A generalized fractional variational problem depending on indefinite integrals: Euler-Lagrange equation and numerical solution

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Abstract

The aim of this paper is to generalize the Euler-Lagrange equation obtained in Almeida et al. (2011), where fractional variational problems for Lagrangians, depending on fractional operators and depending on indefinite integrals, were studied. The new problem that we address here is for cost functionals, where the interval of integration is not the whole domain of the admissible functions, but a proper subset of it. Furthermore, we present a numerical method, based on Jacobi polynomials for solving this problem.

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1 Introduction

Fractional calculus is an old subject of mathematics, and its study goes back to the XVII century, where Leibniz was intrigued about the meaning of the derivative of order 0.5. Since then, it has capture the attention of many authors, namely Riemann, Liouville, Weyl and Hadamard. In the past few years were found many applications to real phenomena, with certain dynamics being described not by integer derivatives but by real order derivatives. One natural application is to variational problems dependent on a fractional operator. Different types have been considered. In particular, the Lagrangian or Hamiltonian approach, with fractional integrals and/or with fractional derivatives, and on the last case, for several types of derivatives. In Almeida et al. (2012) Almeida et al. was studied a new type of functional involving not only fractional operators but indefinite integrals as well. Several cases were considered, namely the fundamental problem with a necessary condition to obtain extremizers of the functional, constrained and unconstrained problems, sufficient conditions and others. In common was the fact that the interval of integration of the cost functional is the whole interval $[a, b]$. The purpose of this paper is to generalize the previous work for functionals whose interval of integration is a proper subset of $[a, b]$. We deduce a general form of a necessary optimality condition, and combining the techniques of this paper with the ones in Almeida et al. (2012), other fundamental problems of fractional variational calculus can be solved. Variational problems where the interval of integration is a proper subset of $[a, b]$ were
studied in Atanacković et al. (2008) and in Almeida et al. (2011), where they use Riemann-Liouville and Caputo fractional derivatives, respectively.

We begin with a short introduction to this subject, following Kilbas et al. (2006). In the following, \( \alpha \in (0,1) \) is a fixed real and \( y : [a, b] \to \mathbb{R} \) is a function. The left and right Riemann–Liouville fractional integrals of order \( \alpha \) are defined by

\[
a^D_a \frac{\alpha}{\Gamma(a)} \int_a^x (x-t)^{a-1} f(t)dt \quad \text{and} \quad b^D_b \frac{\alpha}{\Gamma(a)} \int_x^b (t-x)^{a-1} f(t)dt,
\]

respectively. The left and right Riemann–Liouville fractional derivative of order \( \alpha \) are defined by

\[
a^D_a y(x) = \frac{1}{\Gamma(1-a)} \frac{d}{dx} \int_a^x (x-t)^{-a} y(t)dt \quad \text{and} \quad b^D_b y(x) = \frac{-1}{\Gamma(1-a)} \frac{d}{dx} \int_x^b (t-x)^{-a} y(t)dt,
\]

respectively. For the Caputo operators, we also have two types. The left and right Caputo fractional derivative of order \( \alpha \) are defined by

\[
_a^C D_a^\alpha y(x) = \frac{1}{\Gamma(1-a)} \int_a^x (x-t)^{-a} y'(t)dt \quad \text{and} \quad _b^C D_b^\alpha y(x) = \frac{-1}{\Gamma(1-a)} \int_x^b (t-x)^{-a} y'(t)dt,
\]

respectively. One basic property needed for variational problems is an integration by parts formula. For the Caputo fractional derivative, we have (see e.g. Agrawal (2007b) and Kilbas et al. (2006)):

\[
\int_a^b y_1(x) \cdot _a^C D_a^\alpha y_2(x) dx = \int_a^b y_2(x) \cdot _a^C D_a^\alpha y_1(x) dx + [y_2(x) \cdot _a^C D_a^{1-a} y_1(x)]_a^b.
\]

We remark that we have in the left member the left Caputo derivative, while in the right we have the right Riemann-Liouville derivative. Also, when \( \alpha = 1 \) this formula becomes the usual integration by parts formula.

