRP consists of a set of \( k \leq |V \setminus \{0\}| \) directed cycles \( C_1, C_2, \ldots, C_k \) containing the depot such that:

1. Each vertex \( i \in V \setminus \{0\} \) belongs to exactly one cycle.
2. The total weight of objects located at the vertices of each cycle \( C_j, 1 \leq j \leq k \) is at most \( Q \).

Given a feasible solution \( S \), a travel schedule for the vehicle located at the depot can be constructed as follows. For \( 1 \leq j \leq k \), the vehicle at the depot goes once around cycle \( C_j \) in the preferred direction of traversal, picks up the objects located at the vertices of \( C_j \) and drops them at the depot.

**Cost Function.** We now compute the fuel consumed by schedule \( S \). Since the fuel consumption per unit distance varies linearly with the total vehicle weight, we will compute the fuel consumption due to every component weight separately and add them up. For \( i \in V \setminus \{0\} \), let \( d_i^S \) denote the distance traveled by a vehicle between picking up the object at vertex \( i \) and dropping it at the depot according to the travel schedule given by \( S \). In the following, \( |C_j| \) denotes the length of cycle \( C_j \).

Let us first compute the fuel consumed by vehicle while traversing cycle \( C_j \), where \( 1 \leq j \leq k \). The fuel consumption due to the weight of the empty vehicle is \( a|C_j| \), as the vehicle goes around cycle \( C_j \) once. For each non-depot vertex \( i \in C_j \), the vehicle carries weight \( w_i \) for a total distance \( d_i^S \). Therefore, the contribution of weight \( w_i, i \in C_i \) is \( bw_id_i^S \). The fuel consumed by the vehicle while traversing cycle \( C_j \):

\[
 f(C_j) = a|C_j| + b \sum_{i \in C_j} w_i d_i^S.
\]

The total fuel consumption \( f(S) \) of the travel schedule given by solution \( S \) is \( f(S) = \sum_{j=1}^k f(C_j) = a \sum_{j=1}^k |C_j| + b \sum_{j=1}^k \sum_{i \in C_j} w_i d_i^S \). Since, each vertex occurs on exactly one cycle, we get that:

\[
 f(S) = a \sum_{j=1}^k |C_j| + b \cdot \sum_{i=1}^n w_i d_i^S.
\]

**Objective.** Our objective is to find the solution \( S^* \) with the minimum fuel consumption \( f(S^*) \) among all feasible solutions with less than or equal to \( |V \setminus \{0\}| \) cycles.

1.2. Our contributions

Different variants of cumulative VRPs have been studied in literature. A special case is when there is a single vehicle with infinite capacity and the travel schedule must consist of a single traveling salesperson (TSP) tour \( C \) of graph \( G \). Constant factor approximation algorithms have been given for this case by Blum et. al. [5] who call it the positive-linear time-dependent traveling salesman problem and note the relation of this problem to minimizing fuel consumption.

The main contribution of this paper is a factor 4 approximation algorithm for cumulative VRP when the vehicle as the depot has a finite capacity \( Q \). For our considerations, it will be convenient to consider four different versions of this problem:

1. **EQ-INF-FM**: All objects have equal weights, and the vehicle has infinite capacity.
2. **UNEQ-INF-FM**: The objects have unequal weights, and the vehicle has infinite capacity.
3. **EQ-CAP-FM**: All objects have equal weights, and the vehicle has capacity \( Q \).
4. **UNEQ-CAP-FM**: The objects have unequal weights, and the vehicle has capacity \( Q \).

For the above four problems, we will give approximation algorithms with factors 2.5, 2.5, 3.186, and 4 respectively. To the best of our knowledge, these are the first constant factor approximation algorithms for these problems.

The approximation factors are established by a novel application of the iterated tour partitioning technique of Haimovich and Rinnooy Kan [10] and the travel schedule for the vehicle is computed using a variation of dynamic programming on a traveling salesperson tour [4, 15].

2. Previous Work

We now discuss the algorithms for capacitated minimum distance vehicle routing problem (CVRPs), where the objective is to compute a schedule with minimum total travel distance. We assume that there is a single vehicle with capacity \( Q \) at the depot. Let \( w_i \) be the weight of object at vertex \( i \) and let \( d_i \) be the shortest distance between vertex \( i \) and the depot. The following lower bound on the length of any travel schedule was given by Haimovich and Rinnooy Kan [10].

