Spectrum of prime $L$-submodules

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Abstract

Let $L$ be a complete lattice. We introduce and characterize the prime $L$-submodules of a unitary module over a commutative ring with identity. Finally, we investigate the Zariski topology on the prime $L$-Spectrum of a unitary module, consisting of the collection of all prime $L$-submodules, and prove that for $L$-top modules the Zariski topology on $L$-$Spec(M)$ exists.

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1. Introduction

Let $R$ be a commutative ring with identity and $M$ be an unitary $R$-module. The prime spectrum, $Spec(R)$, and topological space obtained by introducing Zariski topology on the set of all prime ideals of a commutative ring with identity plays an important role in the field of commutative algebra, algebraic geometry and lattice theory. In the last few years a considerable amount of work has been done on fuzzy ideals in general and prime fuzzy ideals in particular, and some interesting topological properties of the spectrum of fuzzy prime ideals of a ring are obtained (see [2,4–8,11]).

Goguen [3] replaced the unit interval by a complete lattice $L$ in the definition of fuzzy sets [18] and introduced the notion of $L$-fuzzy sets. The concept of fuzzy submodules was first introduced by Negoita and Ralescu in 1975 [14] and subsequently studied, among others, by Pan, [15] in 1987. The notion of a fuzzy primary submodules is studied in many papers, (for example see [12,16]). Recently, the notion of prime submodules and Zariski topology on $Spec(M)$, the set of prime submodules of a module $M$ over a commutative ring $R$, are studied by many authors (for example see [9,10,12]). In this paper we introduce the notion of prime $L$-submodules of a module over a commutative ring with identity say $R$, where $L$ is a complete lattice. Note that in [1] the term of “fuzzy prime submodules” is used in different way. In fact our definition is a generalization of the notion of ordinary prime submodules which appears in current literature of algebra (for example see [9,10,12]), also this definition agrees with the definition of a prime $L$-ideal in [7], when we replace $M$ with $R$. We will investigate some basic properties of prime $L$-submodules and characterize the prime $L$-submodules of $M$ (Theorem 3.4) and establish relationships between primeless and $L$-primeless for a given module via the role of the lattice $L$ (Theorems 4.1–4.3 and Example 3.15). Finally, we investigate the Zariski topology on $L$-$Spec(M)$, the set of all prime $L$-submodules of $M$ and show that for $L$-top modules Zariski topology

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on $L$-$Spec(M)$ exists. The paper provides the suitable tools to define and study the properties of Zarisky topology of prime $L$-submodules. And hence it can be considered as an introduction to fuzzy spectral theory.

2. Preliminaries

Throughout this paper by $R$ we mean a commutative ring with identity and $M$ is an unital $R$-modules, $L$ denotes a complete lattice. $L$ is regular if for all $a, b \in L$ such that $a \neq 0, b \neq 0$, then $a \land b \neq 0$. By an $L$-subset $\mu$ of a non-empty set $X$, we mean a function $\mu$ from $X$ to $L$. If $L = [0, 1]$, then $\mu$ is called a fuzzy subset of $X$. $L^X$ denotes the set of all $L$-subsets of $X$. Let $A$ be a subset of $X$ and $y \in L$. Define $y_A \in L^X$ as follows:

$$y_A(x) = \begin{cases} y & \text{if } x \in A, \\ 0 & \text{otherwise.} \end{cases}$$

In special case if $A = \{a\}$, we denote $y_{\{a\}}$ by $y_a$, and it is called an $L$-point of $X$.

For $\mu, v \in L^X$ we say that $\mu$ is contained in $v$ and we write $\mu \subseteq v$ if $\mu(x) \leq v(x)$, for all $x \in X$. For $\mu, v \in L^M$, the intersection and union, $\mu \cap v, \mu \cup v \in L^X$ are defined by

$$(\mu \cap v)(x) = \mu(x) \land v(x) \quad \text{and} \quad (\mu \cup v)(x) = \mu(x) \lor v(x) \quad \text{for all } x \in X.$$  

Also for $\mu \in L^X, a \in L, \mu_a$ is defined by

$$\mu_a = \{x \in M | \mu(x) \geq a\},$$

where $\mu_a$ is called $a$-cut or $a$-level subset of $\mu$.$^*$

Let $f$ be a mapping from $X$ into $Y$ and let $\mu \in L^X, v \in L^Y$. Then $f(\mu) \in L^Y$ and $f^{-1}(v) \in L^X$ are defined as follows:

$$f(\mu)(y) = \begin{cases} \bigvee \{\mu(x)| x \in f^{-1}(y) \} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise} \end{cases}$$

and

$$f^{-1}(v)(x) = v(f(x)) \quad \forall x \in X.$$  

This is the extension principle.

