On well-covered triangulations: Part II

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\textbf{A B S T R A C T}

A graph \(G\) is said to be \textit{well-covered} if every maximal independent set of vertices has the same cardinality. A planar (simple) graph in which each face is a triangle is called a \textit{triangulation}. It was proved in an earlier paper [A. Finbow, B. Hartnell, R. Nowakowski, M. Plummer, On well-covered triangulations: Part I, Discrete Appl. Math., 132, 2004, 97–108] that there are no 5-connected planar well-covered triangulations. It is the aim of the present paper to completely determine the 4-connected well-covered triangulations containing two adjacent vertices of degree 4. In a subsequent paper [A. Finbow, B. Hartnell, R. Nowakowski, M. Plummer, On well-covered triangulations: Part III (submitted for publication)], we show that every 4-connected well-covered triangulation contains two adjacent vertices of degree 4 and hence complete the task of characterizing all 4-connected well-covered planar triangulations. There turn out to be only four such graphs. This stands in stark contrast to the fact that there are infinitely many 3-connected well-covered planar triangulations.

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1. Introduction

In 1969, the fourth author first proposed the study of graphs in which each maximal independent set of vertices has the same size and suggested that the name \textit{well-covered} be applied to them [13]. Although it is now well-known that the independent set problem is \textit{NP}-complete for graphs in general (cf. Karp [9]), for certain interesting subfamilies of graphs, such as those called \textit{claw-free}, the problem becomes polynomially solvable (cf. Minty [12] and Sbihi [16]). Clearly, the independent set problem has a polynomial solution for the class of well-covered graphs, but how does one recognize this class? It was shown independently by Chvátal and Slater [3] and by Sankaranarayana and Stewart [15] that the recognition problem for well-covered graphs is \textit{co-NP}-complete. In contrast, if the graphs are claw-free, then the recognition problem becomes polynomial. (See Tankus and Tarsi [17,18].) For more comprehensive surveys of well-covered graphs, see Plummer [14] and more recently, Hartnell [7].

A widely studied subclass of planar graphs are those which are maximal planar and which are commonly called (planar) triangulations. Clearly, any triangulation (larger than a single triangle) must have vertex connectivity 3, 4 or 5. Lebesgue [11], Kotzig [10], Borodin [1] and Jendrol' [8] have extensively investigated what kind of configurations must always exist in any triangulation. In an earlier paper [4] these results were used to prove that there is no 5-connected planar well-covered triangulation. In the present paper all 4-connected planar well-covered triangulations containing two adjacent vertices of degree 4 are determined. In another paper [5] the authors show that all 4-connected planar well-covered triangulations contain two adjacent vertices of degree 4. Taken together, these two works allow us to completely determine all 4-connected planar triangulations. There are only four such graphs. (See graphs \(R_6\), \(R_7\), \(R_8\) and \(R_{12}\) in Fig. 2.1.) Interestingly, all of these

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graphs were known to the authors some years before they were able to accomplish the much more difficult task of showing that they are indeed the only such graphs. The finiteness of this class also contrasts sharply with the fact that there are known to be infinitely many 3-connected well-covered planar triangulations.

As the complete proof of our main result is quite long, let us present an outline of how we shall proceed. Proposition 2.2 deals with properties of induced 4-cycles in general 4-connected planar triangulations. The remaining results in Section 2 all deal with properties of induced 4-cycles in such graphs which are, in addition, well-covered.

In Section 3 we introduce the concept of a BW-configuration and prove a so-called Extension Lemma based on this idea. BW-configurations and the Extension Lemma, although somewhat technical in nature, serve to eliminate a number of cases which arise in Section 4 and also in a subsequent paper [5]. Hence we deal with them in their own section here. We also point out that this concept applies to general (not necessarily planar) well-covered graphs and hence may prove of assistance in future studies of other families of well-covered graphs.

In Section 4, we apply the aforementioned results to first determine those graphs which contain the induced subgraph $Q_3$ and finally those which contain a second induced subgraph $Q_2$. (The subgraphs $Q_3$ and $Q_2$ are shown in Fig. 2.1.) This then quickly leads to Theorem 4.3, the main result of the paper.

We will adopt the following notation and terminology throughout this paper. In this paper all graphs will be finite and simple. If $v$ is a vertex of a graph $G$, $\alpha(G)$ will denote the cardinality of a maximum independent set in $V(G)$ and $N[v]$ will denote the closed neighborhood of the vertex $v$, namely, $N[v] = N(v) \cup \{v\}$. For $n \geq 2$, $a_1a_2 \ldots a_n$ will denote an n-cycle $C$ with vertex set $\{a_1, a_2, \ldots, a_n\}$ and with edge set $\{a_1a_2, a_2a_3, \ldots, a_{n-1}a_n, a_na_1\}$. As generally accepted, by the term block we will always mean a maximal 2-connected subgraph.

Clearly a graph is well-covered if and only if all its components are. Therefore we shall assume all graphs are connected, unless otherwise specified.

2. Preliminary results

Our first lemma combines several elementary results which will be used repeatedly throughout the remainder of this paper.

Lemma 2.1. Let $G$ be a connected well-covered graph and $x \in V(G)$.
(a) Then $G - N[x]$ is also well-covered.
(b) If $|V(G)| \geq 2$ and $G - x$ is well-covered, then $\alpha(G - x) = \alpha(G)$. 

Fig. 2.1. Eight important graphs.
(c) If \( V(G) = E \cup F, E \cap F = \{x\} \), and there are no edges between \( E - \{x\} \) and \( F - \{x\} \), then either \( G[F] \) or \( G[F - \{x\}] \) is well-covered.

(d) Then the vertex \( x \) cannot have two neighbors of degree 1.

**Proof.** Part (a) is Theorem 2.2 of [2].

To prove part (b) simply note that any maximal independent set in \( S \) which contains a neighbor of \( x \) in \( G \) is also a maximal independent set of \( G \).

Consider part (c). If \( x \) has no neighbor in \( E \), then \( G[F] \) is itself a union of components of \( G \) and we are done. Otherwise let \( y \) be a neighbor of \( x \) in \( E \). Then \( G - N[y] \) contains \( G[F - \{x\}] \) as a union of components. Hence \( G - N[y] \) is not well-covered and therefore \( G \) is not well-covered either.

Finally, to prove part (d), let \( u \) and \( v \) be two neighbors of \( x \) having degree 1 and let \( I \) be a maximal independent set in \( G \) containing \( x \). Then \( J = (I - \{x\}) \cup \{u, v\} \) is independent. But \( |J| < |I| \) and hence \( G \) is not well-covered.

As we begin our consideration of planar graphs, we remind the reader that by a classical result of Whitney [20,21] (see also [19,6] for two alternate proofs), every 3-connected planar graph has a unique embedding in the plane (or equivalently, on the sphere).

**Definition 2.1.** Let \( C \) be a cycle which appears as an induced subgraph of a fixed plane representation of a planar triangulation \( G \), and let \( v \) be a vertex of \( G \) that is not on \( C \). Then \( In(C, v) \) (respectively, \( In(C, -v) \)) is the subset of vertices in \( G - V(C) \) on the side of \( C \) containing \( v \) (respectively, not containing \( v \)). Note that \( In(C, -v) \) could be empty.

In this paper we shall repeatedly refer to the eight graphs shown in Fig. 2.1.

**Proposition 2.2.** Let \( C \) be an induced 4-cycle in a 4-connected planar triangulation \( G \), let \( v \) be a vertex of \( G \) that is not on \( C \) and let \( S \) be the subgraph induced by \( V(C) \cup In(C, v) \).

(a) If \( |In(C, v)| = 1 \), then \( S \) is isomorphic to \( Q_1 \).

(b) If \( |In(C, v)| = 2 \), then \( S \) is isomorphic to \( Q_2 \).

(c) If \( |In(C, v)| = 3 \), then \( S \) is isomorphic to \( Q_3 \) or to \( Q_4 \).

**Proof.** The proof of (a) is trivial.

Consider part (b). Since \( G \) is a triangulation, each edge of \( C \) is on an interior triangular face. If the two vertices in \( In(C, v) \) are \( v \) and \( x \), by the pigeonhole principle one of them, say \( x \), is adjacent to at least three vertices of \( C \), say, without loss of generality, \( a, b \) and \( c \). If \( x \) were adjacent to all four vertices of \( C \), then \( v \) would be inside a separating triangle, contradicting the 4-connectivity of \( G \). Now \( G[\{a, x, c, d\}] \) is isomorphic to a 4-cycle \( C' \) and \( In(C', v) = \{v\} \). Hence applying part (a), we see that \( G[S] \) is isomorphic to \( Q_2 \).

