Large bipartite Cayley graphs of given degree and diameter

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Abstract
Let $BC_{d,k}$ be the largest possible number of vertices in a bipartite Cayley graph of degree $d$ and diameter $k$. We show that $BC_{d,k} \geq 2(k-1)((d-4)/3)^{k-1}$ for any $d \geq 6$ and any even $k \geq 4$, and $BC_{d,k} \geq (k-1)((d-2)/3)^{k-1}$ for $d \geq 6$ and $k \geq 7$ such that $k \equiv 3 \pmod{4}$.

Keywords: Cayley graph; Bipartite graph; Degree; Diameter

The problem of determining the largest graphs of given maximum degree and diameter is known as the degree-diameter problem. For a detailed survey on the problem we refer to [5]. In this note we focus on bipartite Cayley graphs of given degree and diameter. Let $B_{d,k}$ denote the largest order of a bipartite graph of maximum degree $d$ and diameter $k$, and let $BC_{d,k}$ be the largest number of vertices in a bipartite Cayley graph of degree $d$ and diameter $k$.

Biggs [2] showed that the number of vertices in a bipartite graph of degree $d$ and diameter $k$ can not exceed the bipartite Moore bound $M_{d,k} = 2((d-1)^k - 1)/(d-2)$ for $d \geq 3$, and the bound $2k$ for $d = 2$. Exact values of $B_{d,k}$ and $BC_{d,k}$ are available only in rare cases. For example, it is easy to see that
for \( d, k \geq 2 \), we have \( B_{2,k} = 2k \) and \( B_{d,2} = 2d \). The graphs of order \( 2k \) are the \( 2k \)-cycles and the graphs of order \( 2d \) are the complete bipartite graphs \( K_{d,d} \). Since both, cycles and complete bipartite graphs with partite sets of equal size are Cayley graphs, \( BC_{2,k} = 2k \) and \( BC_{d,2} = 2d \) as well. It is also known that \( B_{d,k} \) is equal to the bipartite Moore bound if \( k = 3, 4, 6 \) and \( d - 1 \) is a prime power, see [1], [2].

Improvements on the upper bound for \( B_{d,k} \) can be seen in the recent papers of Pineda-Villavicencio [6] and Delorme et al. [4]. Pineda-Villavicencio [6] proved that there exist no bipartite graphs of order \( M_{d,k} - 2 \) for any \( d \geq 3 \) and \( k \geq 4 \) which yields the bound \( B_{d,k} \leq M_{d,k} - 4 \) for any \( d \geq 3 \) and \( k \geq 5 \) with \( k \neq 6 \). There are no better general upper bounds on \( BC_{d,k} \) than upper bounds for \( B_{d,k} \).

For the largest known bipartite graphs of degree \( d \) and diameter \( k \) for small \( d \) and \( k \), see [8]. The orders of known constructions of bipartite graphs for large \( d \) and \( k \) are significantly lower than the bipartite Moore bound. Bond and Delorme [3] presented large bipartite graphs of given degree and diameter using their concept of a partial Cayley graph. By suppressing edge directions in constructions of directed graphs of Vetrík [7], one has the general lower bound \( BC_{d,k} \geq (k-1)((d-1)/4)^{k-1} \) for any \( k \geq 4 \) and any even \( d \geq 8 \).

We give bipartite Cayley graphs larger than bipartite graphs coming from constructions of [7]. First, let us present a family of bipartite Cayley graphs of degree \( d \equiv 0 \pmod{3} \).

**Theorem 1.** Let \( d \geq 6 \) be a multiple of 3.

(i) For any even \( k \geq 4 \), we have \( BC_{d,k} \geq 2(k-1)(d/3)^{k-1} \).

(ii) For \( k \geq 7 \) such that \( k \equiv 3 \pmod{4} \), we have \( BC_{d,k} \geq (k-1)(d/3)^{k-1} \).

**Proof.** Let \( H \) be a group of order \( m \geq 2 \) with unit element \( e \) and let \( H^{k-1} \) be the product \( H \times H \times \ldots \times H \), where \( H \) appears \( k-1 \) times. Denote by \( \alpha \) the automorphism of the group \( H^{k-1} \) such that \( \alpha(x_1, x_2, \ldots, x_{k-1}) = (x_{k-1}, x_1, x_2, \ldots, x_{k-2}) \). It means that \( \alpha \) shifts coordinates by one to the right. The cyclic group of order \( t \) will be denoted by \( Z_t \) and the semidirect product \( H^{k-1} \rtimes Z_t \) will be denoted by \( G \). (Note that \( G \) is the semidirect product of \( N \) and \( A \), written \( G = N \rtimes A \), if \( N \) is a normal subgroup of \( G \), \( A \) is a subgroup of \( G \), and every element of \( G \) can be written as a unique product of an element of \( N \) and an element of \( A \).) Multiplication in \( G \) is given by \( (x,y)(x',y') = (x \alpha^y(x'), y + y') \), where \( x, x' \in H^{k-1}, y, y' \in Z_t \) and \( \alpha^y \) is the composition of \( \alpha \) with itself \( y \) times. We will write elements of \( G \) in the form \( (x_1, x_2, \ldots, x_{k-1}; y) \), where \( x_1, x_2, \ldots x_{k-1} \in H \) and \( y \in Z_t \).
(i) Let $k \geq 4$ be even and $t = 2(k - 1)$. Let $a_g = (g, \iota, \ldots, \iota; 1)$ for any $g \in H$. Then $a_g^{-1} = (\iota, \ldots, \iota, g^{-1}; -1)$. Further, for any $h \in H$ let $b_h = (\iota, \ldots, \iota, h, \iota, \ldots, \iota; k - 1)$, where $x_{k/2} = h$ and $x_j = \iota$ for $1 \leq j \leq k - 1$, $j \neq k/2$.