Let $M, N$ be $R$-modules and $f : M \rightarrow N$ be an $R$-module homomorphism. A $\mu \in L^M$ is called $f$-invariant if $f(x) = f(y)$, implies that $\mu(x) = \mu(y)$ for all $x, y \in M$.

We recall some definitions and theorems from the book [13], which we need them for development of our paper.

**Definition 2.1.** Let $\mu \in L^R$. Then $\mu$ is called an $L$-ideal of $R$ if for every $x, y \in R$ the following conditions are satisfied:

1. $\mu(x - y) \supseteq \mu(x) \land \mu(y)$;
2. $\mu(xy) \supseteq \mu(x) \lor \mu(y)$ and
3. $\mu(0) = 1$.

The set of all $L$-ideals of $R$ is denoted by $LI(R)$.

**Definition 2.2.** (Extension principle). Let $\mu, v \in LI(R)$. We define $\mu v \in LI(R)$ as follows:

$$\mu v(x) = \bigvee \{\mu(y) \land v(z) | y, z \in R, x = yz\} \quad \text{for all } x \in R.$$  

**Definition 2.3.** Let $R$ be a ring and $\zeta \in LI(R)$. Then $\zeta$ is called prime $L$-ideal of $R$ if $\zeta$ is non-constant and for every $\mu, v \in LI(R), \mu v \subseteq \zeta$ implies that $\mu \subseteq \zeta$ or $v \subseteq \zeta$.

**Definition 2.4.** Let $\mu$ be an $L$-subset of $R$. The radical of $\mu$ is denoted by $\mathcal{R}(\mu)$ is defined by

$$\mathcal{R}(\mu)(x) = \bigvee_{n \in \mathbb{N}} \mu(x^n) \quad \text{for all } x \in R.$$
Theorem 2.5. Let $\zeta \in LI(R)$ be prime, then $\mathfrak{m}(\zeta) = \zeta$.

Definition 2.6. Let $c \in L \backslash \{1\}$. Then $c$:

(i) is called a prime element of $L$ if $a \land b \leq c$, implies that $a \leq c$ or $b \leq c$, for all $a, b \in L$ and

(ii) $c$ is called a maximal element if there does not exist $a \in L \backslash \{1\}$ such that $c < a < 1$.

Theorem 2.7. Let $\zeta \in L^R$. Then $\zeta$ is a prime $L$-ideal $R$ if and only if $\zeta(0) = 1$ and $\zeta = 1_{\ast} \cup c_R$ such that $\zeta$ is a prime ideal of $R$ and $c$ is a prime element of $L$.

Definition 2.8. Let $\zeta \in L^R$ and $\mu \in L^M$. Define $\zeta \cdot \mu \in L^M$ as follows:

$$(\zeta \cdot \mu)(x) = \bigvee \{\zeta(r) \land \mu(y) | r \in R, y \in M, ry = x\} \text{ for all } x \in M.$$

Definition 2.9. An $L$-submodule of $M$ is an $L$-subset $\mu \in L^M$ such that:

(1) $\mu(0) = 1$;
(2) $\mu(rx) \supseteq \mu(x)$ for all $r \in R$ and $x \in M$ and
(3) $\mu(x + y) \supseteq \mu(x) \land \mu(y)$ for all $x, y \in M$.

The set of all $L$-submodules of $M$ is denoted by $L(M)$.

Remark 2.10. If $M = R$, then it is easy to verify that $\mu \in L^R$ is an $L$-submodule of $M$ if and only if $\mu$ is an $L$-ideal of $R$.

Theorem 2.11. Let $\mu \in L^M$. Then $\mu \in L(M)$ if and only if each non-empty level subset of $\mu$ is a submodule of $M$. Moreover if $\mu \in L(M)$ then $\mu_\ast = \{x \in M | \mu(x) = 1\}$ is a submodule of $M$.

