SYNCHRONIZING CONTINUOUS TIME CHAOTIC SYSTEMS OVER NONDETERMINISTIC NETWORKS WITH PACKET DROPOUTS

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The problem of control synthesis for master–slave synchronization of continuous time chaotic systems of Lur’e type using sampled feedback control subject to sampling time random fluctuation and data packet dropouts is investigated. New stability and stabilization conditions are proposed based on Linear Matrix Inequalities (LMIs). The idea is to connect two very efficient approaches to deal with delayed systems: the discretized Lyapunov functional for systems with pointwise delay and the convex analysis for systems with time-varying delay. Simulation examples based on synchronizing coupled Chua’s circuits are used to illustrate the effectiveness of the proposed methodology.

Keywords: Synchronization; Lur’e system; time-varying delay; Lyapunov–Krasovskii functional; linear matrix inequality (LMI).

1. Introduction

In face of the huge success and widespread use of the Ethernet TCP/IP protocol stack, some applications have to be redesigned considering the stochastic nature of the underlying communication protocols used to transmit information over the Internet. As the number of network interconnected systems continues to grow, it seems more realistic to investigate the robustness of current applications to data packet dropouts and time-varying sampling times, instead of proposing new protocols and hardware capable of delivering information with increased reliability [Hespanha et al., 2007; Varutti & Frank, 2009; Kruszewski et al., 2012].

Usually, the solutions for the aforementioned problems involving network-based control systems follow the path of designing specific network hardware and/or protocols to guarantee as small as possible network-induced delays; or, on the other hand, they follow the approach of incorporating information on the unavoidable network delays in the controller synthesis procedure.

Among such applications, one can highlight the task of keeping synchronized interconnected continuous time dynamical systems, particularly nonlinear oscillators, located far apart from each other. Considering the master–slave paradigm of one system (the master) influencing the dynamics of the other
systems (the slaves), such problem can be found in chaos-based secure information transmission systems relying on synchronized chaotic systems (see [Eisencraft et al., 2012; Andrieuxki & Fradkov, 2004] and references therein).

On the other hand, data packet dropouts and time-varying sampling times can be tied together in the same theoretical framework of systems subjected to time-varying delay (see [Yu et al., 2005; Suplin et al., 2007; Souza et al., 2011] and references therein). In this context, the Lyapunov–Krasovskii theory and corresponding Linear Matrix Inequalities (LMIs) are used, in the present work, to establish stability criterion and to design control strategies to attain synchronization between master–slave systems represented as uncertain time-delayed Lur’e systems, similarly to what has been reported in [Yalcin et al., 2001; Liao & Chen, 2003; Huang et al., 2006; Cao & Ho, 2005; Sun, 2004; Souza et al., 2008a, 2008b]. The approach consists in rewriting the synchronization error system, composed of a continuous part and a discrete part, as one continuous time system endowed with an uncertain, bounded and nondifferentiable time-delay.

A very efficient way to derive LMIs conditions based on the Lyapunov–Krasovskii theory is to use the so-called discretized Lyapunov–Krasovskii functional (DLKF) (see Sec. 2 and for more details see [Gu et al., 2003]). In the present paper, the DLKF developed to deal with linear systems subject to pointwise delay is extended to Lur’e systems subject to time-varying delay using convex analysis. The use of convex analysis in the time-varying delay was developed by Shao [2009]. Recently in [Fridman & Tsokik, 2009] the DLKF was extended to linear systems with both, discrete and distributed delays.

The novelty of this paper is the application of two very efficient approaches to deal with delayed systems, that is, the discretized Lyapunov functional and the convex analysis for systems with time-varying delay, in the context of synchronizing continuous time chaotic systems over nondeterministic networks with packet dropouts.

**Notation.** Throughout this paper, the superscript $T$ stands for transposition. A null matrix with appropriate dimensions is referred to as $0$. For a real symmetric matrix $M$, the notation $M > 0$ ($< 0$) means that $M$ is positive (negative) definite. The notation $\text{diag}\{ \cdot \}$ stands for a block-diagonal matrix. The corresponding symmetric term in a matrix is denoted by $\ast$.

