Four functionals fixed point theorem

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Abstract

The Four Functionals Fixed Point Theorem is a generalization of the original, as well as the functional generalizations, of the Leggett–Williams Fixed Point Theorem. In the Four Functionals Fixed Point Theorem, neither the upper nor the lower boundary of the underlying set is required to map below or above the boundary in the functional sense. As an application, the existence of a positive solution to a second-order right focal boundary value problem is considered by applying both standard and nonstandard choices of functionals. An extension to multivalued maps is provided for completeness.

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1. Introduction

In this paper we present a major generalization of the original Leggett–Williams Fixed Point Theorem [13], which in turn is a modification of the fixed point theorem of expansion and compression of norm type. In [13] Leggett and Williams presented criteria which guaranteed the existence of a fixed point for single valued continuous, compact maps that did not require the operator to be invariant on the underlying sets by utilizing a concave functional and the norm. In that sense, the Leggett–Williams fixed point theorem generalized the compression–expansion fixed point theorem of norm type by Guo [9]. All of the fixed point theorems corresponding to expansion and compression, regardless of type [7,3,16,9,10,12], require at least one functional boundary of the underlying set be mapped below or above the boundary for the operator (that is, $\beta(Ax) \leq \beta(x)$ or $\beta(Ax) \geq \beta(x)$, where $A$ is the operator and $\beta$ is the functional). In this paper we remove this condition applying the foundational techniques of Leggett–Williams. The use of functionals in many of the modern multiple fixed point theorems ([2,4–6] to mention a few), as well as the generalizations of the compression–expansion fixed point theorems, provides researchers flexibility in arguing existence of solutions and provides varying information concerning the properties of the solution through proper choice of appropriate functionals. In the application of the Four Functionals Fixed Point Theorem, we utilize both standard and nonstandard choices of functionals, and then follow with an example of our application. An extension,
applying the techniques of Agarwal and O'Regan [1], is provided for completeness of the theory to generalize the fixed point theorem to maps which obey an axiomatic index theory; so, in particular, the results apply to all multivalued maps in the literature which have a well-defined fixed point index; see [1,14,15,17] and the references therein.

2. Preliminaries

In this section, we will state the definitions that are used in the remainder of the paper.

**Definition 1.** Let $E$ be a real Banach space. A nonempty closed convex set $P \subset E$ is called a cone if it satisfies the following two conditions:

(i) $x \in P$, $\lambda \geq 0$ implies $\lambda x \in P$;
(ii) $x \in P$, $-x \in P$ implies $x = 0$.

Every cone $P \subset E$ induces an ordering in $E$ given by

$x \leq y$ if and only if $y - x \in P$.

**Definition 2.** An operator is called completely continuous if it is continuous and maps bounded sets into precompact sets.

**Definition 3.** A map $\alpha$ is said to be a nonnegative continuous concave functional on a cone $P$ of a real Banach space $E$ if

$\alpha : P \to [0, \infty)$

is continuous and

$\alpha(tx + (1 - t)y) \geq t\alpha(x) + (1 - t)\alpha(y)$

for all $x, y \in P$ and $t \in [0, 1]$. Similarly, we say the map $\beta$ is a nonnegative continuous convex functional on a cone $P$ of a real Banach space $E$ if

$\beta : P \to [0, \infty)$

is continuous and

$\beta(tx + (1 - t)y) \leq t\beta(x) + (1 - t)\beta(y)$

for all $x, y \in P$ and $t \in [0, 1]$.

Let $\alpha$ and $\psi$ be nonnegative continuous concave functionals on $P$, and let $\beta$ and $\theta$ be nonnegative continuous convex functionals on $P$; then, for positive real numbers $r$, $\tau$, $\mu$ and $R$, we define the sets:

\[
Q(\alpha, \beta, r, R) = \{ x \in P : r \leq \alpha(x) \text{ and } \beta(x) \leq R \},
\]

\[
U(\psi, \tau) = \{ x \in Q(\alpha, \beta, r, R) : \tau \leq \psi(x) \}
\]

and

\[
V(\theta, \mu) = \{ x \in Q(\alpha, \beta, r, R) : \theta(x) \leq \mu \}.
\]

