On the Limitations of Random Sensor Placement for Distributed Signal Detection

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Abstract—We consider the design of a sensor network for detecting an emitter who if present is known to be located in an interval but whose exact position is unknown. We seek to minimize the total system power consumption subject to detection performance constraints by carefully choosing the thresholds and positions of the sensors. Toward this goal, we propose an iterative algorithm for the optimization problem. Numerical results are given to provide insights into the design of such networks. We show that random sensor placement can perform poorly, in contrast to what many currently believe.

Index Terms—Sensor Networks, distributed signal detection, sensor placement, power consumption.

I. INTRODUCTION

It is believed [1] that sensor networks have the potential to change the way people live and work. One application is to monitor an area for a specific event, where the network consists of a fusion center and a number of physically distributed sensors. Based on the data collected from the sensors, the fusion center makes a decision on whether the event occurred.

The optimal form of sensor decision rules and fusion rule has been studied under a topic called distributed signal detection theory [2][3]. The majority of research has focused on cases with conditional independence and error-free communication links between the fusion center and the sensors. On the other hand, [4] studies the case where the communication links are not ideal but instead are subject to errors. While detection performance is very important to the system design, there are also other important aspects which need to be considered. Minimum power design is highly desirable in sensor networks since individual sensors are often powered by batteries and power consumption determines the lifetime of the system directly. In [5][6], it is shown that sensors should not always transmit data back to the fusion center when there is a constraint on the transmission rate or power consumption.

In this work, we consider the problem of using multiple sensors to detect a radio signal whose emitter, when present, is located somewhere along a line. The amplitude of the radio signal is attenuated over distance as studied in [7]. The occurrence of the emitter is assumed to be rare and unpredictable. We assume no a prior knowledge of the location of the emitter, leading to a particular composite hypothesis testing problem for which research has been lacking. Due to lack of knowledge on the emitter location, we adopt an emitter-location-independent criterion requiring the system to guarantee a detection probability regardless of the actual emitter location.

Recently, networks employing a large number of sensors are often envisioned where the sensors are thrown in a random way to cover a region of interest. One approach is to place sensors randomly, with a uniform distribution over the area of interest and to have all sensors employ a common threshold [7]. Here we provide results which question the efficiency of such an approach. In particular, we analytically demonstrate that, by employing a particular approach, one can decrease the total transmission power of a signal detection system trying to detect an emitter if more sensors are added under reasonable conditions. However, we show random placement will generally not achieve this power reduction.

In our approach, we attempt to minimize the system’s power consumption subject to detection performance constraints. Such an approach is appealing for applications where limiting power consumption is of high priority. It also enables us to formulate an emitter-location-independent approach which is needed to overcome the lack of knowledge about the emitter location statistics. To promote clarity and simplify our discussions, we consider binary sensor decisions. An iterative algorithm is developed to obtain the sensor decision rules and sensor locations for minimum power consumption. Numerical results show our designs have power advantages compared to other alternatives.

The rest of the paper is organized as follows. Section II formulates the problem we consider. Section III discusses the solution to the optimization problem and proposes an iterative algorithm to find locally optimal solutions. Numerical results and discussion are provided in Section IV. Finally, Section V summarizes the paper.

II. PROBLEM FORMULATION

We consider a very basic configuration which captures the essence of the application of interest. We consider using $M$ sensors to detect the presence of a deterministic radio signal...
emitter in the interval $(0, L)$, where $L$ is a positive real number. The occurrence of the signal emitter is assumed to be rare and unpredictable. The system adopts the parallel topology where each sensor sends its observation summary to the fusion center via a wireless link which has an inverse-$n$ power propagation rule. The fusion center is placed at the origin for simplicity. Let $l = (l_1, l_2, \ldots, l_M)^T$ denote the positions of the sensors where $l_1 < l_2 < \cdots < l_M$. To focus on the issues of central importance here, we assume the wireless communication links employ sufficiently large power so we can ignore any transmission errors.