Let $X = \{a_g, a_g^{-1}, b_h; \text{ for all } g, h \in H\}$. Since $b_h^{-1} = b_{h^{-1}}$, it is clear that $X = X^{-1}$. The Cayley graph $C(G, X)$ is of degree $d = |X| = 3m$ and order $|G| = 2(k - 1)m^{k-1} = 2(k - 1)(d/3)^{k-1}$. We prove that the diameter of $C(G, X)$ is $\leq k$, which is equivalent to showing that each element of $G$ can be expressed as a product of at most $k$ elements of $X$.

We show that any element $(x_1, x_2, \ldots, x_{k-1}; y)$, where $x_1, x_2, \ldots x_{k-1} \in H$ and $y \in Z_{2(k-1)}$, $y$ is odd, can be obtained as a product of $k - 1$ elements of $X$. Let $0 \leq r \leq k/2 - 1$. Let

$$S = (\Pi_{i=1}^{k-2-r} a_{x_i})(\Pi_{j=1}^{r} b_{x_{j+k-r-1}} a_{x_{j+k/2-r}})(\Pi_{\ell=k/2+1}^{k-r} a_{x_{\ell}})$$

and

$$S' = (\Pi_{i=1}^{k-2-r} a_{x_i}^{-1})(\Pi_{j=1}^{r} b_{x_{j+k-r-1}}^{-1} a_{x_{j+k/2-r}}^{-1})(\Pi_{\ell=k/2+1}^{k-r} a_{x_{\ell}}^{-1}).$$

It can be checked that $S = (x_1, x_2, \ldots, x_{k-1}; 2k - 2 - r)$ if $r$ is odd, and $S = (x_1, x_2, \ldots, x_{k-1}; k - 1 - r)$ if $r$ is even. Then $S' = (x_{k-1}, x_{k-2}, \ldots, x_1; r)$ if $r$ is odd, and $S' = (x_{k-1}, x_{k-2}, \ldots, x_1; k - 1 + r)$ if $r$ is even.

Any element $(x_1, x_2, \ldots, x_{k-1}; y + 1)$ can be expressed as follows:

$$(x_1, x_2, \ldots, x_{k-1}; y + 1) = (x_1, x_2, \ldots, x_{k-1}; y)a_i.$$ 

None of the elements $(x_1, x_2, \ldots, x_{k-1}; y)$ where all $x_i \neq \iota$, $1 \leq i \leq k - 1$, can be obtained as a product of fewer than $k - 1$ elements of $X$, therefore none of the elements $(x_1, x_2, \ldots, x_{k-1}; y + 1)$ can be obtained as a product of fewer than $k$ elements of $X$. The diameter of $C(G, X)$ is exactly $k$.

Since the last coordinate of any element in the generating set $X$ is odd, no two different vertices $(x_1, x_2, \ldots, x_{k-1} ;y)$ and $(x'_1, x'_2, \ldots, x'_{k-1}; y')$ of $C(G, X)$ are adjacent if either both $y, y'$ are even or both $y, y'$ are odd ($y, y' \in Z_{2(k-1)}$ and $x_i, x'_i \in H, 1 \leq i \leq k - 1$). It follows that the graph $C(G, X)$ is bipartite.

(ii) Let $k \geq 7$ such that $k \equiv 3 \pmod{4}$ and $t = k - 1$. The semidirect product $H^{k-1} \rtimes Z_{k-1}$ will be denoted by $G'$. For any $g, h \in H$ let $a_g = (g, \iota, \ldots, \iota; 1)$ and $b_h = (\iota, \ldots, \iota, h, \iota, \ldots, \iota; (k - 1)/2)$, where $x_s = h$ for $s = (k - 1)/2$ and $s = k - 1$. It is evident that $b_{h^{-1}}^{-1} = b_{h^{-1}}$.

