Linearly-constrained line-search algorithm for adaptive filtering

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We develop a linearly-constrained line-search adaptive filtering algorithm by incorporating the linear constraints into the least squares problem and searching the solution (filter weights) along the Kalman gain vector. The proposed algorithm performs close to the constrained recursive least squares (CRLS) algorithm while having a computational complexity comparable to the constrained least mean square (CLMS) algorithm. Simulations demonstrate its effectiveness.

Introduction: Linearly-constrained adaptive filtering algorithms are powerful signal processing tools with widespread use in several applications such as beamforming, blind multiuser detection in code-division multiple-access systems, and system identification [1]. The linear constraints usually emerge from the prior knowledge about the system, e.g., directions of arrival in array processing, spreading codes in blind multiuser detection, and linear phase in system identification [2].

Several linearly-constrained adaptive filtering algorithms have been proposed, which can be loosely divided into two categories:

1. stochastic-gradient-based algorithms such as the constrained least mean square (CLMS) algorithm [3] and the constrained affine projection (CAP) algorithm [2];
2. least-squares (LS)-based algorithms such as the constrained recursive least squares (CRLS) algorithm [4] and the constrained conjugate-gradient (CCG) algorithm [2].

The algorithms of the former category are generally simpler, more robust, and computationally more efficient than the algorithms of the latter category. However, they typically have slower convergence rate that is further aggravated when the input signal is correlated.

In this letter, we develop a linearly-constrained line-search algorithm by converting the constrained LS problem to an unconstrained one, which incorporates the linear constraints, and then solving it on a linear manifold defined by the Kalman gain vector. The proposed algorithm features CRLS-like performance with CLMS-like complexity.

Algorithm description: Let us consider a linear system described by

$$d_n = \omega^T x_n + r_n$$

where $d_n \in \mathbb{R}$ is the desired signal at time index $n \in \mathbb{N}$, $\omega \in \mathbb{R}^L$ is the vector of unknown system parameters, $x_n = [x_{n,1}, x_{n,2}, \ldots, x_{n,L+1}]^T$ is the input vector, $x_n \in \mathbb{R}$ is the input signal, $r_n \in \mathbb{R}$ is the background noise, and superscript $T$ stands for matrix transposition.

Adaptive filtering algorithms iteratively estimate $\omega$ exploiting the data available up to the current time. The tap weights of the adaptive filter, denoted by $w_n$, are taken as the estimate at iteration $n$. In linearly-constrained adaptive filtering, a set of constraints is imposed upon the filter weights at each time instant as

$$C^T \omega_n = f$$

where $C \in \mathbb{R}^{L \times K}$ is the constraint matrix ($K < L$) and $f \in \mathbb{R}^K$ is the response vector.

A constrained exponentially-weighted LS estimate can be found as

$$w_n = \arg \min \frac{1}{2} \| X_n w - d_n \|^2$$

subject to $C^T w = f$ (1)

where

$$X_n = \begin{bmatrix} x_{n,1}^{1/2} & x_{n,2}^{1/2} & \cdots & \lambda^{(n-1)/2} x_{n,L+1} \end{bmatrix},$$

$$d_n = \begin{bmatrix} d_{n,1}^{1/2} & d_{n,2}^{1/2} & \cdots & \lambda^{(n-1)/2} d_{n,L+1} \end{bmatrix}^T,$$

$\lambda$ denotes the Euclidean norm and $0 < \lambda \leq 1$ is the forgetting factor.

The solution of (1) obtained through the method of Lagrange multipliers is [4]

$$w_n = R_n^{-1} p_n + R_n^{-1} C (C^T R_n^{-1} C)^{-1} (f - C^T R_n^{-1} p_n)$$

where

$R_n = X_n X_n^T = \sum_{i=1}^{n} \lambda^{n-i} x_i x_i^T$

is the exponentially-weighted input autocorrelation matrix and

$$p_n = X_n d_n = \sum_{i=1}^{n} \lambda^{n-i} x_i d_i = X_n p_{n-1} + x_n d_n$$

is the exponentially-weighted cross-correlation vector between the input vector and the desired signal. The CRLS algorithm [4] is a recursive calculation of (2) that avoids the matrix inversions by applying the matrix inversion lemma.

By relaxing the hard constraints, we can convert (1) to an unconstrained problem that incorporates the constraints:

$$w_n = \arg \min \frac{1}{2} \| X_n w - d_n \|^2 + \mu^2 \| C^T w - f \|^2$$

and $\mu \gg 1$ is a weighting parameter that emphasises the certainty of the constraints. Note that this approach is the same as the method of weighting [5] and yields an approximate solution.