Theorem 2.12. Let $x \in M$ and $a \in L$, then

$$< x_a > = I_{\{0\}} \cup (\bigcup \{(rx)_a | r \in R\})$$

Definition 2.13. For $\mu, v \in L^M$ and $\zeta \in L^R$, define $\mu : v \in L^R$ and $\mu : \zeta \in L^M$ as follows:

$$\mu : v = \bigcup \{\eta \in L^R | \eta \cdot v \subseteq \mu\},$$
$$\mu : \zeta = \bigcup \{v \in L^M | \zeta \cdot v \subseteq \mu\}.$$

In [13] it was proved that if $v \in L^M$, $\mu \in L(M)$ and $\zeta \in LI(R)$ then $\mu : v = \bigcup \{\eta | \eta \in LI(R), \eta \cdot v \subseteq \mu\}$ and $\mu : \zeta = \bigcup \{v | v \in L(M), \zeta \cdot v \subseteq \mu\}$.

Theorem 2.14. If $\mu \in L(M), v \in L^M, \zeta \in LI(R)$, then $\mu : v \in LI(R)$ and $\mu : \zeta \in L(M)$.

We recall that in [13] an $L$-submodule $\mu$ of $M$ is called primary if for $\zeta \in LI(R)$ and $v \in L(M)$, $\zeta \cdot v \subseteq \mu$ implies that $v \subseteq \mu$ or $\zeta \subseteq \mathfrak{m}(\mu : 1_M)$.

Let $N$ be a submodule of $M$. As usual by $N : M$ we mean $N : M = \{r \in R | rM \subseteq N\}$. A submodule $N$ of an $R$-module $M$ is called prime if for all $r \in R$ and $x \in M$, if $rx \in N$, then $x \in N$ or $r \in N : M$. In [12] it was proved that if a submodule $N$ of an $R$-module $M$ is prime, then $N : M$ is a prime ideal of $R$. Also, in [9] it was proved that every maximal submodule of a module is prime.

Theorem 2.15. Let $c \in L$ and $N$ be a submodule of $M$. Then

$$(1_N \cup c_M) : 1_M = 1_{N : M} \cup c_R.$$
3. Prime \(L\)-submodules

In this section we introduce the notion of prime \(L\)-submodules and investigate some basic properties of them.

**Definition 3.1.** A non-constant \(L\)-submodule \(\mu\) of \(M\) is said to be prime if for \(\zeta \in LI(R)\) and \(v \in L(M)\) such that \(\zeta \cdot v \subseteq \mu\) then either \(v \subseteq \mu\) or \(\zeta \subseteq 1_M\).

**Notation.** By \(L\)-\(\text{Spec}(M)\), we mean the set of all prime \(L\)-submodules of \(M\).

In the next proposition we show that the notion of prime \(L\)-submodules is a generalization of the notion of prime \(L\)-ideals are given in 2.3.

**Proposition 3.2.** Let \(M = R\). Then an \(L\)-submodule of \(M\) is a prime \(L\)-submodule if and only if it is a prime \(L\)-ideal of \(R\).

**Proof.** The proof immediately follows from Remarks 2.10 and 3.4 in [17]. \(\Box\)

**Remark 3.3.** For \(M\) the following statements hold:

(i) If \(\mu \in L\)-\(\text{Spec}(M)\), then \(\mu\) is primary.

(ii) Note that our definition of prime \(L\)-submodule is different from the definition of prime \(L\)-submodules given in [1] even \(L = [0, 1]\). In fact this definition is a generalization of the notion of prime submodules in module theory, as well as it agrees with definition of prime \(L\)-ideals by Proposition 3.2. Also we remark that the concept of prime \(L\)-submodules in Definition 3.1 implies that the concept of \(s\)-prime \(L\)-submodules in Definition 3.1 in [16].

In the following theorem we give an interesting characterization of prime \(L\)-submodules.

**Theorem 3.4.** Let \(\mu\) be an \(L\)-submodule of \(M\). Then \(\mu\) is prime if and only if \(\mu = 1_{\mu_s} \cup c_M\) such that \(\mu_s\) is a prime submodule of \(M\) and \(c\) is a prime element of \(L\).

**Proof.** The Proof follows from Remark 3.3 and Proposition 3.3 in [16]. \(\Box\)

With the help of the above theorem we give another characterization of prime \(L\)-submodules.

