Simplifying the beamforming optimality region for practical MIMO scenarios

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Abstract—The optimality region for beamforming is redefined in terms of two parameters, the signal to noise ratio and the product of the antenna separation and the angular spread in the transmitter. This is possible by finding an interesting univocal relationship between the two main eigenvalues of the transmitter correlation in terms of this product. This relationship holds for many realistic scenarios, particularly in those where beamforming is a transmission alternative.

I. INTRODUCTION

The relevance of beamforming (BF) in MIMO systems is being widely treated in literature since one-dimensional codes are used to achieve capacity. Assuming that the receiver has perfect channel state information and that the transmitter has channel distribution information based on a zero-mean Gaussian model with one-sided covariance, the achieving-capacity region of BF is being so far characterized by three parameters [1], [2], for a fixed number of antennas in the receiver \( N \): the two principal eigenvalues of the transmitter correlation matrix \( \lambda_1 \) and \( \lambda_2 \) and the signal to noise ratio (SNR). BF is optimal if:

\[
\frac{1}{\lambda_2 \text{SNR}} \geq \frac{1}{1 - (\lambda_1 \text{SNR})^{-1} - 1}
\]

Spatial channel correlation is being studied in terms of different parameters [3], [4], [5]. From a practical point of view, it would be desirable that spatial correlation could be fully characterized by the antenna geometries and the statistical characterization of the angular spread (AS). Assuming separable Power Angular Spread (PAS) in transmitter and receiver [6], the spatial correlation between the \((i,j)\)-th and \((k,l)\)-th entries of the channel matrix \( \mathbf{H} \) is separable:

\[
R_{\mathbf{H}}(i,j;k,l) = E\{ \mathbf{H}_{i,j}(\mathbf{H}_{k,l})^\dagger \} = (\Theta_T)_{i,k}(\Theta_R)_{j,l}
\]

where \( \Theta_T \) and \( \Theta_R \) are the correlation matrices at the transmitter and receiver respectively. Given this, \( \Theta_T \) will depend on the antenna geometry and the PAS at the transmitter, extending this dependence to its eigenvalues \( \lambda_1 \) and \( \lambda_2 \) and under \( \Theta_R = \mathbf{I}_N \) assumption, to the beamforming optimality region. To the authors’ best knowledge there are no previous studies on the behavior of \( \lambda_1 \) and \( \lambda_2 \) under realistic channel correlation scenarios dependent on the antenna geometry and PAS. Thus, a channel model that relies on these parameters is defined, allowing a closed-form formulation for \( \Theta_T \). This channel model is a particularization of the one provided in [7].

This letter proposes a simplification of the BF region that in previous studies was defined in terms of three parameters.

Here, it is shown that there is an univocal dependence between the two main eigenvalues of the transmitter correlation, for certain realistic assumptions that, for example, match many cellular downlink environments. These assumptions are a limited angular spread [6] and antenna size. This result allows redefining the optimal region for BF in terms of just two parameters: SNR and the product of the antenna separation and the angular spread. Thus, a much simpler scenario that besides relies on practical parameters determines BF optimality.

II. CHANNEL AND CORRELATION MODEL

We assume a stationary, flat fading channel with \( M \) transmitters and \( N \) receivers. This \( M \times N \) MIMO channel is described by a channel matrix \( \mathbf{H} \) where element \( h_{nm} \) defines the (fading) channel from the \( m \)-th transmitting antenna to the \( n \)-th receiving antenna [7]:

\[
(\mathbf{H})_{n,m} = \int \int S(k',k)e^{-jkr_m}e^{jkr_n} dk'dk
\]

where \( S(k',k) \) is a scattering function, that relates the planar wave emitted from direction \( k \) impinging on the receiver from direction \( k' \). The position of the antennas is given by vectors \( r_m \) and \( r_n \) and \( k \cdot r \) denotes the dot product of the two vectors.