We have the following simple versus composite hypothesis testing problem:

$$H_0 : \quad y_i = w_i$$
$$H_1 : \quad y_i = Ad_i^{-\gamma} s + w_i$$

where $y_i = (y_{i1}, \ldots, y_{IN})^T$ is the length-$N$ observation vector at sensor $i$, $i = 1, \ldots, M$ and $w_i = (w_{i1}, \ldots, w_{IN})^T$ is the corresponding additive observation noise vector. The noise samples at different sensors are assumed to be independent. In (1), $d_i = |l_i - l_s|$ is the physical distance between sensor $i$ and the emitter location, $l_s \in (0, L)$ denotes the emitter location when the emitter is present, $A$ is the signal amplitude measured at a distance $d_i = 1$ from the emitter, $s = (s_1, \ldots, s_N)$ sets the shape of the emitted signal and $\gamma > 0$ denotes the path loss attenuation exponent of the signal. For simplicity we assume $(s_1, \ldots, s_N)$ is known and that $s$ has normalized power, i.e., $\frac{1}{N} \sum_{j=1}^{N} s_j^2 = 1$.

Based on its observations, each sensor makes a binary decision on whether the emitter is present and informs the fusion center. Each sensor only sends its decision to the fusion center otherwise. Let $P_F$ and $P_D$ denote the false alarm and detection probabilities at sensor $i$. Then $P_F = (P_{F1}, \ldots, P_{FM})^T$ and $P_D = (P_{D1}, \ldots, P_{DM})^T$ denote the sensor false alarm probability vector and the sensor detection probability vector respectively. The fusion center makes the final decision based on the sensor decisions. Since the statistics of $l_s$ are unknown, we adopt an emitter-location-independent detection formulation and define the detection performance measure

$$P_{D_{min}} = \min_{l_s \in (0, L)} P_{D_l}$$

where $P_{D_l}$ denotes the detection probability at the fusion center conditioned on the emitter present at $l_s$. Let $P_F$ denote the false alarm probability at the fusion center. Due to the lack of knowledge of the statistics of $l_s$, we consider symmetrical fusion rules where all the sensor decisions are treated as equally important. Hence the fusion center counts the number of sensors which report detections and reports a final decision for $H_1$ when $k$ or more sensors decide for $H_1$. Such a fusion rule is called a $k-out-of-n$ rule.

Proposition 1 Assume the fusion center adopts a $k-out-of-n$ fusion rule and the Neyman-Pearson criterion with detection probability defined in (2). Then a likelihood ratio test at each individual sensor is optimal for the fusion performance if the sensor decisions are conditionally independent.

Proof: We omit the proof due to space limitations.

The detection probability for sensor $i$, $P_{D_i}$, depends on the threshold used in the sensor likelihood ratio test and the distance from the sensor to the emitter. The sensor detection probability, $P_{D_i}$, can be made arbitrarily close to 1 for any $d_i$ by choosing a small enough threshold. On the other hand, small thresholds for the likelihood test lead to high false alarm probabilities. Consider choosing a threshold for sensor $i$ such that $P_{D_i} \geq 1$ if $d_i \leq r_i$ and $P_{D_i} < 1$ if $d_i > r_i$. Here $r_1 > 0, 0 < \beta < 1$. Let $f(r_i, \beta)$ denote the corresponding false alarm probability $P_F$. In other words, $f(r_i, \beta)$ is the smallest false alarm probability of the sensor to guarantee a minimum sensor detection probability $\beta$ conditioned on $d_i \leq r_i$. If the threshold of sensor $i$ is chosen this way, we say the sensor has coverage radius $r_i$ for detection probability $\beta$ and call $(l_i - r_i, l_i + r_i)$ the coverage area of sensor $i$ for detection probability $\beta$. It is easy to show that for given $\beta$, $f(r_i, \beta)$ is a monotonically increasing, one-to-one, bounded function of $r_i$. For given $r_i$, $f(r_i, \beta)$ is a monotonically increasing one-to-one, bounded function of $\beta$. We use $f_{\beta}^{-1}(P_F)$ to denote the inverse function of $f(r_i, \beta)$ conditioned on $\beta$, i.e., $f(f_{\beta}^{-1}(P_F)), \beta) = P_F$, for any $P_F \in (0, 1)$.