2 The Euler-Lagrange equation

Fractional variational calculus deals with optimization problems, where the functional that we wish to minimize or maximize depends on a fractional derivative or on a fractional integral. We may find on the literature a vast number of research papers for different fractional operators. In this work, we will consider the Caputo fractional derivative. This derivative has some advantages compared with the others, namely the derivative of a constant is zero (the Riemann-Liouville fractional derivative of a constant is not zero) and the Laplace transform of the Caputo fractional derivative is written in terms of the integer derivatives evaluated at the origin (with clear advantages for problems of fractional differential equations). To mention a few works dealing with this type of derivative, we cite Agrawal (2007a), Almeida et al. (2011), Jarad et al. (2010) and Malinowska et al. (2010).

The problem that we consider in this work is formulated in the following way. Let \([A, B] \subseteq [a, b]\) be an interval, and \(J\) be the functional defined by

\[
J(y) = \int_A^B L(x, y(x), _a^C D_a^\alpha y(x), z(x)) dx,
\]

where \(z\) is defined by

\[
z(x) = \int_A^x f(t, y(t), _a^C D_a^\alpha y(t)) dt.
\]

We are assuming that \(_a^C D_a^\alpha y(x)\) exists and is continuous. Also, the functions \((x, y, v, z) \to L(x, y, v, z)\) and \((x, y, v) \to f(t, y, v)\) are at least of class \(C^1\). To simplify, we denote

\[
[y](x) = (x, y(x), _a^C D_a^\alpha y(x), z(x))
\]
and
\[ \{y\}(x) = (x, y(x), D_0^\alpha y(x)). \]
In some cases, we will assume that set of admissible functions for \( J \) is restricted to the conditions
\[ y(a) = y_a, \quad y(A) = y_A \quad \text{and} \quad y(B) = y_B. \] (3)

**Remark 1.** Although we consider functionals depending on the left Caputo fractional derivative only, other fractional operators such as the right Caputo fractional derivative, Riemann-Liouville fractional integrals or derivatives, could be included. Similar techniques would solve the problems in such cases.

**Remark 2.** Allowing \( \alpha \) to be 1, the functional reduces to the standard one
\[ J(y) = \int_A^B L(x, y(x), y'(x), z(x))dx, \]
with
\[ z(x) = \int_A^x I(t, y(t), y'(t))dt. \]
We refer to Gregory (2008), where a special kind of such functionals is studied. We also mention Martins et al. (2011), where similar problems are solved on a time scale.

The main theorem states a necessary condition of optimality that extremizers of \( J \) must satisfy. Such differential equations are called Euler-Lagrange equations.

**Theorem 3.** Let \( y \) be an extremizer of \( J \) as in (1)- (3), under the restrictions (3). Then \( y \) is a solution of the two fractional differential equations:
For all \( x \in [a, A] \),
\[ x D_B^\alpha \left( \frac{\partial L}{\partial y}[y](x) \right) - x D_A^\alpha \left( \frac{\partial L}{\partial y}[y](x) \right) + x D_B^\alpha \left( \int_x^B \frac{\partial L}{\partial y}[y](t)dt \frac{\partial I}{\partial y}\{y\}(x) \right) \]
\[ - x D_A^\alpha \left( \int_x^B \frac{\partial L}{\partial y}[y](t)dt \frac{\partial I}{\partial y}\{y\}(x) \right) = 0, \]
and for all \( x \in [A, B] \),
\[ \frac{\partial L}{\partial y}[y](x) + x D_B^\alpha \left( \frac{\partial L}{\partial v}[y](x) \right) + \left( \int_x^B \frac{\partial L}{\partial z}[y](t)dt \frac{\partial I}{\partial v}\{y\}(x) + x D_B^\alpha \left( \int_x^B \frac{\partial L}{\partial z}[y](t)dt \frac{\partial I}{\partial v}\{y\}(x) \right) = 0. \]

**Proof.** Let \( h : [a, b] \to \mathbb{R} \) be a function and \( \epsilon \) a parameter taking values on a neighborhood of zero. In order to \( y + \epsilon h \) be an admissible function, the conditions \( h(a) = h(A) = h(B) = 0 \) must be satisfied. Consider a new function \( \epsilon \to J(y + \epsilon h) \). Since it takes an extremum value at \( \epsilon = 0 \), differentiating at the origin, we deduce
\[ \int_A^B \left[ \frac{\partial L}{\partial y}[y](x)h(x) + \frac{\partial L}{\partial y}[y](x)D_0^\alpha h(x) + \frac{\partial L}{\partial z}[y](x) \int_A^x \left( \frac{\partial I}{\partial y}\{y\}(t)h(t) + \frac{\partial I}{\partial v}\{y\}(t)D_0^\alpha h(t) \right)dt \right]dx = 0. \]