**Definition 4.** Let $D$ be a subset of a real Banach space $E$. If $r : E \to D$ is continuous with $r(x) = x$ for all $x \in D$, then $D$ is a retract of $E$, and the map $r$ is a retraction. The convex hull of a subset $D$ of a real Banach space $X$ is given by

\[
\text{conv}(D) = \left\{ \sum_{i=1}^{n} \lambda_i x_i : x_i \in D, \lambda_i \in [0, 1], \sum_{i=1}^{n} \lambda_i = 1, \text{ and } n \in \mathbb{N} \right\}.
\]

The following theorem is due to Dugundji and a proof can be found in [8, p. 44].
Theorem 5. For Banach spaces $X$ and $Y$, let $D \subset X$ be closed and let

$$F : D \to Y$$

be continuous. Then $F$ has a continuous extension

$$\tilde{F} : X \to Y$$

such that

$$\tilde{F}(X) \subset \text{conv}(F(D)).$$

Corollary 6. Every closed convex set of a Banach space is a retract of the Banach space.

Note that any cone $P$ of a Banach space $E$ is a retract of $E$.

3. Fixed point index

The following theorem, which establishes the existence and uniqueness of the fixed point index, is from [11, pp. 82–86]; an elementary proof can be found in [8, pp. 58 & 238]. The proof of our main result in the next section will invoke the properties of the fixed point index.

Theorem 7. Let $X$ be a retract of a real Banach space $E$. Then, for every bounded relatively open subset $U$ of $X$ and every completely continuous operator $A : U \to X$ which has no fixed points on $\partial U$ (relative to $X$), there exists an integer $i(A, U, X)$ satisfying the following conditions:

(G1) Normality: $i(A, U, X) = 1$ if $Ax \equiv y_0 \in U$ for any $x \in U$;

(G2) Additivity: $i(A, U, X) = i(A, U_1, X) + i(A, U_2, X)$ whenever $U_1$ and $U_2$ are disjoint open subsets of $U$ such that $A$ has no fixed points on $U - (U_1 \cup U_2)$;

(G3) Homotopy Invariance: $i(H(t, \cdot), U, X)$ is independent of $t \in [0, 1]$ whenever $H : [0, 1] \times U \to X$ is completely continuous and $H(t, x) \neq x$ for any $(t, x) \in [0, 1] \times \partial U$;

(G4) Permanence: $i(A, U, X) = i(A, U \cap Y, Y)$ if $Y$ is a retract of $X$ and $A(U) \subset Y$;

(G5) Excision: $i(A, U, X) = i(A, U_0, X)$ whenever $U_0$ is an open subset of $U$ such that $A$ has no fixed points in $U - U_0$;

(G6) Solution: If $i(A, U, X) \neq 0$, then $A$ has at least one fixed point in $U$.

Moreover, $i(A, U, X)$ is uniquely defined.

4. Main result

Theorem 8. Suppose $P$ is a cone in a real Banach space $E$, $\alpha$ and $\psi$ are nonnegative continuous concave functionals on $P$, $\beta$ and $\theta$ nonnegative continuous convex functional on $P$, and there exist nonnegative numbers $r$, $\tau$, $\mu$ and $R$ such that

$$A : Q(\alpha, \beta, r, R) \to P$$

is a completely continuous operator, and $Q(\alpha, \beta, r, R)$ is a bounded set. If

(i) $\{x \in U(\psi, \tau) : \beta(x) < R\} \cap \{x \in V(\theta, \mu) : r < \alpha(x)\} \neq \emptyset$,

(ii) $\alpha(Ax) \geq r$, for all $x \in Q(\alpha, \beta, r, R)$, with $\alpha(x) = r$ and $\mu < \theta(Ax),$

(iii) $\alpha(Ax) \geq r$, for all $x \in V(\theta, \mu)$, with $\alpha(x) = r$,

(iv) $\beta(Ax) \leq R$, for all $x \in Q(\alpha, \beta, r, R)$, with $\beta(x) = R$ and $\psi(Ax) < \tau$, and

(v) $\beta(Ax) \leq R$, for all $x \in U(\psi, \tau)$, with $\beta(x) = R$,

then $A$ has a fixed point $x$ in $Q(\alpha, \beta, r, R)$. 
Proof. Let

\[ W = \{ x \in Q(\alpha, \beta, r, R) : r < \alpha(x) \text{ and } \beta(x) < R \}. \]

Then \( W \) is an open subset of \( P \) and we have assumed that \( W \) is a bounded set. If \( A \) has a fixed point \( x \in \partial W \), then we are finished. So, without loss of generality, let us assume that \( Ax \neq x \) for all \( x \in \partial W \).