Assuming Gaussian entries and a separable model in the scattering function \( S(k',k) \), the spatial correlation at the transmitter is given by:

\[
(\Theta_T)_{n,m} = \int \mathcal{P}_\alpha(k) e^{-jkr_m} dr_n \]

where \( \mathcal{P}_\alpha(k) \) is the PAS at the transmitter and \( 2\alpha \) is the AS parameter.

A. Correlation at the transmitter

To achieve a tractable correlation formulation at the transmitter, the wave space defined by the space vectors \( k \) is reduced to azimuthal angles \( \theta \in (-\frac{\pi}{2}, \frac{\pi}{2}) \) and the antenna geometry considered is a uniform linear array with \( d_t \) spacing in \( \lambda \) units.

\[
(\Theta_T)_{n,m} = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \mathcal{P}_\alpha(\theta)e^{-j2\pi d_t \sin \theta} d\theta
\]

\[
= \int_{-d_t}^{d_t} \mathcal{P}_\alpha(\arcsin(\frac{\phi}{d_t})) e^{-j2\pi(m-n)\phi} d\phi
\]

Defining for a particular PAS distribution:

\[
a(m-n,d_t,\alpha) = \int_{-d_t}^{d_t} \mathcal{P}_\alpha(\arcsin(\frac{\phi}{d_t})) e^{-j2\pi(m-n)\phi} d\phi
\]

This work has been supported by Projects CYCIT TIC2002-03498 and UC3M-TEC-05-027.
the correlation matrix is given by:

\[
\Theta_T = \begin{bmatrix}
1 & \ldots & a(M - 1, d, \alpha) \\
\ast(1, d, \alpha) & 1 & \ldots & a(M - 2, d, \alpha) \\
& \ddots & \ddots & \ddots \\
a\ast(M - 1, d, \alpha) & \ldots & 1
\end{bmatrix}
\]  

(7)

When \( P_\alpha (\theta) \) is sufficiently narrow (i.e. \( \alpha \) is small), then, assuming broadside direction, \( P_\alpha (\theta) \) has significant values only for small \( \theta \) values and the following approximation can be made:

\[
P_\alpha (\arcsin(\frac{\phi}{d_t})) \approx P_\alpha \left( \frac{\phi}{d_t} \right)
\]

(8)

The particularization of the correlation matrix to two well-known scenarios such as Laplacian angular spread and uniform angular spread leads to closed-form solutions for the coefficients of the transmitter correlation matrix.

For uniform PAS in \( \theta \in (-\alpha, \alpha) \):

\[
(\Theta_T)_{n,m} \approx \int_{-d_t}^{d_t} P_\alpha \left( \frac{\phi}{d_t} \right) e^{-j2\pi(m-n)\phi} d\phi
\]

(9)

For Laplacian PAS:

\[
(\Theta_T)_{n,m} \approx \int_{-d_t}^{d_t} P_\alpha \left( \frac{\phi}{d_t} \right) e^{-j2\pi(m-n)\phi} d\phi
\]

(10)

The interesting result

\[
\lambda_1 = 1 + a(1)
\]

\[
\lambda_2 = 1 - a(1)
\]

(12)

For \( M = 2 \) both eigenvalues are straightforward:

\[
\lambda_1 = 1 + a(1)
\]

\[
\lambda_2 = 1 - a(1)
\]

(13)

For \( M = 3 \), \( M = 4 \) only the first and second eigenvalues solutions are given next; for \( M = 3 \):

\[
\lambda_1 = 1 + \frac{a(2)}{2} + \frac{1}{2} \sqrt{a^2(2) + 8a^2(1)}
\]

\[
\lambda_2 = \left\{ \begin{array}{l}
1 - a(2) \\
1 + \frac{a(2) - 1}{2} \sqrt{a^2(2) + 8a^2(1)} \quad : a(2) \leq a(1)
\end{array} \right.
\]