Now we proceed evaluating each of these integrals. Applying fractional integration by parts, we have
\[ \int_A^B \frac{\partial L}{\partial v}[y](x)D_0^\alpha h(x)dx = \int_a^B \frac{\partial L}{\partial v}[y](x)D_0^\alpha h(x)dx - \int_a^A \frac{\partial L}{\partial v}[y](x)D_0^\alpha h(x)dx \]
\[ = \int_a^B h(x)xD_B^\alpha \left( \frac{\partial L}{\partial v}[y](x) \right)dx - \int_a^A h(x)xD_A^\alpha \left( \frac{\partial L}{\partial v}[y](x) \right)dx + \left[ h(x)I_B^{1-\alpha} \left( \frac{\partial L}{\partial v}[y](x) \right) \right]_a^B - \left[ h(x)I_A^{1-\alpha} \left( \frac{\partial L}{\partial v}[y](x) \right) \right]_a^A. \]
By noting that \( h(a) = h(b) = 0 \), we get
\[
\int_a^B \frac{\partial L}{\partial v} [y](x) D_\alpha^a h(x) \, dx = \int_a^B h(x) D_\alpha^B \left( \frac{\partial L}{\partial v} [y](x) \right) \, dx - \int_a^A h(x) D_\alpha^A \left( \frac{\partial L}{\partial v} [y](x) \right) \, dx.
\]
For the next integral, we apply the usual integration by parts formula
\[
\int_a^B \frac{\partial L}{\partial z}[y](x) \left( \int_a^x \frac{\partial L}{\partial y} [y](t) h(t) \, dt \right) \, dx = \int_a^B \left( - \frac{d}{dx} \int_a^B \frac{\partial L}{\partial z}[y](x) \, dx \right) \left( \int_a^x \frac{\partial L}{\partial y} [y](t) h(t) \, dt \right) \, dx
\]
\[
= \int_a^B \left( \int_a^B \frac{\partial L}{\partial z}[y](x) \, dx \right) \frac{\partial L}{\partial y} [y](x) \, dx.
\]
For the last integral, we first apply the usual integration by parts formula and then the fractional integration by parts. Repeating the same calculations as in the two previous cases, we deduce
\[
\int_a^B \frac{\partial L}{\partial z}[y](x) \left( \int_a^x \frac{\partial L}{\partial y} [y](t) h(t) \, dt \right) \, dx
\]
\[
= \int_a^B h(x) \left( \int_a^B \frac{\partial L}{\partial z}[y](x) \, dx \right) \frac{\partial L}{\partial y} [y](x) \, dx.
\]
Then, combining the last three formulas, we obtain the equality
\[
\int_a^A h(x) \left[ x D_\alpha^B \left( \frac{\partial L}{\partial v} [y](x) \right) - x D_\alpha^a \left( \frac{\partial L}{\partial v} [y](x) \right) + x D_\alpha^B \left( \int_a^B \frac{\partial L}{\partial z}[y](x) \, dx \right) \right] \, dx = 0.
\]

Since \( h \) is an arbitrary function, if we assume that \( h(x) = 0 \), for all \( x \in [a, b] \), by the du Bois-Reymond lemma (see e.g. Brunt (2004)), we obtain the first desired equation. On the other hand, if \( h(x) = 0 \) for all \( x \in [a, A] \), we prove the second condition. \( \square \)

**Remark 4.** When \([A, B] = [a, b]\) we obtain the main result of Almeida et al. (2012), i.e., if \( y \) is an extremizer of
\[
J(y) = \int_a^b L(x, y(x), \frac{\partial}{\partial y} [y](x)) \, dx,
\]
where \( z \) is defined by
\[
z(x) = \int_a^x l(t, y(t), \frac{\partial}{\partial y} [y](t)) \, dt,
\]
then \( y \) is a solution of
\[
\frac{\partial L}{\partial y}(y) + x\mathcal{D}_b^\alpha \left( \frac{\partial L}{\partial v_i} \right) + \left( \int_x^b \frac{\partial L}{\partial y}(t)dt \right) \frac{\partial L}{\partial y}(y) + x\mathcal{D}_b^\alpha \left( \int_x^b \frac{\partial L}{\partial y}(t)dt \right) \frac{\partial L}{\partial v_i}(y) = 0.
\]
for all \( x \in [a, b] \). In fact, observe that if \( A = a \), then in formula \( 1 \) the first integral vanishes and thus we only get one necessary condition.