In order to consider the power in the communications from the sensors, we model the power needed to communicate reliably from sensor $i$ to the fusion center as $P_i = P_0 l_i^\gamma$. Here an inverse-$n$ power propagation law is assumed and $P_0$ is the power needed to insure reliable communications if $l_i = 1$. Collect the powers in a vector $P_l = (P_1, \ldots, P_M)^T$ for convenience. Assume that $H_0$ occurs much more frequently than $H_1$ so that any power used under $H_1$ can be ignored. The expected total transmission power consumption can be approximated as

$$P^{\text{tr}} = \sum_{i=1}^{M} P_F P_i. \quad (3)$$

Our goal is to minimize power consumption subject to detection performance constraints at the fusion center. Toward this goal, we use (2) and (3) to formulate an optimization problem as

$$\text{Minimize } P^{\text{tr}} \text{ s.t. } P_{D_{min}} \geq \beta \text{ and } P_F \leq \alpha \quad (4)$$

for some given $\alpha$ and $\beta$ such that $0 < \alpha < 1, 0 < \beta < 1$.

### III. RESULTS AND DISCUSSIONS

A focus of this work is to guarantee reduced total transmission power for each additional sensor. Next we provide a Theorem which shows that sufficiently smart placement and fusion will guarantee this behavior.

Theorem 1 Consider $P^{\text{tr}}$ for a minimum transmission power uniform sensor placement using an OR fusion rule that satisfies the constraints in (4). Such $P^{\text{tr}}$ decreases with $M$ for...
that increases fast enough with $x$ such that

$$f\left(\frac{t}{2(M+1)}\right) < \min \left\{ \frac{M}{M+1} \left(\frac{M+1}{M}\right)^n \sum_{i=1}^{M+1} (2i-1)^n \right\}$$

and for sufficient large $M$ such that

$$f^{\beta}_{\beta} \left(1 - (1-\alpha)^{1/M}\right) \geq L/2M$$

if the sensors all employ the same thresholds (which can vary with $M$).

Proof: We omit the proof due to space limitations.

The conditions in Theorem 1 require that $f(x, \beta)$ increases with $x$ sufficiently fast. To gain more insight, we consider the case of $n = 2$ which is the appropriate attenuation law in free space. Using $\sum_{i=1}^{2M} (2i-1)^2 = \frac{1}{3} M (4M^2 - 1)$, we have

$$\sum_{i=1}^{2M} (2i-1)^n \geq \frac{M (4M^2 - 1)}{(M+1)(4(M+1)^2 - 1)}$$

$$= \left(\frac{M}{M+1}\right)^3 \frac{4 - 1/M^2}{4 - 1/(M+1)^2}$$

$$\geq \frac{4}{5} \left(\frac{M}{M+1}\right)^3 .$$

Using (7) in (5), we obtain a sufficient condition for (5) as

$$(2i-1)^3$$

Similarly, using $\sum_{i=1}^{2M} (2i-1)^3 = M^2 (2M^2 - 1)$, we can obtain a sufficient condition for (5) as $f\left(\frac{t}{2(M+1)}\right) < \frac{0.57 M}{M+1}$ for $n = 3$. We note that (8) is one of many bounds we can obtain for $n = 2$. For example, one can obtain a bound for which the right-hand side of (8) approaches unity for large $M$ also. Similar comments apply for $n = 3$. We note that numerical evaluations for cases with additive Gaussian noise in (1) imply that the conditions of Theorem 1 hold for many cases with parameters (for example $n, \gamma, M, A, \alpha, \beta$) of practical interest. See Figure 1 for a few examples.

A. Efficient Placement Algorithm

We have just shown that the total transmission power can be reduced by adding more sensors under reasonable conditions when an OR rule is used with uniform sensor placement. It should be clear that further optimization, beyond uniform placement, is possible. We show later that in typical cases, the additional power savings can be significant. Next we propose a specific approach to solve (4) using reasonable simplifications. In particular, we assume an OR fusion rule, which is well suited for the emitter-location-independent measure in (2). This was demonstrated in the proof of Theorem 1. Once the OR rule is employed, it is easy to demonstrate that the constraints in (4) can be enforced by imposing detection constraints on each individual sensor. In particular, these constraints require $P_{F_{i}} \leq 1 - (1-\alpha)^{1/M}$, $\forall i$ and that at least one $P_{D_i} \geq \beta$ for any $l_i$. Using these constraints, we formulate the optimization problem as in (9) where $0 < \alpha < 1$, $0 < \beta < 1$ and $i = 1, \ldots, M$.

