Modified Quasi-OBE Algorithm with Improved Numerical Properties

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Abstract

The quasi-OBE (QOBE) algorithm is a set-membership adaptive filtering algorithm based on the principles of optimal bounding ellipsoid (OBE) processing. This algorithm can provide enhanced convergence and tracking performance as well as reduced average computational complexity in comparison with the more traditional adaptive filtering algorithms such as the recursive least-squares (RLS) algorithm. In this paper, we show that the QOBE algorithm is prone to numerical instability due to the unbounded growth/decay of its internal variables. To tackle this problem, we develop a new set-membership adaptive filtering algorithm by transforming QOBE’s internal variables into a new set of internal variables. The new algorithm, called modified quasi-OBE (MQOBE), can be viewed as an exponentially-weighted RLS algorithm with a time-varying forgetting factor, which is optimized at each iteration by imposing a bounded-magnitude constraint on the a posteriori filter output error. The proposed algorithm delivers the same convergence and tracking performance as the QOBE algorithm but with enhanced numerical properties. We demonstrate the improved numerical behavior of the proposed algorithm by simulation examples for a MIMO channel estimation problem.

Keywords: set-membership adaptive filtering, the quasi-OBE (BEACON) algorithm, numerical stability

1. Introduction

Set-membership (SM) filtering algorithms are set-theoretic estimation methods that, unlike the more traditional methods, e.g., minimum-mean-squared-error (MMSE) or least-squares (LS) filters, estimate sets of feasible solutions rather than single-point solutions. The SM approaches are of particular interest in signal processing applications because they feature two major advantages over their more traditional counterparts. First, they exhibit superior adaptation and tracking properties. Second, they can effectively make use of innovation in the data and improve computational efficiency by establishing a data-discerning update strategy for the parameter estimates. More specifically, unlike the more traditional estimation schemes that implement a continuous update...
process regardless of the usefulness of the data, the SM algorithms assess the potential of the new data to improve the quality of the estimation and weigh the data accordingly. This intelligent update strategy results in discarding the data with unhelpful information content and obviating the expense of updating when the data is redundant. A more detailed background on the SM filtering paradigm can be found in [1-15] and the references therein.

An SM filtering algorithm is typically formulated as a set estimation problem and seeks solutions for a case that a certain constraining assumption is made about the filter output error. A usual assumption is a bounded magnitude for the filter output error. Several techniques have been proposed to estimate the target set of solutions, called membership set, under the bounded-error constraint. The most prominent ones are the optimal bounding ellipsoid (OBE) algorithms that approximate the membership set by tightly outer-bound it with ellipsoids in the parameter space and optimize the size of the ellipsoids in some meaningful sense. Different optimality criteria have led to different OBE algorithms. The first OBE algorithm was introduced in [1]. A thorough review of numerous further works developing the other members of the OBE family can be found in [9, 14].

Among all the OBE algorithms, the quasi-OBE (QOBE) algorithm\(^1\) [7, 14] is particularly attractive since it shares many of the desired features of the various OBE algorithms. Furthermore, it incorporates simple but efficient innovation check and optimal weight calculation processes, which make it computationally more efficient than other OBE algorithms.

In this paper, we show that the internal variables of the QOBE algorithm grow or decay unboundedly as the iterations progress. From a practical standpoint, this drawback limits the applicability of the algorithm to scenarios where the adaptation is performed for a sufficiently small number of iterations or the probability of update is low enough to prevent overflow or underflow of the internal variables. To overcome this problem, we introduce a change of the internal variables in the QOBE algorithm and develop a new SM adaptive filtering algorithm. The new algorithm can be seen as an exponentially-weighted recursive least-squares (EWRLS) algorithm with a time-varying forgetting factor that is optimized within the framework of the SM filtering. In this sense, it differs from most of the OBE algorithms, which can be viewed as weighted recursive least-squares (WRLS) algorithms with time-varying weights. A discussion on the differences between the WRLS and EWRLS algorithms can be found in [16]. Simulation results demonstrate that the numerical behavior of the proposed algorithm is appreciably improved in comparison with the QOBE algorithm while both algorithms have the same complexity and convergence performance.