**Remark 5.** When the functional \( J \) does not depend on the indefinite integral, i.e., if
\[
J(y) = \int_A^B L(x, y(x), \mathcal{C}_a \mathcal{D}_x^\alpha y(x))dx,
\]
then we obtain Theorem 3.1 of Almeida et al. (2011). When the functional does not depend on the right Caputo fractional derivative and if \( y \) is an extremizer of \( J \), then
\[
\begin{align*}
\{ x\mathcal{D}_b^\alpha \left( \frac{\partial L}{\partial y}(y) \right) - x\mathcal{D}_a^\alpha \left( \frac{\partial L}{\partial y}(y) \right) = 0, & \quad \text{for all } x \in [a, A], \\
\frac{\partial L}{\partial y}(y) + x\mathcal{D}_b^\alpha \left( \frac{\partial L}{\partial y}(y) \right) = 0, & \quad \text{for all } x \in [A, B].
\end{align*}
\]

**Remark 6.** Theorem 2.2 in Gregory (2008) can be seen as a particular case of Theorem 3. If \( \alpha = 1 \) and \( [A, B] = [a, b] \), and \( y \) extremizes
\[
J(y) = \int_a^b L(x, y(x), y'(x), z(x))dx,
\]
under the boundary conditions \( y(a) = y_a \) and \( y(b) = y_b \), then \( y \) satisfies the differential equation
\[
\frac{\partial L}{\partial y}(y) - \frac{d}{dx} \left( \frac{\partial L}{\partial y}(y) \right) + \int_x^b \frac{\partial L}{\partial y}(t)dt = 0.
\]
We remark that \( x\mathcal{D}_b^\alpha = -\frac{d}{dx} \) when \( \alpha = 1 \).

The case of dependence on several functions can be easily included. In fact, suppose that \( \overline{y} = (y_1, \ldots, y_n) \) and
\[
\begin{aligned}
H(\overline{y}) &= \int_A^B L(x, \overline{y}(x), \mathcal{C}_a \mathcal{D}_x^\alpha \overline{y}(x), z(x))dx, \\
\text{with } z(x) &= \int_A^x l(t, \overline{y}(t), \mathcal{C}_a \mathcal{D}_x^\alpha \overline{y}(t))dt,
\end{aligned}
\]
and
\[
\mathcal{C}_a \mathcal{D}_x^\alpha \overline{y}(x) = (\mathcal{C}_a \mathcal{D}_x^\alpha y_1(x), \ldots, \mathcal{C}_a \mathcal{D}_x^\alpha y_n(x)),
\]
then following a similar procedure, we obtain a multi-dimensional Euler-Lagrange equation.

**Theorem 7.** Let \( \overline{y} \) be an extremizer of \( H \), under the restrictions \( \overline{y}(a) = \overline{y}_a, \quad \overline{y}(A) = \overline{y}_A \) and \( \overline{y}(B) = \overline{y}_B \).

Then, for all \( i \in \{1, \ldots, n\} \), \( \overline{y}_i \) is a solution of the two following fractional differential equations:
For all \( x \in [a, A] \),
\[
x\mathcal{D}_b^\alpha \left( \frac{\partial L}{\partial v_i} \right) - x\mathcal{D}_a^\alpha \left( \frac{\partial L}{\partial v_i} \right) + \int_x^B \frac{\partial L}{\partial y}(t)dt \frac{\partial L}{\partial v_i} \{\overline{y}(x)\} = 0,
\]
and for all \( x \in [A, B] \),
\[
\frac{\partial L}{\partial y_i}(\overline{y}(x)) + x\mathcal{D}_b^\alpha \left( \frac{\partial L}{\partial v_i} \right) + \int_x^B \frac{\partial L}{\partial y}(t)dt \frac{\partial L}{\partial v_i} \{\overline{y}(x)\} + x\mathcal{D}_b^\alpha \left( \int_x^B \frac{\partial L}{\partial y}(t)dt \right) \frac{\partial L}{\partial v_i} \{\overline{y}(x)\} = 0.
\]
Proof. Consider a variation of type $\eta + \epsilon h$, with $\eta = (h_1, \ldots, h_n)$. To obtain the two conditions with respect to a fixed $i$, consider variations with $h_j \equiv 0$ for $j \neq i$ and follow the proof of Theorem 3.