**Theorem 3.5.** Let \(\mu \in L(M)\), then \(\mu \in L\)-\(\text{Spec}(M)\) if and only if \(\mu\) satisfies the following conditions:

(i) \(\mu_s\) is a prime submodule of \(M\);

(ii) \((\mu : 1_M)(1)\) is a prime element in \(L\) and

(iii) if \(r_a . x_b \subseteq \mu\), \(r \in R\), \(x \in M\) and \(a, b \in L\), then \(r_a \subseteq (\mu : 1_M)\) or \(x_b \subseteq \mu\).

**Proof.** Suppose that \(\mu \in L\)-\(\text{Spec}(M)\). Then, by Theorem 3.4 \(\mu = 1_{\mu_s} \cup c_M\) such that \(\mu_s\) is a prime submodule of \(M\) and \(c\) is a prime element in \(L\). Thus (i) holds.

(ii) \((\mu : 1_M)(1) = (1_{\mu_s} \cup c_M : 1_M)(1) = (1_{\mu_s : M} \cup c_R)(1) = c\), since \(1 \notin \mu_s : M\), and \(c\) is a prime element in \(L\) by Theorem 3.4.

(iii) Suppose that \(r_a . x_b \subseteq \mu\) for \(r \in R\), \(x \in M\), and \(a, b \in L\), but \(r_a \not\subseteq \mu_s : M = 1_{\mu_s : M} \cup c_M\) and \(x_b \not\subseteq \mu\). Then \(r \notin \mu_s : M\) and \(x \notin \mu_s\). Thus \(rx \notin \mu_s\), since \(\mu_s\) is a prime submodule. Also we have

\[
a \wedge b = r_a(r) \wedge x_b(x) \subseteq (r_a . x_b)(rx) \subseteq \mu(rx) = c.
\]

Therefore, \(a \wedge b \leq c\) and since \(r_a \not\subseteq \mu : 1_M\) and \(x_b \not\subseteq \mu\), then \(a \not\subseteq (\mu : 1_M)(r) = c\) and \(b \not\subseteq \mu(x) = c\), which is a contradiction. Then \(r_a \not\subseteq (\mu : 1_M)\) or \(x_b \not\subseteq \mu\). Conversely, suppose that \(\mu\) satisfies in (i)–(iii). Suppose that \(c = (\mu : 1_M)(1)\) is a prime element in \(L\) and \(a = 1\). Let \(x \in M \setminus \mu_s\) and \(b = \mu(x)\), then we have \(1_b . x_a = x_b \subseteq \mu\).
and $x \notin \mu$, then by (iii), $1_b \subseteq \mu : 1_M$, and hence
\[
b = 1_b(1) \leq (\mu : 1_M)(1) = \sqrt{\{\eta(1) | \eta, 1_M \leq \mu\}}
= \sqrt{\{\eta(1) \land 1_M(x) | \eta, 1_M \leq \mu\}}
\leq \sqrt{\{(\eta, 1_M)(x) | \eta, 1_M \leq \mu\}} \leq \mu(x) = b.
\]

Thus $\mu(x) = c = (\mu : 1_M)(1)$, and hence $\mu = 1_{\mu_c} \cup c_M$. This means that $\mu$ is prime. $\Box$

**Theorem 3.6.** If $\mu \in L \text{-} \text{Spec}(M)$, then $\mu : 1_M$ is a prime $L$-ideal of $R$.

**Proof.** Suppose that $\mu \in L \text{-} \text{Spec}(M)$, then $\mu = 1_{\mu_c} \cup c_M$, where $\mu_c$ is a prime submodule of $M$ and $c$ is a prime element in $L$. Then $\mu_c : M$ is a prime ideal of $R$ and
\[
\mu : 1_M = 1_{\mu_c} : M \cup c_R.
\]
Thus by Theorem 2.7, $\mu : 1_M$ is prime. $\Box$

Note that the converse of Theorem 3.6 dose not hold in general. But, in the next result we show that the converse of Theorem 3.6 is true for primary $L$-submodule.