In the same vein as the line-search algorithms [6], we consider a first-order autoregressive model for the adaptive filter weights and search them along the Kalman gain vector, $k_n = R_n^{-1} x_n$, as

$$w_n = w_{n-1} + a_n k_n.$$ 

Hence, the problem turns into finding $a_n$ such that

$$a_n = \arg \min \frac{1}{2} \| X_n(w_{n-1} + a_n k_n) - d_n \|^2 + \mu^2 \| C^T(w_{n-1} + a_n k_n) - f \|^2$$

where

$$\frac{1}{2} \| X_n(w_{n-1} + a_n k_n) - d_n \|^2 + \mu^2 \| C^T(w_{n-1} + a_n k_n) - f \|^2 = \frac{1}{2} \| w_{n-1} - R_n^{-1} X_n x_n - a_n \mu^2 (d_n - X_n p_{n-1}) \|^2 + 2a_n (d_n - X_n p_{n-1})^T (X_n x_n + a_n \mu^2 r_n)$$

and the auxiliary variables are defined as

$$s_n = C^T k_n,$$

$$r_n = X_n w_{n-1} - f.$$ 

Equating the derivative of $\tilde{f}(\alpha)$ with respect to $\alpha$ to zero gives the solution of (3), which is

$$a_n = \frac{\sum_{i=1}^{n} r_i s_i}{\sum_{i=1}^{n} s_i^2}.$$ 

We call the proposed algorithm constrained line-search (CLS) and summarise it in Table 1 where, similar to CRLS, $k_n$ is updated using the stabilised fast transversal filter (S-FTF) algorithm [7]. The proposed algorithm requires $(2K + 13)L + 2K + 18$ multiplications per iteration while the CLMS and CRLS algorithms require $(3K^2 + 5K + 9)L$ and $K^2 + 2K + 16$ and $L^2 + 2L + 1$ multiplications per iteration, respectively.

Simulations: We consider a system identification problem where the filter length is an odd number and the filter weights are constrained to preserve linear phase at each iteration. Thus, the constraint matrix is $C = [I_K, 0, -I_K]^T \in \mathbb{R}^{L \times K}$ and the response vector is $f = 0 \in \mathbb{R}^K$ where $K = (L-1)/2$, $I_K \in \mathbb{R}^{K \times K}$ is the identity matrix and $J_K \in \mathbb{R}^{K \times K}$ is the reversal matrix (identity matrix with all rows in reversed order). The input signal is zero-mean unit-variance Gaussian. The unknown system is an arbitrary finite-impulse-response filter of order $L = 15$ with linear phase and unit energy. The background noise is white Gaussian with a power of 0.1. We set $\lambda = 0.995$ and obtain the results by ensemble-averaging over $10^3$ independent runs.

In Fig. 1, we plot the misalignment curves of the CRLS and CLMS algorithms as well as those of the proposed algorithm with different values of $\mu$. The misalignment is defined as $\| \omega - w_n \|^2$. It is clear that the larger $\mu$ is, the smaller the steady-state misalignment is but the
slower the algorithm converges and vice versa. In Fig. 2, we plot the
constraint mismatch of the proposed algorithm as a function of $\mu$. The
constraint mismatch is defined as $\|C^Tw_n - f\|^2$. As expected, with
larger $\mu$, the constraints are better satisfied.

In this experiment, the CRLS and CLMS algorithms respectively
require 2944 and 256 multiplications per iteration while the CLS
algorithm requires 437 multiplications per iteration.

Conclusion: We have developed a linearly-constrained line-search
(CLS) adaptive filtering algorithm by incorporating the linear
constraints into the LS problem and minimising the resultant cost
function on a linear manifold. The new algorithm performs nearly
as good as the CRLS algorithm with a considerably reduced computational
complexity, which is comparable to the CLMS algorithm.

We observe that a large value of the weighting parameter, $\mu$, results
in a lower misalignment (better estimation) and a lower constraint
mismatch (better satisfaction of the constraints) while it slows down the
convergence. On the other hand, a small $\mu$ results in a faster
convergence but to a higher misalignment and constraint mismatch.
Therefore, there exists a trade-off between estimation accuracy and
satisfaction of the constraints on one hand and the convergence rate of
the proposed algorithm on the other.

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