We now consider the cases where the set of admissible functions is free at $x = a$, $x = A$ or $x = B$.

Theorem 8. Let $y$ be an extremizer of $J$ as in (1) and (2). Then $y$ is a solution of the two fractional differential equations of Theorem 3. Moreover,

1. if $y(A) = y_A$, $y(B) = y_B$ and $y(a)$ is free, then the condition
\[
\left[ x I_B^{-\alpha} \left( \frac{\partial L}{\partial v} [y](x) + \int_x^B \frac{\partial L}{\partial z} [y](t)dt \frac{\partial l}{\partial v} [y](x) \right) - x I_A^{-\alpha} \left( \frac{\partial L}{\partial v} [y](x) + \int_x^A \frac{\partial L}{\partial z} [y](t)dt \frac{\partial l}{\partial v} [y](x) \right) \right]_a = 0
\]
holds;

2. if $y(a) = y_a$, $y(B) = y_B$ and $y(A)$ is free, then the condition
\[
\left[ x I_A^{-\alpha} \left( \frac{\partial L}{\partial v} [y](x) + \int_x^B \frac{\partial L}{\partial z} [y](t)dt \frac{\partial l}{\partial v} [y](x) \right) \right]_A = 0
\]
holds;

3. if $y(a) = y_a$, $y(A) = y_A$ and $y(B)$ is free, then condition
\[
\left[ x I_B^{-\alpha} \left( \frac{\partial L}{\partial v} [y](x) + \int_x^B \frac{\partial L}{\partial z} [y](t)dt \frac{\partial l}{\partial v} [y](x) \right) \right]_B = 0
\]
holds.

Proof. Since $y$ is an extremizer of $J$, if we choose variations of $y$ of type $y + \epsilon h$ for which $h(a) = h(A) = h(B) = 0$, the two conditions of Theorem 3 follows. Then
\[
\left[ h(x) x I_B^{-\alpha} \left( \frac{\partial L}{\partial v} [y](x) + \int_x^B \frac{\partial L}{\partial z} [y](t)dt \frac{\partial l}{\partial v} [y](x) \right) \right]_a - \left[ h(x) x I_A^{-\alpha} \left( \frac{\partial L}{\partial v} [y](x) + \int_x^B \frac{\partial L}{\partial z} [y](t)dt \frac{\partial l}{\partial v} [y](x) \right) \right]_a = 0.
\]
If $y(A) = y_A$, $y(B) = y_B$ and $y(a)$ is free, then the variations must be such that $h(A) = h(B) = 0$, and so
\[
\left[ h(x) x I_B^{-\alpha} \left( \frac{\partial L}{\partial v} [y](x) + \int_x^B \frac{\partial L}{\partial z} [y](t)dt \frac{\partial l}{\partial v} [y](x) \right) \right]_a - \left[ h(x) x I_A^{-\alpha} \left( \frac{\partial L}{\partial v} [y](x) + \int_x^B \frac{\partial L}{\partial z} [y](t)dt \frac{\partial l}{\partial v} [y](x) \right) \right]_a = 0.
\]
Since $h(a)$ may take any values, condition 1 follows. The other two remaining conditions are proved in a similar way.

Remark 9. Repeat again to Theorem 2.2 in Gregory (2008) and Remark 8. If $y$ is an extremizer and $y(b)$ is free, then
\[
\left[ \frac{\partial L}{\partial v} [y](x) \right]_b = 0,
\]
which is the third condition of Theorem 3.
3 Numerical Method

One way to find the extremizers of functional $J$ is to solve the two fractional differential equations of Theorem 3. In practice, analytically solving such equations is usually a very difficult task or even impossible, and so several numerical procedures are being developed. Also, solving numerically these equations is difficult too. So, we use a numerical method in which we don’t need to solve the two fractional differential equations of Theorem 3. In this method, by using the Jacobi polynomials, the main problem is converted to a non-linear programming problem. To start, we present a short introduction on the Jacobi polynomials and their main properties.