2. Preliminaries

Consider the following master–slave synchronization scheme:

$$
\begin{align}
M: & \begin{cases}
\dot{x}(t) = Ax(t) + B\sigma(Cx(t)), \\
\sigma(t) = Hx(t),
\end{cases} \\
S: & \begin{cases}
y(t) = A\sigma(t) + B\sigma(Cy(t)) + u(t), \\
q(t) = Hy(t),
\end{cases} \\
\mathcal{C}: & w(t) = K[p(t_k) - q(t_k)]
\end{align}
$$

where $M$ is the master system, $S$ is the slave system and $\mathcal{C}$ is the sampled feedback control law. The state vectors are given by $x, y \in \mathbb{R}^n$, the output of each system is denoted by $p, q \in \mathbb{R}^l$, the matrices $A \in \mathbb{R}^{n \times n}, B \in \mathbb{R}^{n \times n}, C \in \mathbb{R}^{n \times n}, H \in \mathbb{R}^{l \times n}$, the feedback matrix $K \in \mathbb{R}^{l \times n}$, and the function $\sigma(\cdot)$ is Lipschitz continuous and nondecreasing, with $\sigma_i(\cdot)$

$\dot{t} = 1, 2, \ldots, n_b$ such that $0 \leq \sigma_i(t_k) = \frac{\sigma_i(t_k) - \sigma_i(t_{k-1})}{\sigma_k(t_k) - \sigma_k(t_{k-1})} \leq \kappa$ for all $\sigma_1, \sigma_2 \in \mathbb{R}, \sigma_1 \neq \sigma_2$, $\kappa > 0$ and $t = 1, 2, \ldots, n_b$.

The sampled data, $p(t_k)$ and $q(t_k)$, are discrete measurements of $p(t)$ and $q(t)$ at the sampling instant $t_k$.

Figure 1 shows the scheme for synchronization between the master and slave systems. The sampling period $h$ is assumed to be constant and known, whereas the controller and zero-order hold (ZOH) are event-driven in the sense that they update their outputs as soon as they receive a new sample.

It is worth noticing that one has a monotonic increasing stochastic sequence $\{t_k\}$ of effective sampling times given by

$$
t_{k+1} = t_k + (\Delta_k + 1)h,
$$

where $\Delta_k$ is the number of accumulated data packet dropouts at the effective sampling instant $t_k$, since the last effective sampling instant $t_k$. Assume $0 \leq \Delta_{k+1} \leq \Delta_{\text{max}}$, where $\Delta_{\text{max}}$ is a known maximum number of data packet dropouts. Note that the output period $t_{k+1} - t_k$ of the ZOH in (1c) is variable and uncertain in this context.

Synchronization means agreement or correlation of different processes in time, therefore the design of the controller $\mathcal{C}$ aims to make $\|x(t) - y(t)\| \to 0$ as $t \to \infty$, where $\| \cdot \|$ is a norm defined in $\mathbb{R}^n$. 

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By defining the error signal as $e(t) = x(t) - y(t)$, the error system can be represented as follows

$$
\dot{e}(t) = Ae(t) + Fe(t) + B\sigma(Ce(t), y(t)),
$$

with $F = -K\eta$, and $\eta(Ce(t), y(t)) = \sigma(Ce(t)) - \sigma(\sigma(Ce(t) + Cy(t)) - \sigma(Cy(t)))$.

The next step is to rewrite the above hybrid system as a continuous one. In order to do this, consider the following representation for the hybrid system:

$$
\begin{align*}
t_k &= t - t_k = t - d(t), \\
\text{with } d(t) &= t - t_k \text{ and since } t_k \leq t \leq (\Delta_k + 1)h,
\end{align*}
$$

$d(t)$ belongs to the domain $0 \leq d(t) \leq (\Delta_{\text{max}} + 1)h$. Then the following equivalent continuous time model possessing a time-varying delay will be studied,

$$
\dot{e}(t) = Ae(t) + Fe(t - d(t)) + B\sigma(Ce(t), y[t]),
$$

where one can see that $d(t)$ can be a nondifferentiable function. It is worth mentioning that the variation of the time-delay in an interval has a strong application appeal, which commonly exists in many practical systems. For example, it has been described in [Yue et al., 2005] that the lower bound of the delay in the network control systems is often larger than zero.