**Theorem 3.7.** Let $\mu \in L(M)$ be primary. Then $\mu$ is prime if and only if $(\mu : 1_M)$ is prime $L$-ideal.

**Proof.** If $\mu$ is prime, then by Theorem 3.6 $\mu : 1_M$ is a prime $L$-ideal of $R$. Conversely, suppose that $\mu : 1_M$ is a prime $L$-ideal and for $\zeta \in LI(R)$, $\varphi \in L(M)$, we have $\zeta \cdot \varphi \subseteq \mu$. Now if $\nu \notin \mu$, then $\zeta \subseteq \mathfrak{f}(\mu : 1_M)$, since $\mu$ is primary. Thus $(\mu : 1_M)$ is prime, and hence $\mathfrak{f}(\mu : 1_M) = \mu : 1_M$. Therefore $\zeta \subseteq (\mu : 1_M)$, and so, $\mu$ is prime. $\Box$

**Definition 3.8.** An $L$-submodule $M$ is called maximal if $\mu$ is a maximal element in the set of all non-constant $L$-submodules of $M$, with respect to the set inclusion.

**Theorem 3.9.** If $\mu \in L(M)$ is maximal then $\mu = 1_{\mu_c} \cup c_M$ such that $\mu_c$ is a maximal submodule of $M$ and $c$ is a maximal element of $L$.

**Proof.** The proof is similar to the proof of Theorem 3.4, up to some manipulations. $\Box$

**Corollary 3.10.** If $\mu$ is a maximal fuzzy submodule of $M$, then $\mu$ is a fuzzy prime submodule.

**Proof.** Let $\mu$ be a maximal fuzzy submodule of $M$, then by Theorem 3.9 we obtain that $\mu = 1_{\mu_c} \cup c_M$, such that $\mu_c$ is a maximal submodule of $M$ and $c$ is a maximal element of $L$. Since every maximal submodule is prime by [9] and since every element of $[0,1]$ is prime element, then in virtue of Theorem 3.4 it follows that $\mu$ is prime. $\Box$

**Corollary 3.11.** If $\mu \in L \text{-} \text{Spec}(M)$, then $1_R \cdot \mu = \mu$.

**Proof.** We have
\[
\mu(x) \leq (1_R \cdot \mu)(x) = \sqrt{\{\mu(y) | ry = x \text{ for some } y \in M\}},
\]
and $\mu = 1_{\mu_c} \cup c_M$, where $\mu_c$ is a prime submodule of $M$ and $c$ is a prime element in $L$. If $x \notin \mu_c$, then $\mu(x) = 1$ and hence $\mu(x) = (1_R \cdot \mu)(x)$. If $x \notin \mu_c$ then for all decompositions of $x$ as $ry = x$ for some $y \in M \mu_c$, $r \in R$, we must have $\mu(x) = c$. Thus
\[
\sqrt{\{\mu(y) | ry = x \text{ for some } y \in M \text{ and } r \in R\}} = c.
\]
Therefore in any case, we obtain that $\mu(x) = (1_R \cdot \mu)(x)$, and hence $\mu = 1_R \cdot \mu$, as desired. $\Box$
Theorem 3.12. Let \( v \in L(M) \) and \( \mu \in L-Spec(M) \). Then

(i) If \( v \subseteq \mu \), then \( \mu : v = 1_R \) and

(ii) If \( v \nsubseteq \mu \), then \( \mu : v = \mu : 1_M \).

Proof. (i) If \( v \subseteq \mu \) then for all \( \eta \in LI(R) \), \( \eta \cdot v \subseteq 1_R \cdot \mu = \mu \), then \( \mu : v = \bigcup \{ \eta \mid \eta \in LI(R), \eta \cdot v \subseteq \mu \} = 1_R \), as desired.

(ii) Suppose that \( v \nsubseteq \mu \). Let \( \eta \cdot v \subseteq \mu \) for some \( \eta \in LI(R) \). Then \( \eta \subseteq \mu : 1_M \) since \( \mu \) is prime. Thus

\[
\mu : 1_M = \bigcup \{ \eta \in LI(R) \mid \eta \cdot v \subseteq \mu \} \subseteq \mu : 1_M.
\]

Also since \( \mu : 1_M \subseteq \mu : v \), then \( \mu : v = \mu : 1_M \). □

Theorem 3.13. Let \( \mu \in L(M) \) and \( \zeta \in LI(R) \). If \( \mu \) is prime, then the following statements are satisfied:

(i) if \( \zeta \nsubseteq (\mu : 1_M) \), then \( \mu : \zeta = \mu \) and

(ii) if \( \zeta \subseteq (\mu : 1_M) \), then \( \mu : \zeta = 1_M \).