3.1 Jacobi polynomials

Let $\alpha, \beta > -1$ be real parameters. The Jacobi polynomials $P_n^{(\alpha, \beta)}(t)$ of indices $\alpha$, $\beta$ and degree $n$ are defined by the formula

$$P_n^{(\alpha, \beta)}(t) = \sum_{k=0}^{n} \frac{(-1)^{n-k}(1+\beta)_n(1+\alpha+\beta)_{n+k}}{k!(n-k)!(1+\beta)_k(1+\alpha)_n} \left(\frac{t+1}{2}\right)^k,$$

where

$$(a)_0 = 1, \quad (a)_i = a(a+1) \ldots (a+i-1).$$

The Jacobi polynomials are mutually orthogonal over the interval $(-1,1)$ with respect to the weight function $w^{\alpha, \beta}(t) = (1-t)^\alpha(1+t)^\beta$. The Jacobi polynomials $P_n^{(\alpha, \beta)}(t)$ reduce to the Legendre polynomials $P_n(t)$ for $\alpha = \beta = 0$, and to the Chebyshev polynomials $T_n(t)$ and $U_n(t)$ for $\alpha = \beta = \pm 1/2$, respectively (see Canuto et al. (2007)).

In practice, we can compute the Jacobi polynomials using the following three-term recurrence formula (see Canuto et al. (2007) and Gautschi et al. (2004))

$$P_0^{(\alpha, \beta)}(t) = 1, \quad P_1^{(\alpha, \beta)}(t) = \frac{1}{2}(\alpha + \beta + 2)t + \frac{1}{2}(\alpha - \beta),$$

$$a_{1,k}^{\alpha, \beta} P_{k+1}^{(\alpha, \beta)}(t) = a_{2,k}^{\alpha, \beta} P_k^{(\alpha, \beta)}(t) - a_{3,k}^{\alpha, \beta} P_{k-1}^{(\alpha, \beta)}(t),$$

where

$$a_{1,k}^{\alpha, \beta} = 2(k+1)(k+\alpha+\beta+1)(2k+\alpha+\beta),$$

$$a_{2,k}^{\alpha, \beta}(t) = (2k+\alpha+\beta+1)(2k+\alpha+\beta+2)(2k+\alpha+\beta)t + \alpha^2 - \beta^2]$$

$$a_{3,k}^{\alpha, \beta} = 2(k+\alpha)(k+\beta)(2k+\alpha+\beta+2).$$

This three-term recurrence relation is generally quite stable and can thus be conveniently employed in the numerical computation of orthogonal polynomials. Among many known interesting properties of the Jacobi polynomials, we recall the following theorem from Esmaeili et al. (2011) and a derivative property.

Theorem 10. Esmaeili et al. (2011) Let $\alpha > 0$ be a real number and $x \in [a, b]$. Then,

$$C_a \frac{D_a^\alpha}{\Gamma(k+1)} \left(\frac{2(x-a)}{b-a} - 1\right) = \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)} P_k^{(\alpha, \beta)} \left(\frac{2(x-a)}{b-a} - 1\right).$$

Also, we have

$$\frac{d}{dx} \left[ P_n^{(\alpha, \beta)}(x) \right] = \frac{1}{2} (n + \alpha + \beta + 1) P_{n-1}^{(\alpha+1, \beta+1)}(x).$$
3.2 Presented method

At first, in order to apply the numerical method in a simple manner, we convert equation (2) to the following equivalent form:

\[ \dot{z}(x) = l(x, y(x), C_a D_x^\alpha y(x)), \]
\[ z(A) = 0. \]

So, we consider the equivalent problem:

\[ \min J(y) = \int_A^B L(x, y(x), C_a D_x^\alpha y(x), z(x)) dx, \]
\[ \dot{z}(x) = l(x, y(x), C_a D_x^\alpha y(x)), \]
\[ z(A) = 0, \]
\[ y(a) = y_a, \quad y(A) = y_A \quad \text{and} \quad y(B) = y_B. \]

Now, let \( a \neq A \neq B \), then we approximate \( y(x) \) as:

\[
y(x) \approx y_n(x) = y^*(x) + \sum_{i=2}^n c_i (x-a)^\alpha \left( \frac{P_i^{(0,\alpha)}(2(x-a)/B-a)}{B-a} - 1 \right) - \sum_{i=2}^n c_i (x-a)^\alpha \left( \frac{P_i^{(0,\alpha)}(2(A-a)/B-a) - 1}{A-B} (x-B) \right) - \sum_{i=2}^n c_i (x-a)^\alpha \left( \frac{P_i^{(0,\alpha)}(1)(A-x)}{B-A} \right),
\]

where \( y^*(x) = d_2x^2 + d_1x + d_0 \) and \( d_i, \ i = 0, 1, 2 \) are chosen such that \( y^*(a) = y_a, \ y^*(A) = y_A, \ y^*(B) = y_B \). Moreover, \( P_i^{(0,\alpha)}(x) \), \( i = 2, 3, \ldots, n \) are the Jacobi polynomials of indices 0, \( \alpha \) as defined in [13], and \( c_i, \ i = 2, 3, \ldots, n \) are unknown coefficients that should be determined. It is worthwhile to note that \( y_n(x) \) satisfied all conditions \( y_n(a) = y_a, \ y_n(A) = y_A, \ y_n(B) = y_B \).