Finally, consider part (c). Since \( G \) is a triangulation, each edge of \( C \) is on an interior triangular face. If the interior vertices are \( x, y \) and \( z \), by the pigeonhole principle one of them, say \( x \), is adjacent to at least three vertices of \( C \). If it were adjacent to all four vertices of \( C \), then \( y \), for example, would be inside a separating triangle, a contradiction.

Assume without loss of generality, that \( x \) is adjacent to \( a, b \) and \( c \). Now the 4-cycle \( D = axcd \) has \( In(D, -b) = \{y, z\} \). Applying part (b), the graph induced by \( D \cup In(D, -b) \) is isomorphic to \( Q_2 \) and it follows that \( S \) is isomorphic to \( Q_2 \) or to \( Q_4 \), depending upon the orientation of \( Q_2 \) with respect to \( D \).

**Definition 2.2.** Let \( C \) be a cycle which appears as an induced subgraph of a planar triangulation \( G \), and let \( H \) be a subset of \( V(C) \). Then we say that \( C \) is accessible from \( H \) provided that \( V(C) \) is contained in \( N[I] \) for some independent set \( I \) in \( V(H) \). If \( C \) is not accessible from \( H \), we say that \( C \) is inaccessible from \( H \).

**Proposition 2.3.** Let \( C = abcd \) be an induced 4-cycle in a well-covered 4-connected planar triangulation \( G \), let \( v \) be a vertex of \( G \) that is not on \( C \) and let \( S \) be the subgraph induced by \( V(C) \cup In(C, v) \). If \( C \) is inaccessible from \( In(C, v) \), then \( S \) is isomorphic to \( Q_2 \) or to \( Q_3 \).

**Proof.** Assume first that no interior vertex is a common neighbor of three vertices of \( C \). Let \( x, y, u \) and \( v \) be the interior facial neighbors of the edges \( ab, cd, ad \) and \( bc \), respectively.

Now \( x, y, u \) and \( v \) are all distinct by our assumption (see Fig. 2.2), and since \( C \) is a subset of both \( N[x, y] \) and \( N[u, v] \), it follows that both edges \( xy \) and \( uv \) belong to \( G \). This contradicts the fact that \( G \) is planar.

Hence we may assume that at least one interior vertex, say \( x \), is adjacent to at least three vertices of \( C \), say \( a, b \) and \( c \). (See Fig. 2.3(i).) Note that \( x \) cannot be adjacent to \( d \), for then \( C \) would be accessible from \( In(C, v) \). Furthermore, \( x \) must be adjacent to each neighbor of \( d \) or else \( C \) would be accessible from \( In(C, v) \). But then by 4-connectivity, all the interior vertices of \( C \) other than \( x \) must be adjacent to both \( x \) and \( d \). (See Fig. 2.3(ii).)

Observe now that if there were more than two interior neighbors of vertex \( d \), then at least two of them must be independent by planarity. Let us denote such an independent pair by \( \{u, v\} \). Now let \( I \) be a maximal independent set in \( G \) containing \( b \) and \( d \) and let \( J = (I - \{d\}) \cup \{u, v\} \). Since \( J \) is independent and \(|J| = |I| + 1 \), this is a contradiction because \( G \) is well-covered.

Hence vertex \( d \) can have only one or two interior neighbors. In the first case \(|In(C, v)| = 2 \) and then by Proposition 2.2(b), \( S \) is isomorphic to \( Q_2 \) (see Fig. 2.3(iii)) whereas in the second case \( S \) is isomorphic to \( Q_3 \) (see Fig. 2.3(iv)).
Proposition 2.4. Let $C$ be a cycle of any length which is an induced subgraph of a plane well-covered graph $G$ and let $v$ be a vertex of $G$ that is not on $C$. If $C$ is accessible from $\text{In}(C, v)$ then $G[\text{In}(C, -v)]$ is well-covered.

**Proof.** Suppose $I$ is an independent set in $G[\text{In}(C, v)]$ which is adjacent to all the vertices of $C$. All components of $G[\text{In}(C, -v)]$ are well-covered and hence $G[\text{In}(C, -v)]$ itself is well-covered. ■

Corollary 2.5. Let $C$ be an induced 4-cycle in a 4-connected well-covered triangulation $G$, let $v$ be a vertex of $G$ that is not on $C$ and let $S$ be the subgraph induced by $V(C) \cup \text{In}(C, v)$. If $G[\text{In}(C, -v)]$ is not well-covered, then $S$ is isomorphic to either $Q_2$ or $Q_3$.

**Proof.** See Proposition 2.3. ■

Proposition 2.6. Let $C$ be an induced 4-cycle in a 4-connected well-covered triangulation $G$, let $v$ be a vertex of $G$ that is not on $C$ and let $S$ be the subgraph induced by $V(C) \cup \text{In}(C, v)$. Then $S$ is not isomorphic to any of the six graphs shown in Fig. 2.4.

**Proof.** Note that if $S$ is any of the six configurations shown in Fig. 2.4, then $G(\text{In}(C, v))$ is not well-covered. So by Corollary 2.5, $G(C \cup \text{In}(C, -v))$ is isomorphic to $Q_2$ or to $Q_3$ (note here that since $C$ is induced, $\text{In}(C, -v)$ is nonempty.) Let $p$ be any vertex in $\text{In}(C, -v)$. In Case (i) above, both the sets $\{x, z\}$ and $\{p, r, s\}$ are maximal independent in $G$, thus contradicting the fact that $G$ is well-covered. In Cases (ii) through (vi) above, both the sets $\{a, c\}$ and $\{b, r, d\}$ are maximal independent in $G$ again contradicting the fact that $G$ is well-covered. ■

Corollary 2.7. Let $C = abcd$ be an induced 4-cycle in a 4-connected well-covered triangulation $G$, let $v$ be a vertex of $G$ that is not on $C$ and let $S$ be the subgraph induced by $V(C) \cup \text{In}(C, v)$. If all the neighbors of $a$ that lie in $\text{In}(C, v)$ are adjacent to $c$, then $S$ must be isomorphic to either $Q_1$ or $Q_2$. (And therefore, $\text{In}(C, v) \subseteq N[x]$ for both $x = a$ and $x = c$.)

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Fig. 2.4. Six forbidden configurations.

Proof. Since $G$ is a 4-connected planar triangulation, $G[\text{In}(C, v)]$ must be isomorphic to a path and hence if the path contains more than two vertices, $G[V(C) \cup \text{In}(C, v)]$ must contain a subgraph isomorphic to graph $Q_4$ shown in Fig. 2.1. This however would contradict Proposition 2.6.

Proposition 2.8. Let $C = abcd$ be an induced 4-cycle in a 4-connected well-covered triangulation $G$ and let $v$ be a vertex of $G$ that is not on $C$. Then $|\text{In}(C, v)| \neq 4$.

Proof. Let $A$ be the subgraph induced by $V(C) \cup \text{In}(C, v)$ and suppose $|\text{In}(C, v)| = 4$. Assume first that no interior vertex is a common neighbor of three vertices of $C$. Let $x, y, u$ and $v$ be the interior facial neighbors of the edges $ab, cd, ad$ and $bc$ respectively (see Fig. 2.5(i)).

Now vertices $x, y, u$ and $v$ are all distinct by our assumption, and since $G$ is a triangulation, the edges $xv, vy, uy$ and $ux$ must belong to $G$. (See Fig. 2.5(ii)) Also since $G$ is a triangulation, one of the diagonals of the cycle $xvyu$ is also in $A$ and hence $A$ is isomorphic to graph (i) in Fig. 2.4, contradicting Proposition 2.6.

Next we assume that at least one interior vertex, say $x$, is adjacent to exactly three vertices of $C$, say, $a, b$ and $c$.

So the subgraph $D$ induced by $\{a, x, c, d\}$ is isomorphic to $C_4$ and $|\text{In}(D, -b)| = 3$. Hence applying Proposition 2.2(c), we see that the subgraph induced by $V(D) \cup \text{In}(D, -b)$ is isomorphic to $Q_3$ or to $Q_4$. By Proposition 2.6 it cannot be $Q_4$ and hence we see that in this case, $A$ is isomorphic to one of the graphs (ii), (iii) or (iv) in Fig. 2.6, depending upon the orientation of the $Q_3$ with respect to $D$, a contradiction of Proposition 2.6.

Fig. 2.5. No interior vertex is adjacent to three vertices on the cycle.
3. \textit{BW}-configurations

\textbf{Definition 3.1}. Let $H$ be an induced subgraph of a graph $G$. Suppose $x \in V(H)$ has neighbors $a_1$ and $a_2$ in $V(H)$ and that there is a set $J$ which is maximal independent in $H$ and two vertices $b_1, b_2 \in V(H)$ such that the following properties are satisfied:

(i) vertices $a_1$ and $a_2$ have no common neighbor in $V(G) - (V(H) \cup N_c(x))$, and
(ii) $\{a_1, a_2\} \subseteq J$, and
(iii) $(J - \{a_1, a_2\}) \cup \{x\}$ is maximal independent in $H$, and
(iv) for $i = 1$ and 2, $N_c(b_i) \subseteq H$, and $(J - \{a_i\}) \cup \{b_i\}$ is maximal independent in $H$.