Proof. (i) Let \( \zeta \nsubseteq (\mu : 1_M) \) and \( \zeta \cdot v \subseteq \mu \) for \( v \in L(M) \). Then \( v \subseteq \mu \) and \( \mu : \zeta = \bigcup \{ v \in L(M) \mid \zeta \cdot v \subseteq \mu \} \subseteq \mu \), since \( \mu \) is prime. Also, \( \zeta \cdot \mu \subseteq 1_R \cdot \mu = \mu \) and hence \( \mu : \zeta = \mu \).

(ii) Suppose that \( \zeta \subseteq \mu : 1_M \), then \( \zeta \cdot 1_M \subseteq (\mu : 1_M) \cdot 1_M \subseteq \mu \). Therefore, \( 1_M \in \{ v \in L(M) \mid \zeta \cdot v \subseteq \mu \} \). Thus \( \mu : \zeta = \bigcup \{ v \in L(M) \mid \zeta \cdot v \subseteq \mu \} = 1_M \). □

Theorem 3.14. Let \( M \) and \( N \) be \( R \)-modules and let \( f \) be a homomorphism from \( M \) onto \( N \). Then the following statements are satisfied:

(i) Let \( \mu \in L-Spec(M) \) be \( f \)-invariant, then \( f(\mu) \in L-Spec(N) \),

(ii) If \( v \in L-Spec(N) \), then \( f^{-1}(v) \in L-Spec(M) \).

Proof. The proof follows from Remark 3.3 and Proposition 4.7 in [17]. □

Example 3.15. (1) Consider the ring of integers \( M = \mathbb{Z} \) as \( \mathbb{Z} \)-module and let \( L \) be an arbitrary lattice. Suppose that \( p \in \mathbb{Z} \) is prime. For every prime element \( t \in L \), define \( P(t) \in L(\mathbb{Z}) \) by

\[
P(t)(x) = \begin{cases} 1 & \text{if } x \in < p >, \\ t & \text{if } x \in \mathbb{Z} \setminus < p >. \end{cases}
\]

Then by Theorem 3.4, \( P(t) \) is a prime \( L \)-submodule of \( M \). Thus \( L-Spec(M) = \{ P(t) \mid t \text{ is a prime element of } L \text{ and } p \text{ is prime element or 0 of } \mathbb{Z} \} \), while for \( L = [0, 1] \), then \( L-Spec(M) = \{ P(t) \mid t \in [0, 1] \text{ and } p \text{ is prime element or 0 of } \mathbb{Z} \} \).

(2) Consider \( M = \mathbb{R}[x] \) as \( \mathbb{R}[x] \)-module, where \( \mathbb{R} \) is the field of real numbers. For every \( P \in \mathbb{R}[x] \) and every \( t \in L \), define the fuzzy subset \( P(t) \) of \( \mathbb{R}[x] \) by

\[
P(t)(x) = \begin{cases} 1 & \text{if } x \in < p >, \\ t & \text{otherwise.} \end{cases}
\]

Then by Theorem 3.4, \( P(t) \) is a prime \( L \)-submodule of \( M \) if and only if \( P \) is irreducible and \( t \) is a prime element of \( L \). Moreover, for \( L = [0, 1] \), we have \( L-Spec(M) = \{ P(t) \mid P \text{ is irreducible in } \mathbb{R}[x], t \in [0, 1] \} \).

(3) Suppose \( M \) is an arbitrary \( R \)-module and \( P \) is a prime submodule of \( M \). For every \( t \in L \), define

\[
P(t)(x) = \begin{cases} 1 & x \in P, \\ t & \text{otherwise.} \end{cases}
\]

Then by Theorem 3.4, \( P(t) \) is a prime \( L \)-submodule of \( M \) if and only if \( t \) is a prime element of \( L \). If \( Spec(L) \) denote the set of all prime elements of \( L \), then \( L-Spec(M) = \{ P(t) \mid t \in Spec(L) \text{ and } P \text{ is a prime submodule of } M \} \).
4. L-primeless modules and L-top modules

Recall that $M$ is called primeless if $\text{Spec}(M)$ is empty (see [12]). We say that $M$ is $L$-primeless if $L\text{-Spec}(M)$ is empty, for example the zero module is clearly $L$-primeless.