Let \( x^* \in \{ x \in U(\psi, \tau) : \beta(x) < R \} \cap \{ x \in V(\theta, \mu) : r < \alpha(x) \} \) (see condition (i)), and let

\[ H : [0, 1] \times \bar{W} \to P \]

be defined by

\[ H(t, x) = (1 - t)Ax + tx^*. \]

Clearly, \( H \) is continuous and \( H([0, 1] \times \bar{W}) \) is relatively compact.

Claim: \( H(t, x) \neq x \) for all \((t, x) \in [0, 1] \times \partial W \).

Suppose not; that is, suppose there exists \((t_1, x_1) \in [0, 1] \times \partial W \) such that \( H(t_1, x_1) = x_1 \). Since \( Ax \neq x \), for all \( x \in \partial W \), we have that \( t_1 \in (0, 1] \). Also, since \( x_1 \in \partial W \), we have that \( \beta(x_1) = R \) or \( \alpha(x_1) = r \).

Case 1: \( \beta(x_1) = R \).

Either \( \tau \leq \psi(Ax_1) \) or \( \psi(Ax_1) < \tau \). If \( \psi(Ax_1) < \tau \), then by condition (iv), we have

\[
\begin{align*}
\beta(x_1) &= \beta((1 - t_1)Ax_1 + t_1x^*) \\
&\leq (1 - t_1)\beta(Ax_1) + t_1\beta(x^*) \\
&< R,
\end{align*}
\]

which is a contradiction. If \( \tau \leq \psi(Ax_1) \), then \( x_1 \in U(\psi, \tau) \), since

\[
\begin{align*}
\psi(x_1) &= \psi((1 - t_1)Ax_1 + t_1x^*) \\
&\geq (1 - t_1)\psi(Ax_1) + t_1\psi(x^*) \\
&\geq \tau,
\end{align*}
\]

and hence, by condition (v), we have

\[
\begin{align*}
\beta(x_1) &= \beta((1 - t_1)Ax_1 + t_1x^*) \\
&\leq (1 - t_1)\beta(Ax_1) + t_1\beta(x^*) \\
&< R,
\end{align*}
\]

which is a contradiction. Thus, \( \beta(x_1) \neq R \).

Case 2: \( \alpha(x_1) = r \).

Either \( \theta(Ax_1) \leq \mu \) or \( \mu < \theta(Ax_1) \). If \( \mu < \theta(Ax_1) \), then by condition (ii), we have

\[
\begin{align*}
\alpha(x_1) &= \alpha((1 - t_1)Ax_1 + t_1x^*) \\
&\geq (1 - t_1)\alpha(Ax_1) + t_1\alpha(x^*) \\
&> r,
\end{align*}
\]

which is a contradiction. If \( \theta(Ax_1) \leq \mu \), then \( x_1 \in V(\theta, \mu) \), since

\[
\begin{align*}
\theta(x_1) &= \theta((1 - t_1)Ax_1 + t_1x^*) \\
&\leq (1 - t_1)\theta(Ax_1) + t_1\theta(x^*) \\
&\leq \mu,
\end{align*}
\]

and hence, by condition (iii), we have

\[
\begin{align*}
\alpha(x_1) &= \alpha((1 - t_1)Ax_1 + t_1x^*) \\
&\geq (1 - t_1)\alpha(Ax_1) + t_1\alpha(x^*) \\
&> r,
\end{align*}
\]

which is a contradiction. Thus, \( \alpha(x_1) \neq r \).
Therefore, we have shown that \( H(t, x) \neq x \), for all \((t, x) \in [0, 1] \times \partial W\), and thus by the homotopy invariance property (G3) of the fixed point index,

\[
i(A, W, P) = i(x^*, W, P),
\]

and by the normality property (G1) of the fixed point index,

\[
i(A, W, P) = i(x^*, W, P) = 1.
\]

Therefore by the solution property (G6) of the fixed point index, the operator \( A \) has a fixed point \( x \in W \). \(\square\)

5. Multivalued generalization

In this section, we provide some background material from fixed point theory related to multivalued maps.