Then the 7-tuple $(H, J, a_1, a_2, b_1, b_2, x)$ is called a \textit{BW}-configuration in $G$.

\textbf{Remark 3.1}. Note that in the context of Definition 3.1, $a_i b_i \in E(H)$ and $b_1 \neq b_2$. Also, if $v \in J - \{a_1, a_2\}$, then $v$ is not adjacent to $x$. Finally note that both $b_1$ and $b_2$ are adjacent to $x$. To see this, observe that by Definition 3.1(iv), for $i = 1, 2$, vertex $b_i$ is not adjacent to any vertex of $J - \{a_i\}$. By Definition 3.1(iii), vertex $b_i$ is adjacent to an element of $(J - \{a_1, a_2\}) \cup \{x\}$, so $b_i$ is adjacent to $x$.

\textbf{Lemma 3.1 (The Extension Lemma)}. Let $G$ be a well-covered graph and suppose that $G$ contains a \textit{BW}-configuration $(H, J, a_1, a_2, b_1, b_2, x)$. Then there exist distinct vertices $w_1, w_2$ and $z \in V(G) - [V(H) \cup N(J - \{a_1, a_2\})]$ such that

(a) $xz \in E(G)$, $w_i a_i \in E(G)$, $w_i z \in E(G)$ and $a_i z \notin E(G)$, for $i = 1$ and 2.
(b) $w_1 a_2 \notin E(G)$, $w_2 a_1 \notin E(G)$, and $x w_i \notin E(G)$, for $i = 1$ and 2.

(See Fig. 3.1.)

\textbf{Proof}. For all vertices $g \in V(G)$ let $N'(g) = N(g) \cap (V(G) - V(H))$.

Let $I$ be a maximal independent set in $G$ containing $J$ such that $A = N'(x) \cap I$ is of minimum cardinality. Then by Definition 3.1(iv) above, $(I - \{a_i\}) \cup \{b_i\}$ is independent in $G$, for $i = 1$ and $i = 2$, and hence is maximal in $G$, since $G$ is well-covered. Since $a_1$ and $a_2$ share no neighbor outside $V(H) \cup N(x)$ by Definition 3.1(i), we must have

\[ [N'(a_1) \cup N'(a_2)] - [N'(x) \cap N'(a_1) \cap N'(a_2)] \subseteq N(I - \{a_1, a_2\}). \tag{3.1} \]

Now $K = (I - N(x)) \cup \{x\}$ is independent. If $K$ were a maximal independent set in $G$, then $|K| < |I|$ contradicts the fact that $G$ is well-covered. Hence $K$ is not maximal in $G$.

Next we claim that

\[ K \cap V(H) = (J - \{a_1, a_2\}) \cup \{x\}. \tag{3.2} \]
Thus we have established our claim that $w$ is adjacent to $K$.

On the other hand, since $w \not\in N(x)$ and $x \not\in N(K)$, so $K \cap V(H) = (J - N(x)) \cup \{x\}$ is independent, so $J - \{a_1, a_2\}$ contains no neighbor of $x$. Therefore

$$J - \{a_1, a_2\} - N(x) = J - \{a_1, a_2\}. \quad (3.3)$$

On the other hand, $(J - \{a_1, a_2\}) - N(x) = J - \{a_1, a_2\} \cup N(x) = J - N(x)$. Thus $K \cap V(H) = (J - \{a_1, a_2\}) \cup \{x\}$, and (3.2) is proved.

So $K \cap V(H)$ is maximal independent in $H$ by Definition 3.1(iii).

Now let $w$ be a vertex in $V(G) - N[K] = (V(H) - N[K]) \cup ((V(G) - V(H)) - N[K]) = (V(G) - V(H)) - N[K]$, since $V(H) - N[K] = \emptyset$. Note $w$ lies outside of $H$. We claim that $w \not\in N[A]$.

To see this, assume that $w \not\in N[A]$. First observe that since $w \not\in N[K] \supseteq N[I - N(x)] = N[I - (A \cup \{a_1, a_2\})]$, then $w \not\in N[I - \{a_1, a_2\}]$ (the vertex set on the right-hand side of (3.1)).

On the other hand, since $w \not\in N[I]$ (recall $I$ is a maximal independent set in $G$), we must have $w \not\in N[{a_1, a_2}]$. Thus since $w$ lies outside of $H$, $w \not\in N'(a_1) \cup N'(a_2)$. Also because $x \in K$ we have $w \not\in N'(a_1) \cup N'(a_2)$. But then $w \not\in [N'(a_1) \cup N'(a_2)] - [N'(x) \cap N'(a_1) \cap N'(a_2)]$ (the vertex set on the left-hand side of (3.1)). This is a contradiction and thus we have established our claim that $w \not\in N[A]$.

Next we claim that $w$ must be adjacent to exactly one of $a_1$ and $a_2$. Suppose, first, that $w$ is adjacent to neither $a_1$ nor $a_2$. Consider the set $S = (I - (N(w) \cap A)) \cup \{w\}$. Note that $S \subseteq S$. First we show that $S$ is independent. To show this, it suffices to show that $w$ is not adjacent to any vertex of $I - (N(w) \cap A)$. By way of contradiction, choose a vertex $r \in I - (N(w) \cap A)$ and suppose $r$ is adjacent to $w$. Then $r \not\in K$ by definition of $w$. Also $r \not\in \{a_1, a_2\}$ by our assumption. Therefore $r \in I - K = I - ((I - N(w) \cup \{x\}) \subseteq I - N(w) \cap A)$. Then $r \in (N(x) \cap I) - \{a_1, a_2\} = N'(x) \cap I - (I - N(w) \cap A)$. This contradicts our choice of $r$ and hence establishes that the set $S$ is independent.

Now extend set $S = (I - (N(w) \cap A)) \cup \{w\}$ to a maximal independent set $L$ in $G$. We will obtain a contradiction to the minimum intersection assumption of $I$ with $N'(x)$ by showing that $|L \cap N'(x)| < |L' \cap N'(x)|$. Let $L' = L - S$ and first note that $L' \subseteq V(G) - V(H)$. Also $L = (I - (N(w) \cap A)) \cup (N(w) \cap A)$ and, since $G$ is well-covered, $|L| = |I|$. By the definition of $L$, $L \cap N'(x) = (S \cup L') \cap N'(x) = (I - (N(w) \cap A)) \cup \{w\} \cup L' \cap N'(x)$

$$= ((I \cap N'(x)) - (N(w) \cap A \cap N'(x))) \cup ((w) \cap N'(x)) \cup (L' \cap N'(x))$$

$$\subseteq ((I \cap N'(x)) - (N(w) \cap A)) \cup L',$$

since $\{w\} \cap N'(x) = \emptyset$ and since $A \subseteq N'(x)$. Furthermore, since $|L'| = |L - S|$, it follows that $|L'| = |N(w) \cap A|-1$. Now $L = S \cup L'$ and so $L \cap N'(x) = (S \cap N'(x)) \cup (L' \cap N'(x))$. As $S = (I - (N(w) \cap A)) \cup \{w\}$, and since $w \not\in N'(x)$, we have $S \cap N'(x) = ((I - (N(w) \cap A)) \cap N'(x)) - (N(w) \cap A \cap N'(x)).$ Now $A \cap N'(x) = A$, since $A \subseteq N'(x)$, and so it follows that $S \cap N'(x) = (I \cap N'(x)) - (N(w) \cap A)$. Therefore $L \cap N'(x) = [(I \cap N'(x)) - (N(w) \cap A)] \cup (L' \cap N'(x)) \subseteq [(I \cap N'(x)) - (N(w) \cap A)) \cup L'.$ Thus

$$|L \cap N'(x)| \leq |I \cap N'(x)| - |N(w) \cap A| + |L'|$$

$$= |I \cap N'(x)| - |N(w) \cap A| + |N(w) \cap A| - 1$$

$$= |I \cap N'(x)| - 1,$$

yielding the promised contradiction.