By using Theorem [14] we can obtain \( C_a D_x^\alpha y_n(x) \) as:

\[
C_a D_x^\alpha y(x) \approx C_a D_x^\alpha y_n(x) = C_a D_x^\alpha y^*(x) + \sum_{i=2}^n c_i \left( \frac{I(i+\alpha+1)}{\Gamma(i+1)} P_i^{(0,\alpha)} \left( \frac{2(x-a)/B-a}{B-a} - 1 \right) - \sum_{i=2}^n c_i \left( \frac{P_i^{(0,\alpha)}(2(A-a)/B-a) - 1}{A-B} \Gamma(\alpha+1) ((\alpha+1)x - a\alpha - B) \right) - \sum_{i=2}^n c_i \left( \frac{P_i^{(0,\alpha)}(1)(A-x)}{B-A} ((\alpha+1)x - a\alpha - A) \right). \]

Note that, for obtaining the above equation, we use the following equation:

\[
C_a D_x^\alpha ((x-a)^\alpha(x-A)) = C_a D_x^\alpha ((x-a)^\alpha(x-a+a-A)) = ((\alpha+1)x - a\alpha - A) \Gamma(\alpha+1).
\]

Similarly, \( z(x) \) is approximated to

\[
z(x) \approx z_m(x) = \sum_{i=0}^m g_i (x-A) P_i^{(0,\alpha)}(2x-1).
\]

By using [11], we have

\[
\dot{z}_m(x) = \sum_{i=0}^m g_i P_i^{(0,\alpha)}(2x-1) + \sum_{i=1}^m g_i \left( (i+\alpha+1)(x-A) P_{i-1}^{(1,\alpha)}(2x-1) \right).
\]
By substituting (11)-(13) in $J$ and using a quadrature rule, we can approximate $J(y)$ as:

$$J(y) \approx J_{n,m}(y) = \int_{A}^{B} L(x, y_n(x), c D_x^a y_n(x), z_m(x)) \, dx$$

$$\approx \sum_{j=0}^{k} \omega_j L(\xi_j, y_n(\xi_j), c D_x^a y_n(\xi_j), z_m(\xi_j)),$$

where $\xi_j$ and $\omega_j$ are the nodes and weights of the quadrature rule. In addition, in order to obtain high order accuracy, the Gauss-Legendre quadrature Gautschi et al. (2004) and Trefethen (2000) is used. Note that the above approximation can be considered as a function of the unknown parameters $c_2, \cdots, c_n$ and $g_0, \cdots, g_m$.

Now, if we substitute (11), (12) and (14) in (8), and collocating in distinct nodes $\tau_j \in (A, B)$, $j = 0, \cdots, m$, the following $m + 1$ equation are obtained

$$\sum_{i=0}^{m} g_i P_i^{0, \alpha}(2\tau_j - 1) + \sum_{i=1}^{m} g_i \left[ (i + \alpha + 1)(\tau_j - A)P_{i-1}^{(1,1+\alpha)}(2\tau_j - 1) \right] = l(\tau_j, y_n(\tau_j), c D_x^a y_n(\tau_j)),

j = 0, \cdots, m.$$

It is noted that the above equations are $m + 1$ equations of the unknown parameters $c_2, \cdots, c_n$ and $g_0, \cdots, g_m$. We recall that the proper choice of collocation nodes $\tau_j$ is crucial for the accuracy of the obtained coefficients and its computational stability Gautschi et al. (2004) and Trefethen (2000). As a typically good choice of such collocation points, we can use the well-known Chebyshev-Gauss collocation nodes, i.e.

$$\tau_j = \frac{A + B}{2} - \frac{B - A}{2} \cos \left( \frac{(j + 1)\pi}{m + 1} \right), \quad j = 0, 1, \ldots, m.$$