**Theorem 1.** [10] Let \( C^* \) denote an optimal traveling salesperson tour of the graph \( G(V, E) \). Then, the total distance traveled by a vehicle to bring all objects to the depot is at least:

\[
 \max \left( |C^*|, 2 \sum_{i=1}^n w_id_i \right), \frac{Q}{Q}
\]

Let \( C \) be a traveling salesperson tour of graph \( G(V, E) \). A salesperson tour that visit a subset of vertices in \( V \) is referred to as a subtour. At times when it is clear from the context we will use tour to refer to a subtour. Without loss of generality we will label the depot 0, and number the vertices 1, 2, \ldots, \( n \) in the order in which they are visited by a vehicle following tour \( C \). The length of tour \( C \) will be denoted by \( |C| \). A cluster \([i, j]\) is a contiguous set \( \{i, i+1, \ldots, j\} \) of vertices on tour \( C \). A cluster partition \( P = \)
[1, i_1, i_2, \ldots, i_{k-1}, n], k \geq 2, \text{ and } 1 < i_1 < i_2 < \ldots < i_{k-1} \leq n \text{ is a decomposition of tour } C \text{ into } k \text{ clusters } [1, i_1 - 1], [i_1, i_2 - 1], \ldots, [i_{k-1}, n]. \text{ If the whole tour is a single cluster, the cluster partition is denoted by } [1, n].

In the following, we assume that the maximum total weight of objects in any cluster of a cluster partition is at least the vehicle capacity \(Q\).

Given a cluster partition \(P = [1, i_1, i_2, \ldots, i_{k-1}, n]\) of \(C\), we construct \(k\) subtours \(C_1, C_2, \ldots, C_k\) such that tour \(C_j\) corresponds to the \(j\)th cluster. A vehicle following sub-tour \(C_j\) starts from the depot, visits all vertices in the \(j\)th cluster in increasing order, and returns back to the depot.

The length \(l(P)\) of the cluster partition \(P\) is defined as the sum of the lengths of all the \(k\) subtours \(C_1, C_2, \ldots, C_k\) i.e.,

\[ l(P) = |C_1| + |C_2| + \ldots + |C_k|. \]

The next theorem due to \([10, 2]\) gives an upper bound on the length of the optimal partition for the case when all objects have equal weights.

**Theorem 2.** \([10, 2]\) Suppose all vertices have objects with equal weights. Let \(C\) be a TSP tour of \(G\) and let \(Q^*\) be the maximum number of objects a vehicle can carry at any point of time. Then, there exists a cluster partition \(P\) of \(C\) of total length at most \(2 \sum_{i=1}^{n} \frac{w_i d_i}{Q} + (1 - \frac{1}{Q}) |C|\).

A partition \(P\) satisfying the conditions of Theorem 2 can be computed in \(O(n)\) time \([10]\). Further, if \(C\) is an \(\alpha\)-optimal traveling salesman tour, the lower bound in Theorem 1 shows that the schedule corresponding to \(P\) is of length at most \(1 + \alpha \left(1 - \frac{1}{Q}\right)\) times the optimal schedule \([10, 2]\).

The following theorem due to \([1]\) gives an upper bound on the length of the optimal partition when objects have unequal weights less than or equal to the vehicle capacity \(Q\).

**Theorem 3.** \([1]\) Suppose the objects at vertices have arbitrary integer weights less than or equal to \(Q\). Let \(C\) be a TSP tour of \(G\) and let \(Q\) be the vehicle capacity. Then, there exists a cluster partition \(P\) of \(C\) of total length at most \(4 \sum_{i=1}^{n} \frac{w_i d_i}{Q} + (1 - \frac{2}{Q}) |C|\).

A partition \(P\) satisfying the conditions of Theorem 3 can be computed in \(O(n^2)\) time \([4, 15]\), and if we have an \(\alpha\)-optimal traveling salesman tour, the schedule corresponding to partition \(P\) has length at most \(2 + \alpha \left(1 - \frac{2}{Q}\right)\) times the optimal schedule \([1]\).