Hence $w$ is adjacent to at least one of $a_1$ or $a_2$. Now $w \not\in V(H)$, since $w \in V(G) - N(K)$, and moreover $x \in K$, by definition of $K$. Thus $w \not\in N(x)$ and hence $w \not\in V(G) - (V(H) \cup N(x))$. But by Definition 3.1(i), vertex $w \not\in N(a_1) \cap N(a_2)$. Therefore, $w$ is adjacent to exactly one of $a_1$ and $a_2$. A.S. Finbow et al. / Discrete Applied Mathematics 157 (2009) 2799–2817
Define, for \( i = 1 \) and \( 2 \): \( W_i = \{ w \in V(G) - N[K] | w a_i \in E(G) \} \). Then
\[
W_1 \cap W_2 = \emptyset,
\]
and
\[
W_1 \cup W_2 = V(G) - N[K].
\]

We partition set \( A \) into four subsets: \( A_1 = [A \cap N(W_1)] - N(W_2), A_2 = [A \cap N(W_2)] - N(W_1), A_0 = A - (N(W_1) \cup N(W_2)) \), and \( A_3 = A \cap N(W_1) \cap N(W_2) \). Recall that if \( w \in V(G) - N[K] \) and if \( w \notin I \), then \( w \) is adjacent to some of the vertices removed from \( I \) when forming the set \( K \), and to no other vertex of \( I \). These removed vertices form the set \( N(x) \cap I = (\cup_{i=0}^3 A_i) \cup \{ a_1, a_2 \} \).

Now we show that \( A_3 \neq \emptyset \). Assume, to the contrary, that \( A_3 = \emptyset \). Let \( M = (I - ((A_1 \cup A_2) \cup b_1) \) and let \( M_1 \) be a maximum independent set in the subgraph induced by \( W_1 \). Suppose \( m \in M_1 \). Then since \( m \in W_1 \), the only vertices of \( I \) to which \( m \) can be adjacent are in \( N(x) \cap I = (\cup_{i=0}^3 A_i) \cup \{ a_1, a_2 \} \) and specifically only to vertices of \( A_1 \cup \{ a_1 \} \). But \( M = (I - ((A_1 \cup A_2) \cup b_1) \) contains no vertex of \( A_1 \cup \{ a_1 \} \) and so \( m \) is not adjacent to any vertex belonging to \( M \). Thus \( M \cup M_1 \) is independent. So
\[
|M \cup M_1| = |M| + |M_1| = |I| - |A_1| + |B_1| + |M_1| = |I| - |A_1| + |A_1| = |I| - |A_1| + |A_0| - 1 + |A_2|.
\]

On the other hand, \( G \) is well-covered, so \( |M \cup M_1| \leq |I| \) and hence it follows that \( |M_1| \leq |A_1| \). Similarly, we may obtain a maximum independent set \( M_2 \) in \( W_2 \) with \( |M_2| \leq |A_2| \).

Now let \( K' \) be a maximal independent set in the subgraph of \( G \) induced by \( W_1 \cup W_2 \) and note that \( |K'| \leq |M_1| + |M_2| \leq |A_1| + |A_2| \). As \( K \cup K' \) is a maximal independent set in \( G \) then by definition of \( K \) and \( A \),
\[
|I| = |K \cup K'| = |K| + |K'| = |I| - |A| - 1 + |K'|
\]
\[
\leq |I| - |A| - 1 + |A_1| + |A_2| \leq |I| - 1,
\]
a contradiction. Hence \( A_3 \) is not empty.

Choose \( z \in A_3 \). There are vertices \( w_1 \in W_1 \) and \( w_2 \in W_2 \) with \( \{zw_1, zw_2\} \subseteq E(G) \) and moreover, \( zx \in E(G) \), since \( z \in A \). Furthermore, by the definition of \( K \), \( W_1 \) and \( W_2 \), it follows that \( \{z, w_1, w_2\} \subseteq V(G) - (V(H) \cup N[J - \{a_1, a_2\}) \). Thus part (a) is established.

Since \( a_1 \) and \( a_2 \) have no common neighbor in \( G - (V(H) \cup N(x)) \), part (b) follows. 

4. The main theorem

The proof of our main result (Theorem 4.3) is primarily achieved in two preliminary steps: first we deal with graphs having an induced subgraph isomorphic to \( Q_3 \) (Lemma 4.1); and then we generalize to graphs having an induced subgraph isomorphic to \( Q_2 \) (Lemma 4.2).

**Lemma 4.1.** Let \( G \) be a 4-connected planar well-covered triangulation which contains an induced subgraph which is isomorphic to \( Q_3 \). Then \( G \) is isomorphic to either \( R_8 \) or to \( R_{12} \). (See Fig. 2.1.)

**Proof.** Let \( H \) be an induced subgraph which is isomorphic to \( Q_2 \) and suppose its vertices are labeled as in Fig. 4.1.

**Case 1.** Suppose vertices \( b \) and \( d \) have a common neighbor outside \( H \).

Let \( w \) be such a common neighbor. Let \( C_1 \) be the 4-cycle \( bwdc \), let \( C_2 = bwdc \) and set \( G_1 = G[In(C_1, -f)] \) and \( G_2 = G[In(C_2, -f)] \). Since \( G \) is a triangulation, vertices \( d \) and \( w \) have a common neighbor, say \( t \), in \( G_1 \cup \{a\} \); but then \( G - N[f, t] \) has \( G_2 \) as a component, so \( G_2 \) must be well-covered. Similarly, \( w \) and \( d \) share a common neighbor, say \( s \), in \( G_2 \cup \{c\} \) and then \( G - N[e, s] \) has \( G_1 \) as a component and so \( G_1 \) is also well-covered.
Next, we claim that either $V(G_1)$ or $V(G_2)$ is empty. Suppose, to the contrary, both are nonempty. In this case there are maximal independent sets $J_1$ in $G_1$ containing a neighbor of $w$ and $J_2$ in $G_2$ containing a neighbor of $c$. Now $J_1 \cup J_2 \cup \{e\}$ is a maximal independent set in $G$ and hence:

$$\alpha(G) = \alpha(G_1) + \alpha(G_2) + 1. \quad (4.1)$$

On the other hand, let $I$ be a maximal independent set for $G$ containing vertices $a$ and $c$. Then since $G$ is well-covered, both $(I - \{a\}) \cup \{e\}$ and $(I - \{c\}) \cup \{f\}$ are maximal independent sets. If $w$ is not in $I$, then $I \cap V(G_i)$ is a maximal independent set in $G_i$ for both $i = 1$ and 2, and hence we see that $|I| = \alpha(G_1) + \alpha(G_2) + 2$, contradicting (4.1). Hence $w$ is in every maximal independent set of $G$ which contains $\{a, c\}$.

This now implies that vertex $a$ is adjacent to all of the neighbors of $w$ which lie in $G_1$ and vertex $c$ is adjacent to all of the neighbors of $w$ which lie in $G_2$. By maximal planarity and 4-connectivity, this in turn implies that for both $i = 1$ and 2, all vertices in $G_i$ are neighbors of $w$ and hence $G_1$ must be a path. Now also note that $\{a, c, w\}$ is a maximal independent set in $G$ and thus (4.1) implies that $\alpha(G_i) = 1$ so that $G_i$ is isomorphic to a path containing one or two vertices, for $i = 1$ and 2. It follows that $\{b\}$ is a maximal independent set in $G$, contradicting the fact that $G$ is well-covered. Hence at least one of the $G_i$’s must be empty.

If both are empty, then $G$ is isomorphic to $R_8$. Otherwise, without loss of generality, suppose that $G_1$ is nonempty. We observe that $\ln(G_1, f)$ is a graph with four vertices thus contradicting Proposition 2.8.

This completes the case when vertices $b$ and $d$ have a common neighbor outside $H$.

Case 2. Suppose $b$ and $d$ have no common neighbor outside $H$.

Let $p, r, s$ and $q$ be the exterior facial neighbors of the edges $ab, cd, ad$ and $bc$ respectively. We claim $p, r, s$ and $q$ are all distinct. Indeed, if $q = r, q = s, p = r$ or $p = s$, then $b$ and $d$ would have a common outside neighbor and if $p = q$, we have the separating 4-cycle $padc$ containing exactly four internal vertices $b, e, f$ and $g$, thus violating Proposition 2.8. Similarly, by symmetry, $s \neq r$ (see Fig. 4.2).

**Claim 1.** $qr \in E(G)$.

Let $I$ be a maximal independent set in $G$ containing $b$ and $d$. Now $(I - \{d\}) \cup \{g\}$ and $(I - \{b\}) \cup \{f\}$ are both independent and hence maximal, since $G$ is well-covered.

Since $b$ and $d$ share no neighbor outside $H$, this implies that $N[[b, d]] \cap (V(G) - V(H))$ is a subset of $N[I - \{b, d\}]$. Now $(I - \{b, d\}) \cup \{e\}$ is not maximal (since $G$ is well-covered) and the only vertex that can be missing from $N[(I - \{b, d\}) \cup \{e\}]$ is $c$. Hence $c \notin N[I - \{b, d\}]$ for any maximal independent set $I$ of $G$ containing $\{b, d\}$. We also claim that each neighbor of $c$ in $G - H$ is adjacent to either $b$ or $d$. If not, let $v$ be a neighbor of $c$ such that neither $vb$ nor $vd$ are edges in $G$. Now any maximal independent set $I^*$ in $G$ containing $\{b, v, d\}$ contradicts the statement we just proved for all maximal independent sets $I$ which contain $b$ and $d$.