**Theorem 4.1.** If $M$ is primeless then $M$ is $L$-primeless.

**Proof.** The proof immediately follows from this fact that if $\mu \in L(M)$ is prime, then $\mu_*$ is a prime submodule of $M$ by Theorem 3.4. □

**Theorem 4.2.** If $L$ is regular and $M$ is $L$-primeless then $M$ is primeless.

**Proof.** If $P$ is a prime submodule of $M$, then $0$ is a prime element in $L$, since $L$ is regular. Thus $\chi_P$ is a prime $L$-submodule of $M$. □

By Lemma 1.3 and Proposition 1.4 in [12], we have two next results:

**Theorem 4.3.** Let $R$ be an integral domain. If $M$ is a torsion divisible $R$-module, then $M$ is $L$-primeless.

Recall that if $R$ is a domain then an $R$-module $D$ is called divisible if $D = rD$ for all non-zero $r \in R$. An $R$-module $M$ is called torsion if for any $a \in M$, there exists a non-zero $r \in R$ with $ra = 0$.

**Theorem 4.4.** Let $L$ be regular and $R$ be an one-dimensional Noetherian domain. Then $M$ is $L$-primeless if and only if $M$ is a torsion divisible $R$-module.

For any $L$-submodule $\mu$ of $M$, $V(\mu)$, denotes the set of all prime $L$-submodule of $M$ containing $\mu$, i.e, $V(\mu) = \{P \in L\text{-Spec}(M)|\mu \subseteq P\}$. Obviously, $V(1_M)$ is just the empty set and $V(0)$ is $L\text{-Spec}(M)$. Also it is easy to verify that for any family of $L$-submodules $\{\mu_i\}_{i \in I}$ of $M$ and $\mu, \nu \in L(M)$,

$$\bigcap_{i \in I} V(\mu_i) = V\left(\sum_{i \in I} \mu_i\right), \quad V(\mu) \cup V(\nu) \subseteq V(\mu \cap \nu).$$

Thus if $\xi(M)$ denotes the collection of all subsets $V(\mu)$ of $L\text{-Spec}(M)$, then $\xi(M)$ contains the empty set, and $L\text{-Spec}(M)$ is closed under arbitrary intersection. If also $\xi(M)$ is closed under finite union, i.e, for any $L$-submodules $\mu$ and $\nu$ of $M$, there exists an $L$-submodule $\emptyset$ of $M$, such that $V(\mu) \cup V(\nu) = V(\emptyset)$, for in this case $\xi(M)$ satisfies the axioms of closed subsets of a topological space, which is called Zariski topology. An $R$-module $M$ equipped with Zariski topology is called $L$-top module. An $L$-submodule $\mu \in L(M)$ is called $L$-semiprime if $\mu = \bigcap_{i \in I} \mu_i$ such that $\mu_i$ is a prime $L$-submodule of $M$ for all $i \in I$, and $\mu$ is called $L$-extraordinary if for semiprime $L$-submodules $\mu_1, \mu_2 \in L(M)$ such that $\mu_1 \cap \mu_2 \subseteq \mu$, implies that $\mu_1 \subseteq \mu$ or $\mu_2 \subseteq \mu$. For $\mu \in L(M)$, we define the radical of $\mu$, denotes by $\text{Rad}(\mu)$, is the intersection of all prime $L$-submodules of $M$ contains $\mu$. In other words, $\text{Rad}(\mu) = \bigcap_{P \in V(\mu)} P$ and equal to $1_M$ if $V(\mu) = \emptyset$. Clearly, for $\mu \in L(M)$, we have $\text{Rad}(\mu) \subseteq L(M)$ and $V(\mu) = V(\text{Rad} \mu)$.

**Theorem 4.5.** For an $R$-module $M$ the following statements are equivalent:

(i) $M$ is $L$-top module;
(ii) Every prime $L$-submodule of $M$ is $L$-extraordinary and
(iii) $V(\mu_1) \cup V(\mu_2) = V(\mu_1 \cap \mu_2)$, for $\mu_1, \mu_2 \in L(M)$.