3. Our Results

Let \(C\) be a traveling salesman tour of graph \(G\). Let \(P = [1, i_1, i_2, \ldots, i_{k-1}, n]\) be a cluster partition of tour \(C\). Let \(C_1, C_2, \ldots, C_k\) be the subtours corresponding to partition \(P\). A vehicle can traverse each subtour \(C_j\) in either clockwise or anticlockwise direction, and the fuel consumption for these two directions may be different. We define the fuel consumption \(f(C_j)\) of subtour \(C_j\) as the fuel consumption for the optimum direction of traversal. The fuel consumption of partition \(P\) is defined as \(f(P) = f(C_1) + f(C_2) + \ldots + f(C_k)\).

Next we give a lower bound on the cost of the fuel consumption which is a straightforward and an important extension of the bound in \([10]\).

**Theorem 4.** Let \(C^*\) denote an optimal traveling salesperson tour, and let \(Q\) be the capacity of the vehicle. Let \(w_i\) be the weight of the object at vertex \(i\) and let \(d_i\) be the shortest distance between vertex \(i\) and the depot. Then, the minimum fuel consumed by the vehicle to bring all objects to the depot is at least:

\[ a \cdot \max \left( |C^*|, 2 \sum_{i=1}^{n} \frac{w_i d_i}{Q} \right) + b \left( \sum_{i=1}^{n} w_i d_i \right). \]

**Proof:** Let \(C_1^*, C_2^*, \ldots, C_k^*\) be a set of tours which achieve the optimal fuel consumption for a vehicle of capacity \(Q\). Let \(d_i^*\) be the distance traveled by a vehicle following the optimal set of tours between picking up the object with weight \(w_i\) at vertex \(i\) and dropping it at the depot. Then, the optimal fuel consumption is \(a \cdot \left( \sum_{j=1}^{l} |C_j^*| \right) + b \cdot \sum_{i=1}^{n} w_i d_i^*\).

We note two facts to complete the proof. First, \(\sum_{j=1}^{l} |C_j^*|\) is the distance traveled by a vehicle of capacity \(Q\) in graph \(G(V,E)\) to bring all objects to the depot. By Theorem 1, this is at least \(\max \left( |C^*|, 2 \sum_{i=1}^{n} \frac{w_i d_i}{Q} \right)\). Secondly, for all \(i \in V \setminus \{0\}\), \(d_i^* \geq d_i\), as a vehicle has to travel at least the shortest distance from vertex \(i\) to the depot to bring object \(i\) to the depot. Therefore, the optimum fuel consumption is at least:

\[ a \cdot \max \left( |C^*|, 2 \sum_{i=1}^{n} \frac{w_i d_i}{Q} \right) + b \left( \sum_{i=1}^{n} w_i d_i \right). \]

**Lemma 1.** Let \(C\) be a tour of length \(L\) which visits a subset \(S \subset V\) of vertices of total weight \(W\). Then, a vehicle can visit all vertices in set \(S\) with fuel consumption at most \(aL + bWL/2\).

**Proof:** Without loss of generality suppose the tour visits vertices numbered \(1, 2, \ldots, n'\). We note that the fuel consumption can vary based on whether we take the clockwise or anticlockwise direction around \(C\). Let \(d_1^+\) and \(d_1^-\) be the distances of vertex \(i\) from the depot along tour \(C\) in the clockwise and anticlockwise directions. Clearly, \(d_1^+ + d_1^- = L\).

The fuel consumed by a vehicle following \(C\) in the clockwise direction is \(a|C| + b(\sum_{i=1}^{n'} w_i d_1^+)\), and in the anticlockwise direction is \(a|C| + b(\sum_{i=1}^{n'} w_i d_1^-)\). The sum of the two costs is \(2a|C| + b(\sum_{i=1}^{n'} (w_i \cdot (d_1^+ + d_1^-)))\) which is equal to
2aL + bWL. Hence, the minimum of the two tours will have cost at most half of this quantity.

**Theorem 5 (UNEQ-INF-FM).** Let \( \beta > 0 \) be a positive rational number, \( C \) be a traveling salesperson tour, and assume that the vehicle has infinite capacity. Then, there exists a cluster partition \( P = [1, i_1, i_2, \ldots, i_{k-1}, n] \) of \( C \) with total fuel consumption at most:

\[
(1 + \frac{2}{\beta}) b \left( \sum_{i=1}^{n} w_i d_i \right) + \left( 1 + \frac{\beta}{2} \right) a|C|.
\]

**Proof:**

Since \( \beta \) is a rational number and \( a, b \), and the object weights \( w_i \) are finite-precision numbers on a computer, we can choose positive integers \( t, m \) such that (i) \( \frac{2a}{t} \) is equal to \( \frac{1}{m} \), and (ii) \( mw_i \) is a positive integer for all object weights \( w_i, 1 \leq i \leq n \).