So since $G$ is 4-connected, the only outside neighbors of $c$ are $q$ and $r$. But then since $G$ is a triangulation, $qr$ is in $E(G)$ and Claim 1 is proved.

**Claim 2.**

edge $pq$ is in $G$ if and only if the edge $rs$ is in $G$. \quad (4.2)

Suppose $rs \in E(G)$ and $pq \notin E(G)$. Suppose also that $ps \notin E(G)$. (See Fig. 4.3.) Consider the “outer” triangle based on the edge $as$. Let $v$ be its third vertex. Then $v \neq p, b, q$ by 4-connectivity. So $v$ is a “new” outer vertex. Suppose $q$ is adjacent to $v$. Consider the “outer” triangle based on $bq$. Let $w$ be its third vertex, then $w \notin \{p, a, v\}$ by 4-connectivity. So $w$ is also a “new”
Assume \( rs \in E(G) \) and that neither \( ps \) nor \( pq \) is in \( E(G) \).

Therefore vertices \( q \) and \( v \) are independent and \( G - N[\{q, v\}] \) has a component the subgraph induced by \( \{d, e, f, g\} \). But this subgraph is not well-covered, a contradiction. Hence if \( rs \in E(G) \), then \( pq \in E(G) \).

By symmetry, it is also true that if \( pq \in E(G) \), then \( rs \in E(G) \). This completes the proof of Claim 2.

Claim 3.

\[
\text{if } pq \in E(G) \text{ and if } rs \in E(G), \quad \text{then } ps \in E(G). \tag{4.3}
\]

To see this, suppose by way of contradiction that \( ps \notin E(G) \). Then \( G - N[\{p, s\}] \) has as a component the diamond consisting of \( fegc \) together with \( fg \) which is not well-covered. But this subgraph is not well-covered, a contradiction. Hence if \( rs \in E(G) \), then \( pq \in E(G) \).

By symmetry, it is also true that if \( pq \in E(G) \), then \( rs \in E(G) \). This completes the proof of Claim 2.

Claim 4. \( ps \in E(G) \).
Suppose now that $p$ is not adjacent to $s$. If $p$ is adjacent to $q$, then $s$ is adjacent to $r$ by (4.2) and (4.3) implies that $p$ is adjacent to $s$, a contradiction. So $p$ is not adjacent to $q$. Therefore, again by (4.2), $r$ is not adjacent to $s$. So $G - N[[p, s]]$ has as a component a graph of the form shown in Fig. 4.5.

If both $K_1$ and $K_2$ are nonempty, choose a vertex $k_i \in K_i$ for $i = 1, 2$ such that $k_1$ is adjacent to $q$ and $k_2$ is adjacent to $r$. Then $G - N[[p, s, k_1, k_2]]$ has as a component the diamond consisting of $fecg$ together with $fg$ which is not well-covered, a contradiction.

So we may assume that at least one of $K_1$ and $K_2$ is empty. Without loss of generality, suppose that $K_2 = \emptyset$. This implies that $s$ is adjacent to all vertices of $G_2$. On the one hand, if $G_2 = \emptyset$, then $C_2 \cup G_2$ consists of a 4-cycle $rdsu$ together with the edge $ud$. (Recall that $r$ is not adjacent to $s$.) On the other hand, if $G_2 \neq \emptyset$, then since all vertices of $G_2$ are adjacent to $s$, we have by Corollary 2.7 that $G[V(G_2) \cup V(G_2)]$ is isomorphic to either $Q_3$ or to $Q_2$. We now show that the first option is impossible.

Indeed suppose $G_2 = \emptyset$. Then $u$ is adjacent to $d$ and so $a$ is not adjacent to $u$ by 4-connectivity. Thus there exists a vertex $v$ such that $uv$ is an exterior triangle. (See Fig. 4.6.)

However, then $G - N[[q, v]]$ has a component isomorphic to a diamond consisting of the 4-cycle $edgf$ together with $eg$. Hence $G$ is not well-covered.

So $G_2 \neq \emptyset$. Similarly, $G_1 \neq \emptyset$.

Thus there exist vertices $x_1 \in V(G_1)$, $i = 1, 2$, which form triangles based on edges $hq$ and $dr$ respectively. Note that $x_1 \neq u \neq x_2$. Moreover, if there exists a vertex $z$ such that either $zas$ forms an exterior triangle or $zap$ forms an exterior triangle, then $G - N[[x_1, x_2, z]]$ has as one component the diamond consisting of the 4-cycle $egcf$ together with $fg$ and $G$ is not well-covered contrary to the initial assumption on $G$. So there is no such vertex $z$ and hence we may assume that $a$ is adjacent to $u$.

Note that $u$ is not adjacent to $b$ nor to $d$, or else we would have a separating triangle. Now recalling that each $G[V(G_1) \cup V(G_1)]$ is isomorphic to either $Q_3$ or to $Q_2$, it follows that $u$, $b$, $d$ is a maximal independent set in $G$. But $c, e, x_1, x_2$ is also independent in $G$ and hence $G$ is not well-covered, a contradiction. This completes the proof of Claim 4.

So henceforth we may suppose that $p$ is adjacent to $s$.

Now suppose that $V(G_2)$ is empty. If $pq$ is not in $G$, then the graph $G - N[[p, q]]$ has as a component the diamond consisting of the 4-cycle $dgfe$ together with $eg$ and $G$ is not well-covered, a contradiction. Hence $pq$ is in $G$ and hence by (4.2), the edge $rs$ is also in $G$ and we see that $G$ is isomorphic to $R_{12}$. Similarly, if $V(G_1) = \emptyset$, $G$ is also isomorphic to $R_{12}$.

So finally assume that both $G_1$ and $G_2$ are nonempty. This implies that neither $ud$ nor $ub$ is in $G$. Now if all the neighbors of $u$ that lie in $G_1$ are adjacent to $b$ and all the neighbors of $u$ that lie in $G_2$ are adjacent to $d$, then by the 4-connectivity of $G$, all
vertices of \( G_1 \cup G_2 \) are adjacent to \( u \) and thus \( \{u, d, b\} \) is a maximal independent set in \( G \). But \( \{r, s, g, b\} \) is also independent in \( G \), again contradicting the well-covered property possessed by \( G \). Hence without loss of generality, we may assume that vertex \( u \) has a neighbor \( w \) in \( G_1 \) such that edge \( uw \) does not belong to \( G \).

Let \( I \) be a maximal independent set in \( G \) containing \( \{u, b, d\} \). Then \( (I - \{b\}) \cup \{f\} \) is independent and hence maximal by the well-covered property of \( G \). Hence all the neighbors of \( b \) in \( G_1 \) are in \( N[I] \), where \( f = I \cap V(G_1) \). But then \( G - N[I \cup \{r, s\}] \) has as a component the diamond consisting of \( begf \) together with \( ef \) and yet again \( G \) is not well-covered. Hence the assumption that both \( G_1 \)'s are nonempty leads to a contradiction and the proof is complete. \( \blacksquare \)

**Lemma 4.2.** Let \( G \) be a 4-connected planar well-covered triangulation which contains an induced subgraph isomorphic to \( Q_2 \). Then \( G \) is isomorphic to \( R_7, R_8 \) or \( R_{12} \). (See Fig. 2.1.)

**Proof.** By Lemma 4.1 we may assume that \( G \) contains no induced subgraph isomorphic to \( Q_3 \). So combining Proposition 2.2(c), 2.6 and 2.8, we may assume that:

\[
\text{if } C \text{ is an induced 4-cycle of } G \text{ and if } v \in V(G) - V(C) \text{ then } |\text{int}(C, v)| \neq 3, 4. \tag{4.4}
\]

Let \( H \) be an induced subgraph which is isomorphic to \( Q_2 \) labeled as in Fig. 4.7.

For each vertex \( v \in H \), define \( N'(v) = N(v) - V(H) \).

**Case 1.** Suppose \( b \) and \( d \) have a common neighbor outside \( H \).

Let \( w \) be such a neighbor and let \( C_1 \) be the 4-cycle \( buda \) and let \( C_2 = bdwc \) and set \( G_1 = G[\text{int}(C_1, -f)] \) and \( G_2 = G[\text{int}(C_2, -f)] \).