Let \( X \) be a closed, convex subset of some Banach space \( E = (E, \| \cdot \|) \). Suppose, for every open subset \( U \) of \( X \) and every upper semicontinuous map \( A : \overline{U}^X \rightarrow 2^X \) (here \( 2^X \) denotes the family of nonempty subsets of \( X \)), which satisfies Property (B) (to be specified later), with \( x \not\in Ax \) for \( x \in \partial X \) \( U \) (here \( \overline{U} \) and \( \partial X \) \( U \) denote the closure and boundary of \( U \) in \( X \), respectively), there exists an integer, denoted by \( i_X(A, U) \), satisfying the following properties:

(P1) If \( x_0 \in U \), then \( i_X(\hat{x}_0, U) = 1 \) (here \( \hat{x}_0 \) denotes the map whose constant value is \( x_0 \));

(P2) For every pair of disjoint open subsets \( U_1, U_2 \) of \( U \), such that \( A \) has no fixed points on \( \overline{U} \setminus (U_1 \cup U_2) \),

\[
i_X(A, U_1) = i_X(A, U_2);
\]

(P3) For every upper semicontinuous map \( H : [0, 1] \times \overline{U} \rightarrow 2^X \), which satisfies Property (B), and \( x \not\in H(t, x) \) for \((t, x) \in [0, 1] \times \partial X U \),

\[
i_X(H(1, \cdot), U) = i_X(H(0, \cdot), U);
\]

(P4) If \( Y \) is a closed convex subset of \( X \) and \( A(\overline{U}^Y) \subseteq Y \), then

\[
i_X(A, U) = i_Y(A, U \cap Y).
\]

Also assume the family

\[
X \text{ a closed, convex subset of a Banach space } E,
\]

\[
i_X(A, U) : U \text{ open in } X, \quad \text{and } A : \overline{U} \rightarrow 2^X \text{ is an upper semicontinuous map that satisfies Property (B) with } x \not\in Ax \text{ on } \partial X U
\]

is uniquely determined by the Properties (P1)–(P4).

We note that Property (B) is any property on the map so that the fixed point index is well-defined. Usually in application, Property (B) will mean that the map is compact with convex compact values. Other examples of maps with a well-defined fixed point index (e.g. Property (B) could mean that the map is countably condensing with convex compact values) can be found in the literature.

If the above hold, notice also that

(P5) For every open subset \( V \) of \( U \), such that \( A \) has no fixed points on \( \overline{U} \setminus V \),

\[
i_X(A, U) = i_X(A, V);
\]

and

(P6) If \( i_X(A, U) \neq 0 \), then \( A \) has at least one fixed point in \( U \).

The proof of the following generalization of Theorem 8 to multivalued maps is essentially the same as the proof of Theorem 8 following the techniques applied in [2] and is therefore omitted.

**Theorem 9.** Let \( E = (E, \| \cdot \|) \) be a Banach space and \( X \) a closed, convex subset of \( E \). Suppose for every open subset \( U \) of \( X \) and every upper semicontinuous map \( A : \overline{U} \rightarrow 2^X \), which satisfies Property (B) with \( x \not\in Ax \) for \( x \in \partial X U \), there exists an integer \( i_X(A, U) \) satisfying (P1)–(P4). In addition assume the family

\[
X \text{ a closed, convex subset of a Banach space } E,
\]

\[
i_X(A, U) : U \text{ open in } X, \quad \text{and } A : \overline{U} \rightarrow 2^X \text{ is an upper semicontinuous map that satisfies Property (B) with } x \not\in Ax \text{ on } \partial X U
\]

is uniquely determined by the Properties (P1)–(P4). Let $P \subset E$ be a cone in $E$ and suppose that $\alpha$ and $\psi$ are nonnegative continuous concave functionals on $P$, $\beta$ and $\theta$ are nonnegative continuous convex functional on $P$, and there exist nonnegative numbers $r, \tau, \mu$ and $R$ such that $Q(\alpha, \beta, r, R)$ is a bounded set. Furthermore, suppose

$$F : Q(\alpha, \beta, r, R) \to 2^P$$

is an upper semicontinuous map which satisfies Property (B) such that the following properties are satisfied:

(H1) There exists $x^* \in \{ x \in U(\psi, \tau) : \beta(x) < R \} \cap \{ x \in V(\theta, \mu) : r < \alpha(x) \}$ such that the mapping $H : [0, 1] \times (\{ x \in U(\psi, \tau) : \beta(x) < R \} \cap \{ x \in V(\theta, \mu) : r < \alpha(x) \}) \to 2^P$, given by $H(t, x) = (1-t)F x + tx^*$, satisfies Property (B);

(H2) If $x \in Q(\alpha, \beta, r, R)$, with $\alpha(x) = r$ and $\mu < \theta(y)$ for some $y \in F x$, then $\alpha(y) \geq r$;

(H3) If $x \in V(\theta, \mu)$, with $\alpha(x) = r$, then $\alpha(y) \geq r$ for all $y \in F x$;

(H4) If $x \in Q(\alpha, \beta, r, R)$, with $\beta(x) = R$ and $\psi(y) < \tau$ for some $y \in F x$, then $\beta(y) \leq R$;

(H5) If $x \in U(\psi, \tau)$, with $\beta(x) = R$, then $\beta(y) \leq R$ for all $y \in F x$.