We construct a new graph \( G'(V', E') \) from \( G(V, E) \) as follows. We keep the depot as it is, and replace each vertex \( i, 1 \leq i \leq n \), with a set \( V_i = \{v_1, v_2, \ldots, v_{mw_i}\} \) of \( mw_i \) vertices of weight 1/m each. Any two vertices in the same set \( V_i \) are joined by an edge of cost 0, whereas a vertex in set \( V_j \) is joined to a vertex in set \( V_i, j \neq i \) with an edge of length equal to the distance between vertex \( i \) and vertex \( j \) in graph \( G \). The distance of the depot from a vertex in set \( V_i \), \( 1 \leq i \leq n \), is taken as \( d_j \), the shortest distance from the depot to vertex \( j \) in graph \( G \). Clearly, the edge costs in graph \( G' \) form a metric space.

We now construct a traveling salesperson tour \( C' \) of graph \( G' \) from the given tour \( C \) as follows. We replace each vertex \( i \) on tour \( C \) with a path \( P_i = (v_1, v_2, \ldots, v_{mw_i}) \) on the \( mw_i \) vertices in \( V_i \). Since the cost of an edge between any two consecutive vertices in path \( P_i \) is zero, path \( P_i \) has length 0. The depot will be replaced by a path \( P_0 \) consisting of a single vertex. For each \( 0 \leq i \leq n \), the last vertex of path \( P_i \) is connected to the first vertex of path \( P_{i+1} \) by an edge of cost equal to \( c(i, j) \). Clearly, \( C' \) is a tour on the \( N = \sum_{i=1}^{n} \) vertices in graph \( G' \) and has total length equal to \( |C| \).

Label the depot in graph \( G' \) as 0 and number the remaining vertices, 1, 2, \ldots, \( N \) according to the order in which they occur on tour \( C' \). Let \( d_i' \) denote the shortest distance of vertex \( i \) from the depot 0 in graph \( G'(V', E') \). For all \( i \in V_j \), we have that \( d_i' = d_j \). Therefore, \( \sum_{i=1}^{n} d_i' = \sum_{i=1}^{n} mw_id_i \).

Applying Theorem 2 on tour \( C' \) with \( Q = t \), we get a cluster partition \( P' = [1, i_1, i_2, \ldots, i_{p-1}, N] \) of tour \( C' \) of length at most \( 2\sum_{i=1}^{n} d_i' + \left( 1 - \frac{1}{t} \right) |C'| \) such that each cluster in the partition has at most \( t \) vertices.

Putting \( t/m = \frac{2a}{\beta} \) and \( \sum_{i=1}^{N} d_i' = \sum_{i=1}^{n} mw_id_i \), the length \( L \) of partition \( P' \) is at most \( \frac{\beta}{2} \cdot b \left( \sum_{i=1}^{n} w_i d_i \right) + \left( 1 - \frac{1}{t} \right) |C| \).

We now compute the fuel consumption of partition \( P' \). Let \( C'_1, C'_2, \ldots, C'_p \) be the subtours corresponding to partition \( P' \). Let \( \ell \) be the length of tour \( C'_j \) in graph \( G' \). Since each subtour has at most \( t \) vertices of weight \( 1/m \) each, the total weight of vertices in each subtour is at most \( 1/m \). Therefore, by Lemma 1, either the clockwise or anticlockwise traversal of tour \( C'_j \) will have fuel consumption at most \( aL_j + \frac{b}{m} \cdot \frac{1}{m} \cdot L_j \). Therefore, the total fuel consumption of a vehicle which follows each of the \( p \) subtours in the optimal direction is at most \( \sum_{j=1}^{p} \left( a + \frac{b}{m} \right) L_i = \left( a + \frac{b}{m} \right) L \right) L = \left( 1 + \frac{\beta}{2} \right) aL \).