Note that if both \( V(G_1) \) and \( V(G_2) \) are empty, then the diagonals \( aw \) of \( C_1 \) and \( cw \) of \( C_2 \) are both in \( G \) and then \( G \) is isomorphic to \( R_7 \). If exactly one of \( V(G_1) \) and \( V(G_2) \) is empty, say, without loss of generality, that \( V(G_1) \) is empty, then the diagonal \( aw \) is in \( G \) and thus \( G \) contains an induced subgraph which is isomorphic to \( Q_3 \), a contradiction. Hence we may assume that both \( V(G_1) \) and \( V(G_2) \) are nonempty.

Now consider the 4-cycle \( D = bcdw \). If all the neighbors of \( w \) that lie in \( G_1 \) are adjacent to \( a \), then by Corollary 2.7, the subgraph induced by \( V(C_1) \cup V(G_1) \) is isomorphic to \( Q_1 \) or to \( Q_2 \). In the first case, \( \text{int}(D, a) \) contains four vertices in violation of Proposition 2.8, and in the second case, the subgraph induced by \( V(D) \cup \text{int}(D, a) \) is isomorphic to graph (vi) of Fig. 2.4, which violates Proposition 2.6. Hence there is a vertex \( u \) in \( G_1 \) which is a neighbor of \( w \), but is not adjacent to vertex \( a \). Similarly, there is a vertex \( v \) in \( G_2 \) which is a neighbor of \( w \), but which is not adjacent to vertex \( c \). (See Fig. 4.8.)

Let \( H_1 \) be the subgraph induced by \( V(G_1) \cup \{a\} \) and let \( H_2 \) be the subgraph induced by \( V(G_2) \cup \{c\} \). Then \( G - N[\{a, u\}] \) has \( H_2 \) as a component, so \( H_2 \) is well-covered. Similarly, \( H_1 \) is well-covered. Furthermore, there are maximal independent sets \( j_1 \) in \( H_1 \) containing \( \{a, u\} \) and \( j_2 \) in \( H_2 \) containing \( \{v, c\} \). Then \( j_1 \cup j_2 \) is a maximal independent set in \( G \) and hence:

\[
\alpha(G) = \alpha(H_1) + \alpha(H_2). \tag{4.5}
\]
On the other hand, let \( r \) be the facial neighbor of edge \( dw \) in \( G_1 \). Then \( G - N[[e, r]] \) has \( G_2 \) as a component, so \( G_2 \) is well-covered. Similarly, \( G_1 \) is well-covered. Now let \( I_1 \) be a maximal independent set in \( G_1 \) containing vertex \( r \) and let \( I_2 \) be a maximal independent set in \( G_2 \). Then \( I_1 \cup I_2 \cup \{ e \} \) is a maximal independent set in \( G \) and hence:

\[
\alpha(G) = \alpha(G_1) + \alpha(G_2) + 1. \tag{4.6}
\]

**Lemma 2.1(b)** implies that \( \alpha(G_1) = \alpha(H_1) \) and \( \alpha(G_2) = \alpha(H_2) \) and then by (4.5) and (4.6) we obtain a contradiction. This settles Case 1.

**Case 2.** Suppose that \( b \) and \( d \) have no common neighbor other than \( a \) and \( c \).

Let \( p, r, s \) and \( q \) be the exterior facial neighbors of edges \( ab, cd, ad \) and \( bc \) respectively. (See Fig. 4.9.)

We claim that \( p, r, s \) and \( q \) are all distinct, for if \( q = r, q = s, p = r \) or \( p = s \), then \( b \) and \( d \) would have a common outside neighbor, while if \( p = q \), then there is a separating 4-cycle \( padc \) which separates the three vertices \( b, e \) and \( f \) in violation of (4.4). Similarly, \( r \neq s \).

**Claim 1.** Edges \( ps \) and \( qr \) are present in \( G \).

By way of contradiction, suppose that edge \( ps \) is not in \( G \) and let \( u \) be a neighbor of \( a \) which lies outside \( H \) and is different from \( p \) and \( s \). By 4-connectivity, \( u \) is not adjacent to either \( b \) or to \( d \). (Note \( b \) and \( d \) are independent since \( H \) is induced.) Let \( J \) be the independent set \( \{ u, b, d \} \).

We next claim that \( u \) is not adjacent to \( c \). By way of contradiction, suppose that \( u \) is adjacent to \( c \). Consider the induced 4-cycles \( C_1 = dauc, C_2 = fauc \) and \( C_3 = eauc \). Let \( G_i = G[\text{In}(C_i, r)] \) for \( i = 1, 2 \) and 3. Since \( \text{In}(C_i, r) \) contains at least three vertices and since \( G \) contains no induced \( Q_3 \) by assumption, it follows from **Corollary 2.5** that \( G_i \) is well-covered for \( i = 1, 2 \) and 3. Now on the one hand, **Lemma 2.1(b)** implies that \( \alpha(G_1) = \alpha(G_2) = \alpha(G_3) \), while on the other hand, if \( L \) is any maximal independent set in \( G_1 \), then \( L \cup \{ f \} \) is a maximal independent set in \( G_3 \) and thus \( \alpha(G_1) = \alpha(G_3) + 1 \) and we have a contradiction. Hence as claimed, \( u \) is not adjacent to \( c \).

Observe now that \( (G[V(H) \cup \{ u \}], f, b, d, e, f, c) \) is a BW-configuration in \( G \). So by **Lemma 3.1**, there exist vertices \( w_1, w_2, z_1 \) and \( w_2 \) as shown in Fig. 4.10.

Setting \( f'' = \{ b, d, z_c \} \), we further observe that \( (G[V(H) \cup \{ z_1 \}], f'', b, d, e, f, a) \) is a BW-configuration in \( G \). Hence by **Lemma 3.1**, there exist vertices \( w_3, w_4, z_0 \) in \( G - N[f'' - \{ a'_1, d'_2 \}] = G - N[z_1] \) as shown in Fig. 4.11. Note that \( \{ w_1, w_2, z_c \} \cap \{ w_3, w_4, z_0 \} = \emptyset \). So we have the configuration of Fig. 4.11.

Now consider the induced 4-cycles \( C_1 = bw_1z_c, C_2 = dw_2z_c, C_3 = bw_3z_a, C_4 = dw_4z_a \) and set \( G_i = G[\text{In}(C_i, r)] \), for \( i = 1, \ldots, 4 \). Since \( \text{In}(C_i, f) \) contains at least three vertices and since \( G \) contains no induced \( Q_3 \), it follows from **Corollary 2.5** that \( G_i \) is well-covered for \( i = 1, \ldots, 4 \). Note also that \( p, q, r \) and \( s \) lie in \( G_3, G_1, G_2 \) and \( G_4 \) respectively,
and so $V(G_i) \neq \emptyset$ for $i = 1, \ldots, 4$. Let $G_5$ be the subgraph of $G$ induced by $V(G) \setminus (V(G_1) \cup V(G_2) \cup V(G_3) \cup V(G_4) \cup V(H) \cup \{z_r, w_1, w_2, z_0, w_3, w_4\})$.

Next, for $i = 1, \ldots, 4$, choose $x_i$ in $G_i$ as follows: $x_1$ is the facial vertex for the edge $w_1z_r$, $x_2$ is the facial vertex for $w_1d$, $x_3$ is the facial vertex for $w_2z_0$, and $x_4$ is the facial vertex for $w_4d$. Now $L = \{e, x_1, x_2, x_3, x_4\}$ is independent and $G - N[L]$ contains $G_5$ which is well-covered since each component of $G_5$ is well-covered.

Observe that by extending $L$ to a maximal independent set $L'$ in $G$, we obtain:

$$\alpha(G) = \alpha(G_1) + \alpha(G_2) + \alpha(G_3) + \alpha(G_4) + \alpha(G_5) + 1. \quad (4.7)$$

The proof of Claim 1 will now be completed by exhibiting a maximal independent set in $G$ with cardinality differing from that obtained in Eq. (4.7). We distinguish two cases.

Case (a). Suppose there exist vertices $y_1 \in V(G_1) \cup V(G_2)$ and $y_2 \in V(G_3) \cup V(G_4)$ such that edges $y_1z_r$ and $y_2z_0$ are both in $G$ and such that none of the four edges $y_ib$ or $y_id, i = 1, 2$, is in $G$. Note that $y_1 \neq c$ by the definition of $G_1$ and $G_2$ and similarly $y_2 \neq a$.