Then $F$ has at least one fixed point $x$ in $Q(\alpha, \beta, r, R)$.

6. Application

A standard technique to verify the existence of solutions, by applying a fixed point theorem to a boundary value problem, is to assume that the nonlinearity is bounded by a constant on intervals in order to verify certain inequalities, in which case, choosing the minimum of a function over an interval (concave functional) and the maximum of a function over an interval (convex functional) often simplify the arguments. In this application, we will not only demonstrate the standard technique, but we will also choose alternative functionals involving integrals with an upper bound that is linear.

Consider the second-order nonlinear focal boundary value problem

$$y''(t) + f(y(t)) = 0, \quad t \in (0, 1),$$

$$y(0) = 0 = y'(1),$$

where $f : \mathbb{R} \to [0, \infty)$ is continuous. If $x$ is a fixed point of the operator $A$ defined by

$$Ax(t) := \int_0^1 G(t, s) f(x(s))ds,$$

where

$$G(t, s) = \begin{cases} t : t \leq s, \\ s : s \leq t, \end{cases}$$

is the Green’s function for the operator $L$ defined by

$$Lx(t) := -x'',$$

with right focal boundary conditions

$$x(0) = 0 = x'(1),$$

then it is well-known that $x$ is a solution of the boundary value (4) and (5). Throughout this section of the paper we will use the facts that $G(t, s)$ is nonnegative, and for each fixed $s \in [0, 1]$, the Green’s function is nondecreasing in $t$.

Define the cone $P \subset E = C[0, 1]$ by

$$P := \{ x \in E : x \text{ is nonnegative, nondecreasing and concave} \}.$$

Define the concave functionals $\alpha$ and $\psi$ by

$$\alpha(x) := \min_{t \in [1/4, 1]} x(t) = x(1/4),$$

$$\psi(x) := \int_{1/4}^1 \frac{s^2}{2} x(s)ds.$$
and the convex functionals $\theta$ and $\beta$ by

$$\theta(x) := \max_{t \in [0,1]} x(t) = x(1),$$

$$\beta(x) := \int_0^1 x(s) \, ds.$$ 

In the following theorem we demonstrate how to apply the Four Functionals Fixed Point Theorem, Theorem 8, to prove the existence of at least one positive solution to (4) and (5).

**Theorem 10.** Suppose there exist positive real numbers $M$, $B$, $r$ and $R$, with $\frac{320B}{3(128-131M)} \leq R$, $\frac{16r}{3} \leq Mr + B$ and $r < \frac{3R}{128} < 4r$, and a continuous function $f : [0, 2R] \to [0, \infty)$, such that,

(a) $f(z) \leq Mz + B$ for all $z \in [0, 2R]$, and

(b) $f(z) \geq \frac{16r}{3}$ for all $z \in [r, 4r]$.

Then, the operator $A$ has at least one positive solution $x^*$ with

$$r \leq \alpha(x^*) \quad \text{and} \quad \beta(x^*) \leq R.$$ 

**Proof.** Let $\tau = \frac{3R}{128}$ and $\mu = 4r$. By the properties of $G$ and $f$ we have that

$$A : Q(\alpha, \beta, r, R) \to P$$

is completely continuous. Applying a standard calculus argument, we have that the set $Q(\alpha, \beta, r, R)$ is bounded, since

$$\frac{x(1) - x(0)}{2} \leq \int_0^1 x(s) \, ds \leq R.$$ 

Also, it can easily be shown that the constant function $\frac{\tau + 4r}{2} \in \{ x \in U(\psi, \tau) : \beta(x) \leq R \} \cap \{ x \in V(\theta, \mu) : r < \alpha(x) \}$ and hence the set is nonempty.

Claim 1: $\alpha(Ax) \geq r$ for all $x \in Q(\alpha, \beta, r, R)$ with $\alpha(x) = r$ and $\mu < \theta(Ax)$.