By replacing the value of \( L \) from above, we get that the total fuel consumption \( f(P') \) of partition \( P' \) is at most:

\[
\left( 1 + \frac{2}{\beta} \right) b \left( \sum_{i=1}^{n} w_i d_i \right) + \left( 1 + \frac{\beta}{2} \right) a|C|.
\]

To complete the proof, we will now construct a partition of tour \( C \) in \( G(V, E) \) with the same fuel consumption as that of partition \( P' \) in \( G'(V', E') \). Consider a vehicle which brings all objects to the depot in graph \( G' \) by following the subtours \( C'_1, C'_2, \ldots, C'_p \) given by partition \( P' \). Take a vertex \( i \in V \setminus \{0\} \). For \( 1 \leq j \leq mw_i \), let \( d_i' \) denote the distance traveled by the vehicle between picking the object of weight \( 1/m \) at vertex \( v_i \) and dropping it at the depot. Then, \( f(P') = a \sum_{i=1}^{n} \left| C'_i \right| + b \cdot \left( \sum_{i=1}^{n} w_i d_i' \right) \).

We will now modify the weights of vertices in graph \( G' \). Let \( n(i), 1 \leq i \leq n \) denote the index of the vertex in set \( V_i \) with minimum \( d_i', 1 \leq j \leq mw_i \). We will set the weight of the object at vertex \( v_i \) to \( w_i \), and set the weights of objects at all other vertices in set \( V_i \) to 0. The fuel consumption after transferring all the weight in set \( V_i \) to the special vertex \( v_i(n(i)) \) for each \( i \in [1, n] \) is given by \( F = a \sum_{i=1}^{n} \left| C'_i \right| + b \cdot \left( \sum_{i=1}^{n} w_i d_i' \right) \). Since \( d_i(n(i)) \leq d_i', 1 \leq j \leq mw_i \), we get that \( F \leq a \sum_{i=1}^{n} \left| C'_i \right| + b \cdot \left( \sum_{i=1}^{n} mw_i d_i' \right) \left( \frac{1}{m} \right) \cdot d_i(n(i)) \leq f(P') \). Hence the fuel consumption of partition \( P' \) is at most \( f(P') \).

We now take the subtours \( C'_1, C'_2, \ldots, C'_p \), with the new weights on vertices and short-circuit all vertices of weight 0. If a subtour has no vertex of non-zero weight, we delete the tour from the collection. The resulting set of tours \( C_1, C_2, \ldots, C_p \) will consist of only \( n \) vertices \( v_i(n(i)), i \in [1, n] \) and will have fuel consumption less than or equal to \( f(P') \). The required partition \( P \) of tour \( C \) can now be obtained by identifying vertex \( v_i(n(i)) \) in graph \( G' \) with vertex \( i \) in graph \( G \).

**Theorem 6 (UNEQ-CAP-FM).** Let \( \beta > 0 \) be a positive rational number, \( C \) be a traveling salesperson tour, and \( Q \) be the vehicle capacity. Then, there exists a cluster partition \( P = [1, i_1, i_2, \ldots, i_{k-1}, n] \) of \( C \) with total fuel consumption at most \( \left( 1 + \frac{2}{\beta} \right) b \left( \sum_{i=1}^{n} w_i d_i \right) + \left( 1 + \frac{\beta}{2} \right) a|C| + 4a \sum_{i=1}^{n} w_i d_i' \).

Further, if all vertices have unit weights, the fuel con-
sumption of partition $P$ can be reduced to:

$$(1 + \frac{2}{\beta}) \cdot b \cdot \left( \sum_{i=1}^{n} d_i \right) + 4a + 2a \sum_{i=1}^{n} \frac{d_i}{Q}.$$

Proof: By Theorem 5, there exists a partition $P = \{1, i_1, i_2, \ldots, i_k\}$ of tour $C$ with fuel consumption $f(P)$ at most:

$$(1 + \frac{2}{\beta}) \cdot b \cdot \left( \sum_{i=1}^{n} w_i d_i \right) + \left( 1 + \frac{\beta}{2} \right) a|C| + 2a \sum_{i=1}^{n} \frac{d_i}{Q}.$$

Let $C_1, C_2, \ldots, C_k$ be the subtours corresponding to partition $P$. Take a particular tour $C_j$. Let $W_j$ be the total weight of objects picked by the vehicle in tour $C_j$. If $W_j \leq Q$, then $C_j$ satisfies the capacity constraint and hence we keep the cluster corresponding to subtour $C_j$.