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\(^1\) This algorithm was originally published as the \textit{Bounding Ellipsoidal Adaptive CONstrained least-squares} (BEACON) algorithm [7].
2. Set-membership adaptive filtering

Let us consider the affine-in-parameter model

\[ d_n = \boldsymbol{\omega}^* \boldsymbol{x}_n + \nu_n \]

where \( d_n \in \mathbb{C} \) is the reference signal at time index \( n \in \mathbb{N} \), \( \boldsymbol{\omega} \in \mathbb{C}^L \) is the column vector of the underlying system parameters to be identified, \( \boldsymbol{x}_n \in \mathbb{C}^L \) is the input vector, \( L \) is the system order, \( \nu_n \in \mathbb{C} \) accounts for measurement noise, and superscript * denotes complex-conjugate transpose.

If we take \( \boldsymbol{w} \in \mathbb{C}^L \) as an estimate of \( \boldsymbol{\omega} \), constraining the magnitude of the estimation error to be smaller than a pre-determined threshold, \( \gamma \in \mathbb{R}^+ \), yields a specification on \( \boldsymbol{w} \). Consequently, there will be a set of feasible solutions for \( \boldsymbol{w} \) rather than a single estimate. The set of all vectors satisfying the error constraint for all possible input-desired output pairs in the model space of interest, \( \mathcal{S} \), is called the feasibility set and is defined as

\[ \Theta = \bigcap_{(\boldsymbol{x},d) \in \mathcal{S}} \{ \boldsymbol{w} \in \mathbb{C}^L : |d - \boldsymbol{w}^* \boldsymbol{x}| \leq \gamma \}. \]

Even with perfect knowledge of \( \mathcal{S} \), which is not usually available, direct calculation of \( \Theta \) is formidable and computationally prohibitive. Hence, the adaptive SM filtering algorithms seek solutions that belong to a membership set, \( \Psi_n \), which is a superset of \( \Theta \) and is devised to be the minimal set estimate for \( \Theta \) at time instant \( n \). The membership set is defined as

\[ \Psi_n = \bigcap_{i=1}^{n} \Phi_i \]

where \( \Phi_n \) is the constraint set, which contains all the possible estimates that are consistent with the data observed at time instant \( n \). The set \( \Phi_n \) is also called the observation-induced set and is defined as

\[ \Phi_n = \{ \boldsymbol{w} \in \mathbb{C}^L : |d_n - \boldsymbol{w}^* \boldsymbol{x}_n| \leq \gamma \}. \]

The membership set \( \Psi_n \) is an \( L \)-dimensional convex polytope and still not easy to compute. Therefore, the OBE algorithms estimate a sequence of ellipsoids that tightly outer-bound \( \Psi_n \) instead. Different optimization approaches and optimality criteria regarding the bounding ellipsoids have given rise to the emergence of several OBE algorithms [9].

3. The QOBE Algorithm

3.1 Algorithm derivation

The QOBE algorithm is a quasi-OBE algorithm with an unusual optimization criterion (in the realm of OBE processing) that endows it with attractive features like enhanced tracking ability and reduced computational complexity. Although QOBE is built upon the OBE premises, it can be
regarded as a WRLS algorithm with a time-varying weighting factor, $\ell_n$, where the input autocorrelation matrix, $\tilde{R}_n$, and the filter coefficients are updated via

$$\tilde{R}_n = \tilde{R}_{n-1} + \ell_n x_n x_n^*$$  \hspace{1cm} (1)$$

and

$$w_n = w_{n-1} + \ell_n \tilde{P}_n x_n e_n^*$$  \hspace{1cm} (2)$$

with the \textit{a priori} estimation error being defined as

$$e_n = d_n - w_{n-1}^* x_n.$$

In practice, $\tilde{P}_n = \tilde{R}_n^{-1}$ is updated rather than $\tilde{R}_n$ by applying the matrix inversion lemma to (1), yielding

$$\tilde{P}_n = \tilde{P}_{n-1} - \frac{\ell_n \tilde{P}_{n-1} x_n x_n^* \tilde{P}_{n-1}}{1 + \ell_n x_n^* \tilde{P}_{n-1} x_n}.$$  \hspace{1cm} (3)$$