In this case let $J'$ be a maximal independent set for $G$ containing the independent set $\{y_1, y_2, b, d\}$. Now $(J' - \{b\}) \cup \{e\}$ is also maximal independent in $G$. Thus $(J' \cap V(G_1))$ and $(J' \cap V(G_2))$ are maximal independent sets in $G_1$ and $G_2$ respectively. Furthermore, $N[J' \cap V(G_3)] \cup V(G_4) = V(G_3) - N[d]$. Similarly, $(J' - \{d\}) \cup \{f\}$ is maximal in $G$ and thus $(J' \cap V(G_2))$ and $(J' \cap V(G_4))$ are maximal in $G_2$ and $G_4$ respectively and $N[J' \cap V(G_3)] \cup V(G_4) = V(G_3) - N[b]$. We have $N[J' \cap V(G_3)] \supseteq (V(G_3) - N[d]) \cup (V(G_3) - N[b]) = V(G_3) - (N[d] \cap N[b]) = V(G_5)$, since $b$ and $d$ have no common neighbor. Thus $J' \cap V(G_5)$ is a maximal independent set in $G_5$.

Hence $J' = \bigcup_{i=1}^5 (J' \cap V(G)) \cup \{b, d\}$, and therefore $|J'| = \sum_{i=1}^5 \alpha(G_i) + 2$, a contradiction of (4.7). Thus Case (a) cannot occur.

Case (b). So without loss of generality, we may suppose that all vertices in $G_1$ adjacent to $z_r$ are also adjacent to $b$, and all vertices in $G_2$ adjacent to $z_0$ are also adjacent to $d$.

Now $z_r$ has neighbors in $G_1$, for otherwise $wbc$ would be a separating triangle, and similarly, $z_0$ has neighbors in $G_2$. Hence all such neighbors of $z_r$ in $G_1$ are also adjacent to $b$ and all such neighbors of $z_0$ in $G_2$ are also adjacent to $d$.

Thus by Corollary 2.7, $G[V(G_1) \cup V(C_1)]$ is isomorphic to either $Q_1$ or $Q_2$, and if isomorphic to $Q_2$, the interior path of length three is suspended between vertices $w_1$ and $c$. Similarly, $G[V(G_2) \cup V(C_2)]$ is isomorphic to either $Q_2$ or $Q_3$, and if isomorphic to $Q_2$, the interior path of length three is suspended between vertices $w_2$ and $c$.

Now choose a facial vertex $y_3$ in $G_3$ for the edge $w_3z_0$ and a facial vertex $y_4$ in $G_4$ for $dw_4$. Now let $J''$ be a maximal independent set for $G$ containing $\{y_3, y_4, e, z_r\}$. Then $J'' \cap V(G_i)$ is a maximal independent set in $G_i$, for $i = 3$ and $4$, since $\{e, y_3\}$ prevents any vertices of $C_3$ from being in $J''$ and $\{e, y_4, y_4\}$ prevents any vertices of $C_4$ from being in $J''$. Furthermore, $J'' \cap V(G_3)$ is a maximal independent set for $G_3 - N[z_r]$ since the presence of $\{y_3, y_4, e, z_r\}$ as a subset of $J''$ prevents any vertex in $\{w_1, b, w_3, z_0, w_4, d, w_2\}$ from being in $J''$.

Thus $J'' = \{e, z_r\} \cup (J'' \cap V(G_3)) \cup (J'' \cap V(G_4)) \cup (J'' \cap V(G_5))$ and this is a disjoint union. Moreover, $|J'' \cap V(G_5)| \leq \alpha(G_5)$. Thus since $\alpha(G_1) = \alpha(G_2) = 1$,

$$|J''| = 2 + \alpha(G_3) + \alpha(G_4) + \alpha(G_5) = \alpha(G_1) + \alpha(G_2) + \alpha(G_3) + \alpha(G_4) + \alpha(G_5)$$

and again (4.7) is contradicted.

Hence $ps$ is in $G$. Similarly, $qr$ is in $G$ and Case (b) (and hence Claim 1) is established.

To continue, we first note that if the edge $pq$ is in $G$ and $rs$ is not, then $G - N[r, s]$ contains as a component the non-well-covered graph generated by $b, e$ and $f$. Furthermore, if both $pq$ and $rs$ are in $G$, then $G - N[s]$ contains as a component a
subgraph $S$ generated by the disjoint sets $E$ and $F$ where $F = \{q, c, b, e, f\}$ and where the only edges between $E$ and $F$ have $q$ as an endvertex. Since the subgraphs generated by both $F$ and $F - \{q\}$ are not well-covered, by Lemma 2.1(c) the subgraph $S$ is also not well-covered, contradicting the fact that $G$ is well-covered and hence neither the edge $pq$ nor the edge $rs$ belongs to $G$.

Next, let $u$ be the outside facial neighbor for the edge $qr$ and let $v$ be the outside facial neighbor for $qs$. We claim that $u$ and $v$ are both “new” vertices. Indeed, if $u = d$ or $u = b$, we get a separating triangle. Moreover, if $u = s$ then $rs$ is in $G$ and if $u = p$, $pq$ is in $G$. But we just proved above that neither of these two edges belongs to $G$. Similarly, vertex $v$ is “new”.

Claim 2. $u \neq v$.

Indeed, let us assume $u = v$. (See Fig. 4.12.)

Consider the 4-cycle $C_1 = uqbq$. If $ub$ is in $G$, and if $u$ is adjacent to $d$, then $G - N[u]$ contains as a component the diamond consisting of the 4-cycle $afce$ together with $ef$ which is not well-covered and we have a contradiction. On the other hand, if $u$ is not adjacent to $d$, $G - N[u]$ contains a subgraph $S$ which consists of two subgraphs $E$ and $F$ where $F$ is the subgraph generated by $\{a, c, d, e, f\}$ and where the only edges between $E$ and $F$ have vertex $d$ as an endvertex. Since both $F$ and $F - \{d\}$ are not well-covered, then by Lemma 2.1(c), $S$ is also not well-covered contradicting the fact that $G$ is well-covered. Thus $ub$ is not in $G$.

Suppose now that there is an external neighbor $x$ of $b$ such that edge $ux$ is not in $G$. If $u$ is adjacent to $d$, then $G - N[\{u, x\}]$ contains as a component the subgraph generated by $\{a, c, e, f\}$ which is not well-covered. If $u$ is not adjacent to $d$, then $G - N[\{u, x\}]$ contains a subgraph $S$ which consists of two subgraphs $E$ and $F$ having only the vertex $d$ in common where $F$ is the subgraph generated by $\{a, c, d, e, f\}$. Again, since both $F$ and $F - \{d\}$ are not well-covered, then by Lemma 2.1(c), $S$ is also not well-covered contradicting the fact that $G$ is well-covered. Thus $u$ is adjacent to every vertex in $N'(b)$. Hence by Corollary 2.7, we may conclude that the subgraph induced by $V(C_1) \cup In(C_1, -f)$ is isomorphic to either $Q_1$ or $Q_2$.

Similarly, considering the induced 4-cycle $urds$, we may conclude that the subgraph induced by $C_2 \cup In(C_2, -f)$ is isomorphic to either $Q_1$ or $Q_2$.

Now let $y$ be a vertex in $In(C_1, -f)$ and let $z$ be a vertex in $In(C_2, -f)$. Then both $\{u, a, c\}$ and $\{y, z, a, c\}$ are maximal independent sets in $G$ contrary to the fact that $G$ is well-covered. Hence $u \neq v$. (See Fig. 4.13.) So Claim 2 is proved.

Claim 3. The edge $uv$ is in $G$.

Suppose, to the contrary, that $uv \notin E(G)$ and assume that $b$ has a neighbor $x$ which is adjacent to neither $u$ nor $v$ (note $x = u$ or $x = u$ are possible). Then, if at least one of $x$, $u$ or $v$ is adjacent to $d$, $G - N[\{u, v, x\}]$ generates a diamond subgraph consisting of the 4-cycle $afce$ together with $ef$ which is not well-covered, a contradiction to the fact that $G$ is well-covered. On
the other hand, if none of $x$, $u$ or $v$ is adjacent to $d$, then $G - N[[u, v, x]]$ contains as a component the subgraph $S$ which consists of two subgraphs $E$ and $F$ which have only the vertex $d$ in common and where $F$ is the subgraph induced by $\{a, c, d, e, f\}$. Since both $F$ and $F - \{d\}$ are not well-covered, again by Lemma 2.1(c), $S$ is also not well-covered, contradicting once more our assumption that $G$ is well-covered.

So $ub$ is not in $G$ and similarly, none of $ud$, $vb$ or $vd$ is in $G$. Moreover,

\[
each \text{outside neighbor of } b \text{ or } d \text{ is itself a neighbor of one of } u \text{ or } v. \tag{4.8}
\]

Next note that $v$ cannot be adjacent to both $q$ and $r$ by 4-connectivity, so that without loss of generality we may assume that $v$ is not adjacent to $r$. This implies that $v$ cannot be adjacent to all outside neighbors of $b$, for otherwise $G - N[[r, v]]$ has as a non-well-covered component the induced diamond consisting of $f e b a$ together with $a e$.

Let $z_1$ be the outside facial neighbor of $qb$. (See Fig. 4.14.)