Now assume that $W_j > Q$. By Theorem 3, there exists a refined cluster partition $P_j$ of $C_j$ such that (i) the total weight of objects in each cluster is at most $Q$, and (ii) the total length of $P_j$ is at most $|C_j| + 4a \sum_{i \in C_j} w_i d_i$. Let the subtours corresponding to partition $P_j$ be $C_{j1}, C_{j2}, \ldots, C_{jk}$. We will show below that the fuel consumption of partition $P_j$ is at most $f(C_j) + 4a \sum_{i \in C_j} w_i d_i$. Let $P'$ be the final partition obtained which includes intact as well as refined clusters.

Then, the total fuel consumption of partition $P'$, $f(P') \leq \sum_{j=1}^{k} f(C_j) + 4a \sum_{i \in C_j} w_i d_i$, which is equal to $f(P) + 4a \sum_{i \in C_j} w_i d_i$. Putting the upper bound on $f(P)$ given by Theorem 5, we have shown that $P'$ is a partition satisfying the conditions of the theorem.

We now prove the upper bound on the fuel consumption $f(P_j)$. Without loss of generality, assume that $f(P_j)$ is given by a tour which visits the vertices of $C_j$ in clockwise order (i.e., increasing order of labels). The proof for the anticlockwise case is symmetric.

We will compute the total fuel consumed by a vehicle which traverses every subtour $C_{jl}, 1 \leq l \leq k_j$ in partition $P_j$, in clockwise order. Let $S_j$ be the set of vertices covered by tour $C_j$. For each vertex $i \in S_j$, let $d_i^1$ and $d_i^2$ be the distance traveled by the vehicle between picking object $i$ and dropping it at the depot in tour $C_j$ and partition $P_j$ respectively.

Since partition $P_j$ is a refinement of tour $C_j$, a vehicle after picking the object at vertex $i \in S_j$ will turn to the depot at the last vertex of the cluster containing $i$ rather than going all the way along tour $C_j$ before arriving at the depot. Hence we get that $d_i^2 \leq d_i^1$.

Therefore, $f(P_j) = \sum_{l=1}^{k_j} f(C_{jl}) = \sum_{l=1}^{k_j} \left( a|C_{jl}| + b \sum_{i \in C_{jl}} w_i d_i^2 \right)$. This is equal to $a \sum_{l=1}^{k_j} |C_{jl}| + b \sum_{i \in C_{jl}} w_i d_i^2 \leq a \cdot l(P_j) + b \sum_{i \in C_{jl}} w_i d_i^2$.

Putting $l(P_j) = |C_j| + 4a \sum_{i \in C_j} w_i d_i$, we get that

$$f(P_j) \leq \left( a|C_j| + b \cdot \sum_{i \in C_j} w_i d_i^2 \right) + 4a \cdot \frac{\sum_{i \in C_j} d_i}{Q},$$

as required.

The improved bound for unit weights can be obtained by applying Theorem 2 on each subtour for computing the refined partition $P''$.

Theorem 7. There exists a factor 4 polynomial-time approximation algorithm for UNEQ-CAP-FM problem. Further, the approximation factors achievable in polynomial time for EQ-INF-FM, UNEQ-INF-FM, and EQ-CAP-FM are 2.5, 2.5, and 3.186 respectively.

Proof: We will describe a factor 4 approximation algorithm for UNEQ-CAP-FM problem. The derivation of approximation factors for the other three problems is similar and is left to the reader.

The approximation algorithm is as follows:

1. Use Christofides’ algorithm [6] to get a TSP tour $C$ whose length is within 1.5 factor of the optimal TSP tour $C^*$.
2. Compute a partition $P^*$ of tour $C$ with minimum fuel consumption using dynamic programming in $O(n^2)$ time [4, 15] (see section 3.1).
3. Output the subtours $C_1^*, C_2^*, \ldots, C_k^*$ corresponding to partition $P^*$ as the solution.

We now show that there exists a partition $P$ of $C$ with $f(P) \leq 4 \cdot OPT$, where $OPT$ is the optimum fuel consumption. Since $P^*$ is the optimal partition of $C$, we get that $f(P^*) \leq f(P) \leq 4 \cdot OPT$.

Applying Theorem 6 on tour $C$ with $\beta = 2/3$, we get a partition $P$ of tour $C$ with fuel consumption at most

$$4 \cdot b \cdot \left( \sum_{i=1}^{n} w_i d_i \right) + (4/3)a|C| + 4a \sum_{i=1}^{n} \frac{w_i d_i}{Q}.$$

Since $|C| \leq 1.5|C^*|$, the fuel consumption of $P$ is at most

$$4 \cdot b \left( \sum_{i=1}^{n} w_i d_i \right) + a \cdot \max \left( |C^*|, 2 \sum_{i=1}^{n} \frac{w_i d_i}{Q} \right).$$

By Theorem 6, this quantity is less than or equal to $4 \cdot OPT$.