At time instant $n$, if $|e_n| \leq \gamma$, it is interpreted that $w_{n-1}$ is inside the constraint set $\Phi_n$, so there is no need for any update, i.e., $w_n = w_{n-1}$ and $\tilde{P}_n = \tilde{P}_{n-1}$. Conversely, $|e_n| > \gamma$ means that $w_{n-1}$ is outside $\Phi_n$; thus, it needs to be updated to a new vector, $w_n$, that lies inside $\Phi_n$. In this case, an update is carried out via (3) and (2) while the optimum value of the weighting factor, $\ell_n$, is found by satisfying the bounded-error-magnitude constraint

$$|d_n - w_n^* x_n| = \gamma,$$  \hspace{1cm} (4)$$

which ensures that $w_n$ is a member of $\Phi_n$ and consequently $\Psi_n$.

Multiplying both sides of (3) by $x_n^*$ from the left and by $x_n$ from the right, we find

$$x_n^* \tilde{P}_n x_n = \frac{\tilde{g}_n}{1 + \ell_n \tilde{g}_n}$$  \hspace{1cm} (5)$$

where

$$\tilde{g}_n = x_n^* \tilde{P}_{n-1} x_n.$$  \hspace{1cm} (6)$$

Multiplying the complex-conjugate transpose of both sides of (2) by $x_n$ from the right and subtracting the resulting equation from $d_n$ together with substitution of (5), we obtain

$$d_n - w_n^* x_n = \frac{1}{1 + \ell_n \tilde{g}_n} e_n.$$  \hspace{1cm} (7)$$

Substituting (7) into (4) results in

$$\ell_n = \frac{1}{\tilde{g}_n} \left( \frac{|e_n|}{\gamma} - 1 \right).$$  \hspace{1cm} (8)$$

This completes the derivation of the QOBE algorithm, which is summarized in Table 1.
3.2 Numerical behavior

With each update of the QOBE algorithm, the trace of $\tilde{R}_n$ increases (the trace of $\tilde{P}_n$ decreases). Growth of $\tilde{R}_{n-1}$ (decay of $\tilde{P}_{n-1}$) decreases $\tilde{g}_n$ and so increases $\ell_n$ while a larger $\ell_n$ can in turn accelerate the growth of $\tilde{R}_n$ (decay of $\tilde{P}_n$). This positive feedback can eventually lead to a blowup of $\tilde{R}_n$ and $\ell_n$ and vanishing of $\tilde{P}_n$ as the adaptation progresses. Theoretically, with a persistently exciting input, the filter coefficients of the QOBE algorithm will converge to their true values, no matter how large or small the internal variables become [14]. However, in practical finite-precision implementations, the algorithm may suffer from instability and ensuing underflow/overflow of its internal variables, $\tilde{P}_n$ and $\ell_n$. To be specific, $\tilde{P}_n$ may ultimately become a zero matrix and consequently adaptation may cease. In the aftermath, the filter coefficients may diverge from the true values in the presence of the background noise and/or if the target system is not stationary. In the following, in order to substantiate our intuitions, we investigate the limiting behavior of QOBE’s internal variables.

At the updating time instants, we have $|e_n| > \gamma$ or $|e_n|/\gamma - 1 > 0$. Besides, assuming a persistently exciting input signal, $\tilde{P}_{n-1}$ will be positive-definite; hence, $\tilde{g}_n > 0$. Therefore,

$$\ell_n > 0. \quad (9)$$

Calculating the trace of both sides of (1) gives

$$\text{tr}\{\tilde{R}_n\} = \text{tr}\{\tilde{R}_{n-1}\} + \ell_n \|x_n\|^2 \quad (10)$$

where $\text{tr}\{\cdot\}$ denotes the matrix trace. Let us make the following assumption.

A1: In addition to being persistently exciting, the input signal is stationary and has finite mean and variance. Hence, $0 < \|x_n\| < \infty$ where $\|\cdot\|$ denotes the Euclidean norm.