Let $X' = \{a_g, a_g^{-1}, b_h; g, h \in H\}$. The Cayley graph $C(G', X')$ is of degree $d = |X'| = 3m$ and order $|G'| = (k - 1)m^{k-1} = (k - 1)(d/3)^{k-1}$. In order
to prove that the diameter of $C(G', X')$ is equal to $k$ it is enough to show that any element $(x_1, x_2, \ldots, x_{k-1}; y)$, where $x_i \in H$, $1 \leq i \leq k-1$ and $y \in Z_{k-1}$, $y$ is even, can be expressed as a product of $k-1$ elements of $X'$. If $r = 0, 2, \ldots, (k-3)/2$, we have

$$(x_1, x_2, \ldots, x_{k-1}; k - 1 - r) = (\Pi_{i=1}^{(k-1)/2-r} a_{x_1}) (\Pi_{j=1}^{r/2} a_{x_2j + (k-3)/2 - r} b_{x_2j + k - r - 2}) a_{x_2j+k-r-1} b_{x_2j+(k-1)/2-r} (\Pi_{i=1}^{(k-r-1)/2} a_{x_i}).$$

Elements of $G'$ with the last coordinate $r$, where $r \in \{2, 4, \ldots, (k-3)/2\}$, can be obtained as inverses of the above ones. The diameter of $C(G', X')$ is $k$ and it can be easily checked that $C(G', X')$ is bipartite. □

We modify the generating set given in the previous proof to get lower bounds on $BC_{d,k}$ for any $d \geq 6$.

**Theorem 2.** Let $d \geq 6$ be an integer.

(i) If $k \geq 4$ is even, then $BC_{d,k} \geq 2(k-1)((d-4)/3)^{k-1}$.

(ii) If $k \geq 7$ such that $k \equiv 3 \pmod{4}$, then $BC_{d,k} \geq (k-1)((d-2)/3)^{k-1}$.

**Proof.** We use the notation of the proof of Theorem 1.

(i) By Theorem 1, $BC_{d,k} \geq 2(k-1)(d/3)^{k-1}$ for $d = 3m$, $m \geq 2$ and for any even $k \geq 4$. Let $u, v$ be two different elements of $G$ with an odd last coordinate such that $u, v \notin X$, $u \neq u^{-1}$ and $v \neq v^{-1}$. It is clear that such elements exist. Let $X_1 = X \cup \{u, u^{-1}\}$ and $X_2 = X \cup \{u, u^{-1}, v, v^{-1}\}$. Then, the Cayley graph $C(G, X_1)$ is bipartite, $C(G, X_1)$ has degree $d = |X_1| = 3m + 2$, diameter at most $k$ and order $|G| = 2(k-1)m^{k-1} = 2(k-1)((d-2)/3)^{k-1}$. The bipartite Cayley graph $C(G, X_2)$ is of degree $d = |X_2| = 3m + 4$ and order $2(k-1)((d-4)/3)^{k-1}$.

Moreover, if $m$ is even, the group $G$ must contain an involution other than the identity, say $z$, not appearing in $X$. Let $X_3 = X \cup \{z\}$. The Cayley graph $C(G, X_3)$ is bipartite of degree $d = |X_3| = 3m + 1$, diameter at most $k$ and order $|G| = 2(k-1)((d-1)/3)^{k-1}$. Thus, $BC_{d,k} \geq 2(k-1)((d-4)/3)^{k-1}$ for any $d \geq 6$ and any even $k \geq 4$.

(ii) From Theorem 1 it follows that $BC_{d,k} \geq (k-1)(d/3)^{k-1}$ if $d \geq 6$ is a multiple of 3 and $k \equiv 3 \pmod{4}$, $k \geq 7$. Let $z, u$ be elements of $G'$ with an odd last coordinate such that $z, u \notin X'$, the order of $z$ is 2 and the order of $u$ is greater than 2. Note that $z$ must be of the form $(x_1, x_2, \ldots, x_{(k-1)/2}, x_1^{-1}, x_2^{-1}, \ldots, x_{(k-1)/2}^{-1}, (k-1)/2)$, where $x_1, x_2, \ldots, x_{(k-1)/2}$ are elements of $H$. 4
Let $X'_1 = X' \cup \{z\}$ and $X'_2 = X' \cup \{u, u^{-1}\}$. Then, the Cayley graph $C(G', X'_1)$ is bipartite, $C(G', X'_1)$ has degree $d = |X'_1| = 3m + 1$, diameter at most $k$ and order $|G'| = (k-1)m^{k-1} = 2(k-1)((d-1)/3)^{k-1}$. The bipartite Cayley graph $C(G', X'_2)$ is of degree $d = 3m+2$ and order $(k-1)((d-2)/3)^{k-1}$. $BC_{d,k} \geq (k-1)((d-2)/3)^{k-1}$ for any $d \geq 6$ and any $k \geq 7$ such that $k \equiv 3 \pmod{4}$.

To the best of our knowledge there is no construction of bipartite graphs of order greater than the order of our graphs. Hence, for sufficiently large $d$ and $k$, our graphs appear to be the largest known bipartite Cayley graphs of degree $d \geq 6$ and diameter $k \geq 4$, where $k \not\equiv 1 \pmod{4}$.

References