Applying Theorem 6 on an $\alpha$-approximate traveling salesperson tour gives us a factor

$$\max \left( 1 + \frac{\alpha}{2}, 2 + \alpha \left( 1 + \frac{\alpha}{2} \right) \right)$$

approximation for UNEQ-CAP-FM. The minimum factor is achieved at a value $\beta > 0$ which makes $1 + \frac{\alpha}{2}$ equal to $2 + \alpha \left( 1 + \frac{\alpha}{2} \right)$.

This gives us a factor $f(\alpha) = 1 + \frac{\sqrt{4\alpha^2 + 24\alpha + 4} + 2(2\alpha^2 + 2\alpha + 1) - (2\alpha^2 + 2\alpha + 1) \cdot \sqrt{4\alpha^2 + 24\alpha + 4}}{4\alpha}$, and $1 + \sqrt{4\alpha^2 + 24\alpha + 4}$ can be achieved for EQ-INF-FM, UNEQ-INF-FM, EQ-CAP-FM, and UNEQ-CAP-FM respectively. Further, if we have access to a $1 + \epsilon$ approximate traveling salesperson tour, the approximation factors can be improved to $2 + \epsilon, 2 + \epsilon, 2 + \epsilon,$...
2.618 + \epsilon$, and $3.414 + \epsilon$ respectively, and this indeed is the case for the Euclidean and planar versions of our problem where PTASes exist to approximate the length of the traveling salesperson tour \cite{14, 3, 13}.

### 3.1. Computing the optimal partition

Given a TSP tour $C$, we can compute the minimum length partition of $C$ in $O(n^2)$ time using dynamic programming \cite{4, 15}. In this section, we give an $O(n^2)$ time algorithm to compute a partition of tour $C$ with minimum fuel consumption based on the same ideas.

As before, number the vertices $1, 2, \ldots, n$ according to the order in which they occur on tour $C$. For a cluster $[i, j]$, we define $A(i, j)$ as the fuel cost of traversing the cluster in the clockwise direction (i.e., in increasing order of labels). Similarly, define $B(i, j)$ as the fuel cost of traversing the same cluster in the anticlockwise direction. Define the fuel consumption $F(i, j)$ of cluster $[i, j]$ as

$$A(i, j) \cdot \min(A(i, j), B(i, j)).$$

For a vertex $i$, let $d_i$ denote the shortest distance from $i$ to the depot. Let $d_i^1$ and $d_i^2$ denote distance traveled by a vehicle between vertex $i$ and depot while traversing tour $C$ in the clockwise and anticlockwise direction respectively. For $i \in V \setminus \{0\}$, define $W_i = \sum_{j=1}^{n} w_j$. Given tour $C$, the arrays $d_i, d_i^1, d_i^2$, and $W_i$ can be computed in $O(n)$ time.

The weight of the tour is $W = \sum_{i=1}^{n} W_i$.

Given tour $C$, let $\Delta_i$ be the part of tour $C \setminus \{\{i\}\}$ that is traversed by a vehicle between vertex $i$ and depot. The fuel cost $F(i, j)$ of the tour $[i, j]$ is computed by storing for each $i$ a partition of vertices $F(i)$ of traversing the same cluster $[i, j]$ as

$$F(i, j) = \min_{1 \leq j' \leq i} (Q(j - 1) + F(j, i)) .$$

We can compute $Q(1), Q(2), \ldots, Q(n)$ in this order, spending $O(n)$ time for computing each $Q(i)$. Thus, we can compute the fuel cost $Q(n)$ of the optimal partition in $O(n^2)$ time. The partition corresponding to $Q(n)$ can be computed by storing for each $Q(i)$, the index $J_i^*$, $1 \leq J_i^* \leq i$, which achieves the minimum in the above recurrence.

### 4. Conclusion

We give constant factor approximations for cumulative VRPs with an arbitrary number of offloads allowed at the depot. Our algorithms can be compared with the algorithm of Blum et al. \cite{5} for cumulative VRP where a single offload is allowed at the depot. The approximability of cumulative VRP when the number of offloads is given as part of input is an open question.

### References

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