Next, in the following proposition, we provide some insight on the sensor placements satisfying (9). To facilitate discussions in the rest of the paper, we use $r_i$ to denote the coverage radius of sensor $i$ for detection probability $\beta$.

Proposition 2 Consider the optimization problem in (9).

For any given $\alpha$ and $\beta$, the optimum sensor coverage areas do not overlap.

Proof: We omit the proof due to space limitations.

Now we describe an efficient sensor placement algorithm. First, we note that the non-overlapping condition for optimality in Proposition 2 implies that $l_i = l_{i-1} + r_{i-1} + r_i$ for $i = 2, \ldots, M$. Further the constraints on $P_{E_i}$ and $P_{D_i}$ in (4) require that (1) $\max_{i} r_i < r^* = f^{\beta}_{\beta}(1 - (1-\alpha)^{1/M})$ (2) $(0, L)$ is completely covered. So we can recast the optimization problem in (9) as in (10).

Note that $M$ must be at least $L/2r^*$ for (10) to have at least one feasible solution. Due to the fact that $f$ is nonlinear and nonconvex, in general (10) is still a hard optimization problem. We turn to numerical methods to find the optimal solutions. We propose to solve Eqn. (10) via iterative local optimization. We start with a feasible solution $(l_1, l_2, \ldots)$, then obtain $(l_1^*, l_2^*, \ldots)$ sequentially such that the sequence of solutions provide decreasing $P^{tr}$. Since $P^{tr}$ is lower-bounded by 0, the algorithm converges to a locally optimal stationary point.

The following outline describes how to perform the update of a given feasible solution.

1. Set $i = 1$.
2. Choose $l_1$ and $r_1$ to minimize $f(r_1, \beta) l_1^2 + f(r_2, \beta) l_2^2$, subject to the constraints in (10) while sensors 1 and 2 still cover $[0, l_2 + r_2]$.
3. $i = i + 1$.
4. Choose $r_i$, subject to (10) while sensors $i$ and $i + 1$ still cover $[l_i - r_{i-1}, l_{i+1} + r_i]$.
5. Repeat steps 3 and 4 till $i$ reaches $M - 1$.
6. Repeat steps 1-5 till convergence is observed.

The proposed algorithm optimizes the coverage radii of two neighboring sensors while keeping their combined coverage area unchanged. Note that the computation of $f(x, \beta)$ is not very complex, thus allowing us to perform an exhaustive search for the local minimum in Step 2 and 4. To help improve the performance of the proposed algorithm, we start the search from several randomly generated initial solutions. Our numerical study finds that they tend to converge to the same or similar solutions.

IV. NUMERICAL RESULTS

In this section we provide numerical results obtained via the proposed algorithm. First we need to obtain $f(x, y)$ explicitly. We assume that the i.i.d. observation noise samples in (1) are Gaussian, each with zero mean and unit variance which is probably the most well accepted and important example. The likelihood ratio test at sensor $i$ is equivalent to comparing the test statistic $T(y_i) = \frac{1}{N} \sum_{j=1}^{N} y_{ij} s_j$ to a threshold $\tau$. Under
\[
\frac{f(L - 2(M + 1), \beta)}{f(2M, \beta)} < \min \left\{ \frac{M}{M + 1}, \left( \frac{M + 1}{M} \right)^2 \frac{4}{5} \left( \frac{M}{M + 1} \right)^3 \right\} = \frac{4M}{5M + 1}.
\]

Minimize \(P_{tr}^r\) s.t. \(\min_i \max_j P_{D_i} \geq \beta\) and \(\max_i P_{F_i} \leq 1 - (1 - \alpha)^{1/M}\) (9)

Minimize \(P_{tr}^r\) s.t. \(r_i \in [0, r^*], l_i = l_{i-1} + r_{i-1} + r_i, l_1 \leq r_1, l_M + r_M \geq L\). (10)