Considering (9) and A1, it is obvious from (10) that

$$\text{tr}\{\tilde{R}_n\} > \text{tr}\{\tilde{R}_{n-1}\}. \quad (11)$$

This means $\text{tr}\{\tilde{R}_n\}$ is monotone increasing. Hence, since the probability of update is normally non-zero at the steady state (e.g., with Gaussian noise), we can infer that

$$\lim_{n\to\infty} \text{tr}\{\tilde{R}_n\} = \infty. \quad (11)$$

In consideration of A1, we can conclude from (11) that

$$\lim_{n\to\infty} \text{tr}\{\tilde{P}_n\} = 0. \quad (12)$$

Rewriting (6) as

$$\tilde{g}_n = \text{tr}\{x_n x_n^* \tilde{P}_{n-1}\}$$
and invoking Coope’s inequality, i.e., \( \text{tr}(AB) \leq \text{tr}(A)\text{tr}(B) \) for positive-semidefinite matrices of the same order A and B [17], we have

\[
\tilde{g}_n \leq \|x_n\|^2 \text{tr}(\tilde{P}_n^{-1}).
\]

(13)

In view of A1 and (12), (13) implies that \( \lim_{n \to \infty} \tilde{g}_n = 0 \) and consequently

\[
\lim_{n \to \infty} \ell_n = \infty.
\]

(14)

The limiting results of (11) and (14) are sufficient to prove that the internal variables of the QOBE algorithm are unstable in nature.

It is worth mentioning that periodic resetting of the inverse autocorrelation matrix, \( \tilde{P}_n \), is a possible way to avoid the abovementioned numerical problems. However, due to frequent removal of past information from \( \tilde{P}_n \), such a solution would be far from optimal [18].

4. The MQOBE Algorithm

4.1 Algorithm derivation

Let us define

\[
R_n = \ell_n^{-1} \tilde{R}_n,
\]

(15)

\[
P_n = R_n^{-1},
\]

(16)

and

\[
\lambda_n = \frac{\ell_{n-1}}{\ell_n}.
\]

(17)

Multiplying both sides of (1) by \( \ell_n^{-1} \) and using (15) and (16) in (2), we find

\[
R_n = \lambda_n R_{n-1} + x_n x_n^*
\]

(18)

and

\[
w_n = w_{n-1} + P_n x_n e_n^*.
\]

(19)

Applying the matrix inversion lemma to (18) yields

\[
P_n = \lambda_n^{-1} \left( P_{n-1} - \frac{P_{n-1} x_n x_n^* P_{n-1}}{\lambda_n + x_n^* P_{n-1} x_n} \right).
\]

(20)

Substituting (8) into (17) gives

\[
\lambda_n = \ell_{n-1} \frac{\tilde{g}_n}{\|e_n\|^2}.
\]

(21)

By rewriting (6) as
\[ \tilde{g}_n = x_n^* P_{n-1}^\ell \ell_{n-1}^{-1} P_{n-1} x_n \]

and substituting it into (21), we get

\[ \lambda_n = \frac{g_n}{|e_n|} - 1 \]  
(22)

where

\[ g_n = x_n^* P_{n-1} x_n. \]  
(23)

As with the QOBE algorithm, at time instant \( n \), an update occurs only if \( |e_n| > \gamma \); otherwise, \( w_n = w_{n-1} \) and \( P_n = P_{n-1} \).

We call the resultant algorithm \textit{modified quasi-OBE} (MQOBE) and summarize it in Table 2. The new algorithm is in fact an EWRLS algorithm with a time-varying forgetting factor that is optimized to satisfy the set-membership-induced error bound in (4).

If we multiply both sides of (20) by \( x_n^* \) from the left and by \( x_n \) from the right, we get

\[ x_n^* P_n x_n = \frac{g_n}{\lambda_n + g_n}. \]  
(24)

By multiplying the complex-conjugate transpose of both sides of (19) by \( x_n \) from the right and subtracting the resulting equation from \( d_n \) together with using (24), we can verify that

\[ d_n - w_n^* x_n = \frac{1}{1 + \lambda_n^{-1} g_n} e_n. \]  
(25)

Therefore, we can alternatively attain (22) by substituting (25) into (4) and solving it with respect to \( \lambda_n \).

\subsection*{4.2 Numerical behavior}

The QOBE and MQOBE algorithms compute the same filter coefficients but in different ways because of having different internal variables, namely \( \tilde{P}_n \) and \( \ell_n \) in QOBE versus \( P_n \) and \( \lambda_n \) in MQOBE. In QOBE, \( \ell_n \) is a weighting factor, whereas in MQOBE, \( \lambda_n \) acts as a forgetting factor and serves to stabilize the internal variables by means of a negative feedback.