**Proof.** Clearly, if $M$ is $L$-primeless then (i)–(iii) hold. Thus we assume that $M$ is not $L$-primeless. 

(i)$\Rightarrow$(ii) Suppose that $P \subseteq L(M)$ is a prime $L$-submodule of $M$ and $\mu_1, \mu_2 \in L(M)$ are semiprime, such that $\mu_1 \cap \mu_2 \subseteq P$. Since $M$ is $L$-top module, then there exists an $L$-submodule $\mu$ of $M$, such that $V(\mu_1) \cup V(\mu_2) = V(\mu)$. Since, $\mu_1$ is semiprime then $\mu_1 = \bigcap_{i \in I} v_i$ such that $v_i$ is prime $L$-submodule of $M$, for all $i \in I$. Therefore, for each
$i \in I$ if $\mu_i \subseteq v_i$, then $v_i \in V(\mu_1) \subseteq V(\mu)$. Thus $\mu \subseteq \bigcap_{i \in I} v_i = \mu_1$ for all $i \in I$. Similarly, for $\mu \subseteq \mu_2$, we obtain that $\mu \subseteq \mu_1 \cap \mu_2$. Also,

$$V(\mu_1) \cup V(\mu_2) \subseteq V(\mu_1 \cap \mu_2) \subseteq V(\mu) = V(\mu_1) \cup V(\mu_2).$$

Thus

$$V(\mu_1) \cup V(\mu_2) = V(\mu_1 \cap \mu_2).$$

Now

$$P \in V(\mu_1 \cap \mu_2) \implies P \in V(\mu_1) \cup V(\mu_2)$$

$$\implies P \in V(\mu_1) \text{ or } P \in V(\mu_2)$$

$$\implies \mu_1 \subseteq P \text{ or } \mu_2 \subseteq P.$$ 

It means that $P$ is $L$-extraordinary.

(ii)$\implies$(iii) Suppose that $\mu_1, \mu_2 \in L(M)$ are semiprime. Clearly

$$V(\mu_1) \cup V(\mu_2) \subseteq V(\mu_1 \cap \mu_2).$$

Suppose that $P \in V(\mu_1 \cap \mu_2)$. Then $\mu_1 \cap \mu_2 \subseteq P$, and hence by hypothesis $\mu_1 \subseteq P$ or $\mu_2 \subseteq P$. Thus $P \in V(\mu_1)$ or $P \in V(\mu_2)$.

Therefore $P \in V(\mu_1) \cup V(\mu_2)$, and hence

$$V(\mu_1 \cap \mu_2) \subseteq V(\mu_1) \cup V(\mu_2).$$

Now from (1) and (2) it follows that $V(\mu_1) \cup V(\mu_2) = V(\mu_1 \cap \mu_2)$.

(iii) $\implies$ (i). Let $\mu_1, \mu_2 \in L(M)$. If $V(\mu_1)$ is empty then

$$V(\mu_1) \cup V(\mu_2) = V(\mu_2).$$

Thus we assume that $V(\mu_1)$ and $V(\mu_2)$ are both non-empty. Then

$$V(\mu_1) \cup V(\mu_2) = V(\text{Rad}\, \mu_1) \cup V(\text{Rad}\, \mu_2),$$

and by (iii) since $\text{Rad}(\mu_1)$ and $\text{Rad}(\mu_2)$ are semiprime then

$$V(\text{Rad}\, \mu_1) \cup V(\text{Rad}\, \mu_2) = V(\text{Rad}\, \mu_1 \cap \text{Rad}\, \mu_2).$$

By letting $v = \text{Rad}(\mu_1) \cap \text{Rad}(\mu_2)$ it conclude that

$$V(\mu_1) \cup V(\mu_2) = V(v).$$

This proves (i). \qed

5. Conclusion

Letting $L_{\zeta'}(M) = \{V(\eta_1M)|\eta \in LI(R)\}$. It is easy to verify that this set always induces a topology, say, $\tau'$ on $L\text{-Spec}(M)$, because $L_{\zeta'}(M)$ is closed under finite union: $V'(\eta_1M) \cup V'(\eta_2M) = V'((\eta_1, \eta_2)M)$. While, by above discussion, $L_{\zeta}(M)$ induces the topology $\tau$ on $L\text{-Spec}(M)$ if and only if $M$ is a top module. Note that in the above we provided the basic notions and results to define the Zariski topology of prime $L$-submodules, in particular for fuzzy submodules. We hope that this paper encourages the researchers to study the relationship between fuzzy topology and fuzzy modules theory as well as the topological properties of $L\text{-Spec}(M)$.

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