In summary, the problem in (11)-(13) is converted to a mathematical programming problem with the unknown parameters $c_2, \cdots, c_n$ and $g_0, \cdots, g_m$, which minimize (15) subject to (16), i.e.,

$$\min I(c_2, \cdots, c_n, g_0, \cdots, g_m) = \sum_{j=0}^{k} \omega_j L(\xi_j, y_n(\xi_j), c D_x^a y_n(\xi_j), z_m(\xi_j)),$$

subject to:

$$\sum_{i=0}^{m} g_i P_i^{0, \alpha}(2\tau_j - 1) + \sum_{i=1}^{m} g_i \left[ (i + \alpha + 1)(\tau_j - A)P_{i-1}^{(1,1+\alpha)}(2\tau_j - 1) \right] - l(\tau_j, y_n(\tau_j), c D_x^a y_n(\tau_j)) = 0, \quad j = 0, \cdots, m.$$

In the above problem, $y_n(\xi_j), c D_x^a y_n(\xi_j)$ and $z_m(\xi_j)$ are defined in (11), (12) and (13), respectively.

### 4 Illustrative examples

To show the validity and efficiency of the procedure, we provide two examples motivated from Almeida et al. (2012) where the exact solution is given. In both cases it is easy to verify that the given $y_{\text{exact}}$ is the solution for the problem since in the examples $J(y) \geq 0$ for every admissible function and $J(y_{\text{exact}}) = 0$. Also, we mention that the extremizers are solutions of the two fractional differential equations that appear in Theorem 3.

**Example 11.** We will minimize the functional

$$J(y) = \int_{0.5}^{1} \left[ (\frac{d}{dx})^a y(x) - 1 \right]^2 + z(x) \, dx,$$
where
\[ z(x) = \int_{0.5}^{x} (y(t) - \frac{t^\alpha}{\Gamma(\alpha + 1)})^2 dt, \]
under the conditions
\[ y(0) = 0, \quad y(0.5) = (0.5)^\alpha \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)}, \quad y(1) = 1. \]

It is easy to verify that the exact solution is \( y_{\text{exact}}(x) = \frac{x^\alpha}{\Gamma(\alpha + 1)}. \)

Comparison of the exact solution and the numerical solution for some values of \( n \) and \( \alpha = 0.5 \) are shown in Fig. 1 (left). In Fig. 1 (right) the error between the exact solution and the numerical solution \( E(n) = y_n(x) - y_{\text{exact}}(x) \) for some values of \( n \) and \( x \in [0,1] \) are shown. In Fig. 2 the error between the exact solution and the numerical solution for \( n = m = 3,5 \) and \( n = m = 10,12 \) for \( x \in [0,1] \) and for \( x \in [0.5,1] \) are shown.

Example 12. We wish to minimize the functional
\[ J(y) = \int_{0.5}^{1} [C^i D_x^\alpha y(x) - \Gamma(\alpha + 2)x]^2 + z(x)] dx, \]
where
\[ z(x) = \int_{0.5}^{x} (y(t) - t^\alpha)^2 dt, \]
under the constraints
\[ y(0) = 0, \quad y(0.5) = (0.5)^\alpha \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)}, \quad y(1) = 1. \]

In this example, the exact solution is \( y_{\text{exact}}(x) = x^{\alpha+1}. \)

Comparison of the exact solution and the numerical solution for some values of \( n \) and \( \alpha = 0.5 \) are shown in Fig. 3 (left). In Fig. 3 (right) the error between the exact solution and the numerical solution \( E(n) = y_n(x) - y_{\text{exact}}(x) \) for some values of \( n \) and \( x \in [0,1] \) are shown. In Fig. 4 the error between the exact solution and the numerical solution for \( n = m = 3,5 \) and \( n = m = 10,12 \) for \( x \in [0,1] \) and for \( x \in [0.5,1] \) are shown.

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Figure 2: Error between the exact solution and the numerical solution for $n = m = 3, 5$ and $n = m = 10, 12$ for $x \in [0, 1]$ and for $x \in [0.5, 1]$ in Example 11.

Figure 3: Comparison of the exact solution and the numerical solution (left) and $E(n)$ (right) for some values of $n$ and $\alpha = 0.5$ in Example 12.
Figure 4: Error between the exact solution and the numerical solution for $n = m = 3, 5$ and $n = m = 10, 12$ for $x \in [0, 1]$ and for $x \in [0.5, 1]$ in Example 12.
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### References