Note that the fact that none of $ub$, $vb$ or $pq$ is in $G$, together with the 4-connectivity of $G$, combine to show that $z_1$ is a “new” vertex. Now $vz_1$ cannot be in $G$, for if it were, then there must be an outside neighbor of $b$, say $x$, which is not adjacent to $v$ and then by planarity, $ux$ would also not be in $G$, violating (4.8). Thus, by (4.8), $uz_1$ is in $G$. Since planarity now prevents the edge $vq$ from being in $G$, we can repeat the argument in the preceding paragraph to obtain in $G$ an outside facial neighbor $z_4$ of $rd$ along with an edge $uz_4$ and such that $vz_4$ cannot be in $G$. Observe that the 4-connectivity of $G$ guarantees that $z_1 \neq z_4$. (See Fig. 4.15.)

Next we claim that the edge $z_1 z_4$ must be in $G$. If not, then $\{v, z_1, z_4\}$ is independent and $G - N[[v, z_1, z_4]]$ contains the diamond component consisting of $afce$ together with $ef$ which is not well-covered, a contradiction.

By symmetry we can repeat the arguments in the preceding paragraphs to obtain in $G$ outside facial neighbors $z_2$ of $bp$ and $z_3$ of $sd$ together with the edges $vz_2$, $vz_3$ and $z_2 z_3$. Moreover, neither $uz_2$ nor $uz_3$ can belong to $G$ and finally $z_2 \neq z_3$. Observe that the adjacency behavior with respect to $u$ and $v$ ensures that $\{z_1, z_4\} \cap \{z_2, z_3\} = \emptyset$. (See Fig. 4.16.)

Planarity, together with (4.8), dictates that $b$ and $d$ have no additional neighbors (i.e., neighbors not pictured in Fig. 4.16). The fact that $G$ is a triangulation, together with the fact that $b$ and $d$ have no common neighbors, forces edges $z_1 z_2$ and $z_3 z_4$ to belong to $G$. Let $C = z_1 z_2 z_3 z_4$. Now both $\{r, e, p\}$ and $\{u, v, a, c\}$ are maximal independent sets in $\ln(C, f)$ showing that $G(\ln(C, f))$ is not well-covered. Hence by Corollary 2.5, the graph induced by $C \cup \ln(C, -f)$ is isomorphic to $Q_3$ or to $Q_3$, but then both $\{z_1, z_3, r, e, p\}$ and $\{z_1, z_3, a, c\}$ are maximal independent sets in $G$ showing that $G$ is not well-covered, a contradiction. Thus $uv$ must belong to $G$. This completes the proof of Claim 3.

Claim 4. None of the edges $vq$, $vr$, $up$ or $us$ is in $G$. 

![Fig. 4.14](image1)

Fig. 4.14. Assume $z_1 v \in E(G)$.

![Fig. 4.14](image2)

Fig. 4.15. The basic configuration in the midst of Claim 3.
By symmetry it suffices to prove that $vq$ is not in $G$. So let us suppose that $vq$ belongs to $G$. (See Fig. 4.17.) Then by 4-connectivity, $vr$ is not in $G$ and this implies that $v$ cannot be adjacent to every outside neighbor of $b$, for otherwise $G - N([r, v])$ has as a non-well-covered component the diamond consisting of $afeb$ together with $ae$. So let $x$ be an outside neighbor of $b$ that is not adjacent to $v$.

Now we claim that $v$ is adjacent to every outside neighbor of $r$. Indeed, if $z$ were an outside neighbor of $r$ which were not adjacent to $v$, then $G - N([v, z, x])$ contains a component with a cutvertex $d$ and two blocks $E$ and $F$ which share only vertex $d$ and such that $F$ is the subgraph generated by $\{a, c, e, f, d\}$ which is not well-covered.

Note that $vb$ is not in $G$ by 4-connectivity and in the component of $G - N([b, v])$ which contains $d$, vertex $d$ has two neighbors, (namely $r$ and $f$, of degree 1). Hence by Lemma 2.1(d), this component (and therefore $G$) is not well-covered. This contradiction establishes that none of the edges $vq, vr, up$ or $us$ is present in $G$. So Claim 4 is proved.

Claim 5. None of the edges $vd, vb, ud$ or $ub$ is in $G$ either.

Again by symmetry it will suffice to prove that $vb$ is not in $G$, so suppose $vb$ is in $G$. Then $vd$ is not in $G$, since $b$ and $d$ have no common neighbors.

Let $w$ be an outside facial neighbor of $vs$ and note that since both $us$ and $rs$ are not in $G$, $w$ is a “new” vertex. (See Fig. 4.18.) Then $G - N([q, w])$ contains as a component a subgraph $S$ where $S$ consists of two blocks $E$ and $F$ sharing only a cutvertex $d$ and where $F$ is the subgraph generated by $\{a, p, e, f, d\}$. Both $F$ and $F - \{d\}$ are non-well-covered and so by Lemma 2.1(c), $S$ is also not well-covered, contradicting the fact that $G$ is well-covered. This contradiction, together with symmetry, establishes that none of the edges $vd, vb, ud$ or $ub$ is present in $G$. Thus Claim 5 is proved.

Finally, by the non-adjacencies established in Claim 5 there are “new” outside facial neighbors $y_1, y_2, z_1$ and $z_2$ of edges $ds, bp, rd$ and $bq$ respectively where we note that for at least one of the values of $i = 1, 2$, it is possible that $y_i = z_i$. (See Fig. 4.19.)

To complete the proof, it suffices to establish the following claim, for if all the edges $wy_i$ and $vz_i$, $i = 1, 2$, are in $G$, then by planarity we would have $y_i = z_i$ for $i = 1, 2$. This, in turn, would imply that $G - N([z_1, z_2])$ contains as a component the “diamond” consisting of $ace$ together with $ef$. But this subgraph is not well-covered and hence neither is $G$, a contradiction.

Claim 6. All four of the edges $wy_i$ and $vz_i$ are in $G$. 
Assume $v_b \in E(G)$.

Fig. 4.18. Assume $v_b \in E(G)$.

Note that, by symmetry it is enough to prove that $vz_1$ is in $G$. So assume that $vz_1$ is not in $G$. We claim that each neighbor of $q$, different from $u$, $r$, $c$ and $b$, is adjacent to $v$. Suppose, to the contrary, that $w$ is an external neighbor of $q$.

If $w$ is adjacent to $b$, then $G - N[v, w, z_1]$ contains the non-well-covered component spanned by $[a, c, e, f]$. So assume that $w$ is not adjacent to $b$. Then $G - N[v, w, z_1]$ has as a component the subgraph $S$ consisting of two blocks $E$ and $F$ sharing only $b$ as a cutvertex and where $F$ is the “wheel” subgraph generated by $[a, c, b, e, f]$. Now both $F$ and $F - \{b\}$ are not well-covered and so once again by Lemma 2.1(c) neither is $S$, and this contradicts the assumption that $G$ is well-covered. Thus each neighbor of $q$, different from $u$, $r$, $c$ or $b$, is adjacent to $v$.

Hence in the component of $G - N[\{d, v\}]$ containing $b$, vertex $b$ has two neighbors $q$ and $e$ each of degree 1 and hence by Lemma 2.1(d), this component is not well-covered, a contradiction. Thus we have established that $vz_1$, and hence also edges $vz_2$, $uy_1$ and $uy_2$, are present in $G$; that is, Claim 6 is true. ■

Now we are in a position to show that there are exactly four 4-connected well-covered triangulations having two adjacent vertices of degree 4.

**Theorem 4.3.** If $G$ is a 4-connected well-covered plane triangulation and has two adjacent vertices of degree 4, then $G$ is isomorphic to $R_6$, $R_7$, $R_8$ or $R_{12}$. (See Fig. 2.1.)

**Proof.** Let $u$ and $v$ be two adjacent vertices of degree 4. Let $x$ and $y$ be two common neighbors of $u$ and $v$. Let $w$ be the fourth neighbor of $u$. Then $w$ is not adjacent to $u$ by 4-connectivity, so $w$ is adjacent to both $x$ and $y$. Let $z$ be the fourth neighbor of $u$. Then $z \neq w$, but $w$ is adjacent to both $x$ and $y$; and $z$ is adjacent to both $x$ and $y$. Now $x$ and $y$ are not adjacent by 4-connectivity. Suppose $z$ is adjacent to $w$. Then, since $G$ is 4-connected, $|V(G)| = 6$ and, in fact, $G$ is isomorphic to $R_6$. On the other hand, if $z$ is not adjacent to $w$, then we have an induced 4-cycle $C = zxwy$ such that $G[V(C) \cup N(C, v)]$ is isomorphic to $Q_2$. So by Lemma 4.2, $G$ is isomorphic to $R_7, R_8$ or $R_{12}$. ■

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