The recursion of (18) represents a first-order autoregressive process with a variable forgetting factor \( (\lambda_n) \). Under A1, it is known that such a process is surely stable if the forgetting factor is always smaller than one, whereas it is surely unstable if the forgetting factor is always greater than one [19]. However, the latter can never be the case since \( \lambda_n \) being always greater than one results in

\[ \lim_{n \to \infty} \text{tr}(R_n) = \infty \]

or
\[
\lim_{n \to \infty} \text{tr}\{P_n\} = 0
\]

which in turn leads to

\[
\lim_{n \to \infty} \lambda_n = 0,
\]

contradicting the initial assumption. In fact, it can be deduced from (14) and (17) that for most of the time instants we have \(\lambda_n < 1\). More specifically, since \(\ell_n\) grows in time (though not monotonically), at most of the time instants \(\ell_n > \ell_{n-1}\), which means \(\lambda_n < 1\).

In order to study the numerical behavior of MQOBE, one should carefully examine the inter-variable relations in the algorithm, especially the algorithm’s innate negative feedback mechanism that plays a key role in maintaining the stability of the internal variables. As an example, assuming A1 and convergence of the algorithm in the mean-squared error sense, one can verify that decrease of \(P_{n-1}\) (increase of \(R_{n-1}\)) decreases \(g_n\) and hence decreases \(\lambda_n\), whereas a smaller \(\lambda_n\) leads to a greater \(P_n\) (smaller \(R_n\)); on the other hand, increase of \(P_{n-1}\) (decrease of \(R_{n-1}\)) increases \(g_n\) and hence increases \(\lambda_n\), whereas a larger \(\lambda_n\) leads to a smaller \(P_n\) (greater \(R_n\)).

To demonstrate the mechanism of the abovementioned negative feedback and its stabilizing effect, we consider the special case of uncorrelated input signal by adopting the following assumption.

A2: The elements of the input signal vector are stationary correlation-ergodic and uncorrelated with zero mean and variance \(\sigma_x^2\); hence, \(E[x_n x_n^*] = \sigma_x^2 I_L\) where \(I_L\) is the \(L \times L\) identity matrix.

From (18), we have

\[
R_n = \prod_{i=1}^{n-1} \lambda_i R_0 + \sum_{i=1}^{n-1} \left( \prod_{j=i+1}^{n} \lambda_j \right) x_i x_i^* + x_n x_n^*
\]

where \(R_0 = \delta I_L\) is the initial covariance matrix with \(\delta\) set to a small positive number. For the sake of simplicity, let us assume that an update occurs at every iteration. Since the input signal is correlation-ergodic, as \(n \to \infty\), \(R_n\) can be approximated by

\[
\lim_{n \to \infty} R_n \approx \lim_{n \to \infty} E[R_n]
\]

\[
\approx \lim_{n \to \infty} E \left[ \prod_{i=1}^{n} \lambda_i R_0 + \sum_{i=1}^{n-1} \left( \prod_{j=i+1}^{n} \lambda_j \right) x_i x_i^* + x_n x_n^* \right]
\]

\[
\approx \sigma_x^2 \left( \lim_{n \to \infty} \sum_{i=1}^{n-1} E \left[ \prod_{j=i+1}^{n} \lambda_j \right] + 1 \right) I_L.
\]
where \( \prod_{i=1}^{\infty} \lambda_i \approx 0 \) by virtue of the fact that \( \lambda_n \) cannot remain larger than one for many successive update iterations (see Fig. 3 for an illustration of this). Consequently, \( r_\infty \) is assured to be finite. Note that \( r_\infty \) cannot possibly diverge to infinity as this would result in \( \lambda_\infty \to 0 \), thereby preventing any divergence in the first place. Equation (26) suggests that, at the steady state and under A2, \( R_n \) can be approximated as a multiple of the identity matrix.

5. Simulations

We compare the performance of the MQOBE algorithm with the QOBE algorithm and the conventional RLS algorithm in an application of flat-fading MIMO channel estimation studied in [20]. For this purpose, a MIMO communication system with four transmitter and four receiver antennas is considered. The sub-channels between all the transmitter and receiver pairs are independent Rayleigh-fading and vary in time based on Jakes model [21] with a normalized Doppler frequency of \( f_D T_s = 0.01 \) where \( f_D \) is the maximum Doppler frequency shift and \( T_s \) is the transmission symbol period. A sudden random change in the channel taps is also introduced halfway through the simulations.