\(H_0, T(y_i)\) follows a Gaussian distribution with zero mean and variance \(\frac{1}{N}\). Hence

\[
P_{F_i} = Pr(T(y_i) > \tau|H_0) = \frac{1}{2} \text{erfc}(\sqrt{\frac{N}{2}} \tau) (11)
\]

where \(\text{erfc}\) is the complementary error function defined as

\[
\text{erfc}(x) = \frac{2}{\pi} \int_x^\infty e^{-t^2} dt.
\]

Under \(H_1, T(y_i)\) follows a Gaussian distribution with mean \(\gamma Ad_i\) and variance \(\frac{1}{N}\). Hence

\[
P_{D_i} = Pr(T(y_i) > \tau|H_1) = \frac{1}{2} \text{erfc}(\sqrt{\frac{N}{2} (\tau - \gamma Ad_i)}).
\]

Given the constraint \(P_{D_i} = y\), we have \(\tau = \sqrt{\frac{N}{2}} \text{erfc}^{-1}(2y) + \gamma Ad_i\). Plugging \(\tau\) in (11) yields \(P_{F_i} = \frac{1}{2} \text{erfc}(\sqrt{\frac{N}{2} Ad_i^{-\gamma}})\). Hence we have an explicit expression for \(f(x, y)\) which is

\[
f(x, y) = \frac{1}{A} \text{erfc}(\sqrt{\frac{N}{2} A x^{-\gamma}}) + \sqrt{\frac{N}{2} A x^{-\gamma}}
\]

where \(0 \leq x < y\). Setting \(P_{F_i}\) equal to this last equation and solving for \(x\), \(f_y^{-1}(P_{F_i})\) can be computed accordingly as

\[
f_y^{-1}(P_{F_i}) = \left( \frac{1}{A} \sqrt{\frac{N}{2} } \text{erfc}^{-1}(2P_{F_i}) - \text{erfc}^{-1}(2y) \right)^{-1/\gamma}
\]

where \(0 < P_{F_i} < 1\).

In the simulations reported in this work, we assume \(A = 1.0, \gamma = 1, n = 2, N = 10, \alpha = 0.01, \beta = 0.9\) and \(P_0 = 1.0\). It is easy to verify that the conditions in Theorem 1 are satisfied here. For illustration purpose, we plot \(\frac{f(L - 2(M + 1), \beta)}{f(2M, \beta)}\) as a function of \(M\) for different \(L\) in Figure 1 along with the function \(\frac{4M}{5M + 1}\). From the plot we can see that the condition in (5) is satisfied for reasonably large \(M\). In Table I, we list the sensor locations and coverage radii generated by our algorithm, where \(L\) is set to be 5. From Table I, it can be easily seen that \(P_{tr}^r\) decreases as \(M\) increases, which is in agreement with our finding in Theorem 1. The false alarm probability \(P_{F}\) also decreases as \(M\) increases. Note we only list \(l_1\) here, the rest of the sensor locations can be obtained from the constraints in (10) easily from \(l_1\) and \(r\).

Next we compare the total transmission power and false alarm probability achieved by our approach with that achieved by other sensor placement schemes. One alternative is the “uniform sensor placement” studied in Theorem 1, where each sensor has the same coverage radius \(L/2M\) for detection probability \(\beta\) and sensor \(i\) is placed at \((2i - 1)L/2M\). Another alternative is “random sensor placement” where the sensor locations are generated independently according to the uniformly distribution in \((0, L)\). Once the sensor positions are fixed, one can have all the sensors employ a common threshold which is just enough to guarantee that \(P_{D_{min}} \geq \beta\). We call it “random placement with common thresholds”. This approach has been considered by previous researchers [7]. Further optimization is possible if the thresholds of the sensors are chosen in an optimal way. For given sensor positions, such optimization is generally very hard. Here we adopt a simpler approach where we start with sensor 1 and choose the threshold such that the origin is just within its coverage area for detection probability \(\beta\). Then we find the next sensor outside the coverage area of sensor 1 and choose the threshold such that the interval between this sensor and the boundary of the coverage area of sensor 1 is covered with detection probability \(\beta\). We do this until \((0, L)\) is covered with detection probability \(\beta\). We can perform a similar procedure that starts at the other end of the interval \((0, L)\) and works toward the origin. We try both procedures and choose the one with smaller total transmission power. We call such a scheme “random placement with adjusted thresholds”. Note in random placements, there is no guarantee that the false alarm probability requirements in (4) will be met.