Let $x \in Q(\alpha, \beta, r, R)$, with $\alpha(x) = r$ and $\mu < \theta(Ax)$. Then since $4G(1/4, s) \geq G(1, s)$, for all $s \in [0, 1]$, we have

$$\alpha(Ax) = \int_0^1 G(1/4, s)x(s) \, ds \geq \frac{\int_0^1 G(1, s)x(s) \, ds}{4} = \theta(Ax) > \frac{\mu}{4} = r.$$ 

Claim 2: $\alpha(Ax) \geq r$, for all $x \in V(\theta, \mu)$, with $\alpha(x) = r$.

Let $x \in V(\theta, \mu)$, with $\alpha(x) = r$. Thus $r \leq x(s) \leq 4r$, for $s \in [1/4, 1]$, and hence $f(x(s)) \geq \frac{16r}{3}$, for $s \in [1/4, 1]$, and therefore

$$\alpha(Ax) = \int_0^1 G(1/4, s)f(x(s)) \, ds$$

$$\geq \int_{1/4}^1 G(1/4, s)f(x(s)) \, ds$$

$$\geq \int_{1/4}^1 G(1/4, s) \left( \frac{16r}{3} \right) \, ds = r.$$ 

Claim 3: $\beta(Ax) \leq R$, for all $x \in Q(\alpha, \beta, r, R)$, with $\beta(x) = R$ and $\psi(Ax) < \tau$.

Since

$$\tau > \psi(Ax) = \int_{1/4}^1 \frac{s^2}{2} A_x(s) \, ds \geq \frac{1}{32} \int_{1/4}^1 A_x(s) \, ds$$ 

and

$$3 \int_0^{1/4} A_x(s) \, ds \leq \int_{1/4}^1 A_x(s) \, ds,$$
we have
\[32\tau > \int_{1/4}^{1} Ax(s)\,ds\]
and
\[\frac{32}{3}\tau > \left(\frac{1}{3}\right)\int_{1/4}^{1} Ax(s)\,ds \geq \int_{0}^{1/4} Ax(s)\,ds.\]

Therefore
\[R = \frac{128\tau}{3} > \int_{0}^{1/4} Ax(s)\,ds + \int_{1/4}^{1} Ax(s)\,ds = \int_{0}^{1} Ax(s)\,ds = \beta(Ax).\]

Claim 4: \(\beta(Ax) \leq R\), for all \(x \in U(\psi, \tau)\), with \(\beta(x) = R\).

Let \(x \in U(\psi, \tau)\) with \(\beta(x) = R\). Thus \(\psi(x) \geq \tau = \frac{3R}{128}\), and hence
\[
\beta(Ax) = \int_{0}^{1} \int_{0}^{1} G(t, s) f(x(s))\,ds\,dt \\
= \int_{0}^{1} \int_{0}^{1} sf(x(s))\,ds\,dt + \int_{0}^{1} \int_{t}^{1} tf(x(s))\,ds\,dt \\
= \int_{0}^{1} \int_{s}^{1} sf(x(s))\,ds\,dt + \int_{0}^{1} \int_{0}^{s} tf(x(s))\,dr\,ds \\
= \int_{0}^{1} \left(1 - \frac{s^2}{2}\right) f(x(s))\,ds \\
\leq \int_{0}^{1} \left(1 - \frac{s^2}{2}\right) (Mx(s) + B)\,ds \\
= \frac{5B}{6} + M\beta(x) - M\psi(x) \\
\leq \frac{5B}{6} + MR - M\tau \leq R.
\]

Therefore, the hypotheses of Theorem 8 have been satisfied; thus the operator \(A\) has at least one positive solution \(x^*\) such that
\[r \leq \alpha(x^*) \quad \text{and} \quad \beta(x^*) \leq R. \quad \square\]

Example 11. The boundary value problem
\[
x'' + 25 + 2 \ln(1 + x) + \frac{x \sin x}{10} = 0, \quad (6) \\
x(0) = 0 = x'(1), \quad (7)
\]
has a positive solution \(x^*\) such that
\[1 \leq x^*\left(\frac{1}{4}\right) \quad \text{and} \quad \int_{0}^{1} x^*(s)\,ds \leq 100\]
which can easily be verified by invoking Theorem 10, with
\[r = 1, \quad R = 100, \quad M = \frac{48}{131} \quad \text{and} \quad B = 75.\]
References