Four finite impulse response (FIR) filters each having four taps constitute the MIMO channel estimator. Similar to [20], in the MQOBE and QOBE algorithms, the Euclidean norm of the error vector composed by the output errors of all the filters is used for the considered MIMO case in place of the absolute of the scalar error in the SISO case. The transmitted symbols are modulated using QPSK scheme and grouped into packets of data each containing 80 symbol vectors. These vectors are the common input to the filters of the MIMO channel estimator. For the RLS algorithm, a fixed forgetting factor of 0.9 is used regarding the assumed normalized Doppler frequency. For the MQOBE and QOBE algorithms, the error threshold is set to \( \gamma = 0.1 \). The energy per bit to noise power spectral density ratio is also \( E_b/N_0 = 12 \) dB.

In Fig. 1, we compare the estimation performance of the algorithms in terms of mean squared error (MSE), ensemble-averaged over \( 10^4 \) independent trials. Fig. 2 shows the trace of the inverse input autocorrelation matrix versus time for different algorithms. Fig. 3 also shows time evolution of the optimal weighting factor of the QOBE algorithm and the optimal forgetting factor of the MQOBE algorithm together with the fixed forgetting factor of the RLS algorithm. It should be noted that, in this experiment, both QOBE and MQOBE updated in average at 80 percent of the iterations. Figs. 4-6 correspond to Figs. 1-3, respectively, when the simulations are performed for 3000 iterations without any sudden change in the channel taps.

We observe that the MQOBE and QOBE algorithms have the same estimation performance as long as QOBE’s internal variables are in the realizable range, which is \( 4.94 \times 10^{-324} \) to \( 1.79 \times 10^{308} \) for the double-precision floating-point format. However, when its internal variables hit the
numerical limits, the QOBE algorithm stalls. The simulation results emphasize that the QOBE algorithm is prone to numerical instability, in particular, when the update frequency is high. It is also evident that QOBE’s internal variables, $\ell_n$ and $\tilde{P}_n$, exponentially grow/decay in time. This makes their dynamic range extremely wide and consequently QOBE’s practicable run-time very limited. In Figs. 2 and 3, QOBE’s optimal weight grows from about 1 to more than $10^{15}$ and the trace of its inverse autocorrelation matrix drops from about 10 to less than $10^{-15}$ in only 80 iterations. On the other hand, the internal variables of MQOBE fluctuate around their steady-state values and are of much smaller dynamic range rendering MQOBE more suitable than QOBE for practical applications.

6. Conclusion

It was shown that the set-membership-based QOBE algorithm, despite all its merits, suffers from numerical instability of its internal variables. To resolve this drawback, an appropriate transformation of the internal variables was introduced leading to a new set-membership adaptive filtering algorithm, called MQOBE. Unlike most of the OBE algorithms that are known as set-membership weighted RLS algorithms, the proposed algorithm is a set-membership exponentially-weighted RLS algorithm with a time-varying forgetting factor, which is optimized within the context of the set-membership filtering. The proposed algorithm exhibits a good convergence and tracking performance, identical to QOBE, while providing a dramatically improved numerical behavior.

Acknowledgment

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References


Table I, The QOBE algorithm.

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<th>Initialization:</th>
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<td>$\tilde{P}_0 = \delta^{-1} I_L$, where $\delta$ is a small positive number</td>
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<td>$w_0 = \mathbf{0}$</td>
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Table II, The MQOBE algorithm.

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Fig. 1, Mean squared error performance of different algorithms.

Fig. 2, Trace of the inverse autocorrelation matrix versus time for different algorithms.
Fig. 3. Time evolution of the optimal weight of the QOBE algorithm and the optimal forgetting factor of the MQOBE algorithm together with the fixed forgetting factor of the RLS algorithm.

Fig. 4. Mean squared error performance of different algorithms.
Fig. 5, Trace of the inverse autocorrelation matrix versus time for different algorithms.

Fig. 6, Time evolution of the optimal weight of the QOBE algorithm and the optimal forgetting factor of the MQOBE algorithm together with the fixed forgetting factor of the RLS algorithm.