(14)

and for \( M = 4 \):

\[
\lambda_1 = 1 + a(3) + \frac{a(1) - a(3)}{2} + \frac{1}{2} \sqrt{(a(1) - a(3))^2 + 4(a(1) + a(2))^2}
\]

\[
\lambda_2 = 1 - a(3) + \frac{a(1) - a(3)}{2} + \frac{1}{2} \sqrt{(a(1) - a(3))^2 + 4(a(1) - a(2))^2}
\]

(15)

III. TRANSMITTER CORRELATION EIGENVALUES

The correlation matrix is in general an Hermitian matrix, however, symmetry in the PAS shape and in the antenna geometry leads to a symmetric Toeplitz matrix. The PAS shapes and antenna geometry assumed in previous section leads to this situation, the correlation matrix is symmetric Toeplitz, i.e. \((\Theta_T)_{n,m} = (\Theta_T)_{m,n}\).

Symmetric Toeplitz matrices have symmetry properties with respect to its eigenvectors [8] that allows a simple derivation of the eigenvalues: \( \Theta_T \) will have \( \lfloor \frac{M+1}{2} \rfloor \) symmetric eigenvectors and \( \lfloor \frac{M-1}{2} \rfloor \) skew symmetric eigenvectors. Applying that, a relatively simple equation system is solved to obtain the eigenvectors and eigenvalues in terms of the real correlation matrix coefficients \( a(i) = (\Theta_T)_{n,n+i} \).

The analytical relationship between both eigenvalues is given in Figs. 1 and 2 for the specified values of \( M \). This result is obtained by plotting \( \lambda_1 \) and \( \lambda_2 \) in eqs. (12)-(14) for the correlation coefficients given in eqs. (10) and (11) that exclusively depend on the parameter \( \alpha d_t \). The interesting result

![Fig. 1. Main eigenvalue of the transmitter correlation \( \lambda_1 \) vs second eigenvalue of the transmitter correlation \( \lambda_2 \) with Laplacian PAS. \( M = \{2, 3, 4\} \).](image-url)
is that the dependence of the eigenvalues with the product of parameters $\alpha d_t$ leads to a unique relationship between both eigenvalues, i.e., given $\lambda_1$ there is an unique value for $\lambda_2$. Simulated results have been obtained evaluating numerically the eigenvalues of the correlation matrix coefficients given by eq. (5) for the values of $2\alpha \leq 30^\circ$ and $d_t \leq 6$ in the case of Laplacian PAS and for $2\alpha \leq 20^\circ$ and $d_t \leq 3.5$ for the case of uniform PAS. In these range of values, the approximation made in section II-A still holds and all possible values of $\lambda_1$ (1 $\leq \lambda_1 \leq M$) are covered.

From equations (10)-(11) and (12)-(14), $\lambda_1$ and $\lambda_2$ depend on $\alpha d_t$. Substituting $\lambda_1$ and $\lambda_2$ in eq. (1) by the corresponding $\alpha d_t$ values, we obtain a simplified BF region where for a given $M$ and $N$ the BF optimality region (BO) is given in terms of SNR and $\alpha d_t$. This region is presented in fig. 3. As correlation decreases ($\alpha d_t$ increases), SNR range where BF is optimal decreases. Another observation for the simplified region is that there exists a SNR value that decreases with $N$, for which BF is always optimal independently of the correlation scenario.

V. CONCLUSION

We present a relationship between the two main eigenvalues of the correlation matrix at the transmitter. This relationship is valid for low angular spread values and a wide range of antenna separation values. The parameter that governs the behavior of both eigenvalues is the product $\alpha d_t$. This leads to a much simpler characterization of the BF optimality region in terms of SNR and measurable parameters such as the angular spread and the antenna separation. We observe also that BF is optimal when SNR is below 5 dB for $N = 1$ receive antennas if $2\alpha d_t \geq 0.5$ which is the case for many practical angular spreads and antenna element separation.

ACKNOWLEDGMENT

We would like to thank D. Chizhik and A. Lozano for their useful discussions. Also to the reviewers for their suggestions.

REFERENCES