In Figure 2, we compare the expected transmission power of our proposed algorithm, uniform sensor placement, and the two random sensor placements we discussed. We also show the false alarm probabilities at the fusion center for all four types of sensor placements in Figure 3. The performance of the random sensor placements is obtained by averaging \(10^6\) independent trials. In those two figures, \(L\) is set to be 15. The results show that the random sensor placement schemes perform poorly both in terms of expected transmission power and false alarm probability. The random sensor placement with adjusted thresholds has performance gain over random
placement with common thresholds. However, the power and false alarm probability of both schemes do not benefit from adding more sensors when compared to the other two schemes. The uniform sensor placement scheme provides the best false alarm probabilities. In Figure 2, our proposed algorithm offers 40% – 60% power savings compared to the uniform sensor placement scheme, which is quite significant. On the other hand, it can be seen that the uniform sensor placement captures a significant portion of power reduction from adding more sensors. Considering its simplicity and good false alarm probability performance, it is a very attractive sensor placement scheme.

A. Optimal sensor density for minimum power design

So far, we have shown that increasing the sensor density reduces the expected transmission power. Besides the transmission power, sensor nodes also consume power in taking measurements. If the measurement power is assumed constant, which appears reasonable, the total measurement power increases linearly with the number of sensors. The total power consumption under $H_0$ can be computed as

$$P = P^{tr} + P^m$$

$$= P^{tr} + MP_N$$

where $P^m$ denotes the total measurement power and $P_N$ denotes the power needed for a single sensor to obtain $N$ samples. The optimum $M$ which results in minimal total power consumption is finite and can be easily obtained once $P^{tr}$ is obtained as a function of $M$. In Figure 4, we show how the total power varies with the number of sensors. Here we assume $P_N = 0.002$ and find the optimal number of sensors is 14 in this case. When system parameters change, the optimal number of sensors can be obtained likewise.

B. Extensions

So far, we have focused on a one dimensional system. The analysis is more involved for systems for two or three dimensions. However, it appears that the total transmission power can still be reduced by adding more sensors, although the conditions for that to happen are generally different from the one dimensional case we have studied. To illustrate this, we provide an example of a two dimensional system, where
the emitter appears at an unknown location within a square whose four corner points are \((0, 0), (0, L), (L, 0),\) and \((L, L)\). The fusion center is placed at the origin \((0,0)\). We assume the fusion center uses an OR rule and place the \(M\) sensors at \(\left(\frac{2i}{\sqrt{M} - 1}, \frac{2j}{\sqrt{M} - 1}\right)\) where \(i = 0, \ldots, \sqrt{M} - 1\) and \(j = 0, \ldots, \sqrt{M} - 1\). Each sensor has the same coverage radius of \(\frac{L}{\sqrt{M} - 1}\) for detection probability \(\beta\). It is easy to verify that with such a configuration we have \(P_{D_{\min}} \geq \beta\). We plot the total transmission power and false alarm probability for such a uniform sensor placement in Figure 5 and Figure 6 respectively. Here we assume i.i.d. Gaussian observation noise and set \(L = 5\). The other parameters are the same as in the previous subsection. From these two plots, it can be seen that both \(P_{tr}\) and \(P_F\) decrease reasonably fast with \(M\).

V. SUMMARY

In this work, we studied the design of sensor networks for detecting a radio signal with unknown emitter location. The design objective is to minimize the system power consumption subject to detection performance constraints. Toward such a goal, we adopted an emitter-location-independent detection approach and formulated an optimization problem. We showed that the expected transmission power consumption decreases with an increasing number of deployed sensors under certain conditions if proper sensor placement and fusion is employed. We illustrate that this is not generally true for random sensor placement. An iterative algorithm is proposed to find optimal sensor placements and thresholds for signal detection. Design examples are provided along with numerical results, which show that our proposed algorithm generates system configurations with power advantages. Finally, we discuss how to obtain the optimum number of sensors if sensor measurement power is also considered.

REFERENCES