From the Lipschitz continuity and nondecreasing assumptions for $\sigma(\cdot)$, one has that

$$
\begin{align*}
0 &\leq \eta(c_i^T e, y) \\
&= \eta(c_i^T e + c_i^T y) \\
&= \sigma(c_i^T x) - \sigma(c_i^T y) \\
&\leq \kappa,
\end{align*}
$$

where $c_i^T$ denotes the $i$th row vector of the $C$ matrix. This inequality is equivalent to the fact that the nonlinearity $\eta(Ce, y)$ belongs to the sector $[0, \kappa]$ in the sense that

$$
\begin{align*}
\eta(c_i^T e, y)\left(\eta(c_i^T e, y) - \kappa c_i^T e\right) \\
&\leq 0, \quad \forall e, y; \quad i = 1, 2, \ldots, n_h.
\end{align*}
$$

In the present context, the master–slave synchronization scheme in (1) synchronizes as long as the origin of the error system (3) is an asymptotically stable equilibrium point. Henceforth error system (3) of master–slave synchronization for Lur’e systems using the time-varying delay feedback control is considered.

The new key idea for the proposed stability analysis criterion is to connect the approaches presented in [Souza et al., 2008b; Shao, 2009]. The former approach was developed in the context of master–slave synchronization for Lur’e systems using a pointwise feedback control law, which is based on the discretization of the Lyapunov–Krasovskii functional. In [Gu et al., 2003] it is shown that refining the discretization, the results obtained to stability analysis of delayed linear systems tends to the analytical results. In [Souza et al., 2008b] a functional in the following form was considered:

$$
V^+(c) = c^T(t)Pe(t) + 2c^T(t)\int_{-\tau}^{0} Q(t)\xi(t + \xi)d\xi
$$

$$
+ \int_{-\tau}^{0} \int_{-\tau}^{0} c^T(s + \xi)R(s, \xi)d\xi e(t + \xi)d\xi
$$

$$
+ \int_{-\tau}^{0} c^T(t + \xi)S(t)\xi(t + \xi)d\xi.
$$

The discretization consists in dividing the interval of integration $[-\tau, 0]$ in the Lyapunov–Krasovskii functional into $N$ segments $[\theta_i, \theta_{i-1}]$, $i = 1, \ldots, N$, of equal length $r = \tau/N$, where $\theta_i = -ir$. The continuous matrix functions $Q(t)$ and $S(t)$ are chosen to be linear within each segment and the continuous...
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matrix function $R(s, \xi)$ is chosen to be linear within each triangle. Since these matrices are piecewise linear, they can be expressed in terms of their values at the boundary points of the intervals using an interpolation formula, i.e.

\begin{align*}
Q(\theta + \alpha r) &= (1 - \alpha)Q_1 + \alpha Q_{-1}, \\
S(\theta + \alpha r) &= (1 - \alpha)S_1 + \alpha S_{-1}, \\
R(\theta + \alpha r, \theta_j + \beta r) &= \begin{cases} 
(1 - \alpha)R_{ij} + \beta R_{-1,j-1} & \alpha \geq \beta, \\
(1 - \beta)R_{ij} + \alpha R_{-1,j-1} & \alpha < \beta,
\end{cases}
\end{align*}

for $0 \leq \alpha \leq 1$, $0 \leq \beta \leq 1$ and $i, j = 1, \ldots, N$.

Therefore, the terms of the functional (5) are completely determined by the constant matrices $P_i$, $Q_i$, $R_{ij}$ and $S_j$.

The approach in [Shao, 2009] was developed in the context of stability analysis of delayed linear systems subject to time-varying delay and, here, it is used to obtain a less conservative bound to the time derivative of the term

\begin{equation}
V^\mu(\xi) = \tau \int_0^\tau \int_{t-\tau}^{t+\tau} e^T(\alpha)Z(\alpha) \, d\alpha \, ds,
\end{equation}

which has the time derivative

\begin{equation}
\dot{V}^\mu(\xi) = \tau^2 e^T(t)Z(\xi(t)) - \tau \int_0^\tau \int_{t-\tau}^{t+\tau} e^T(\alpha)Z(\alpha) \, d\alpha \, da.
\end{equation}

Following the same steps presented in [Shao, 2009], based on the fact that $\int_{t-\tau}^{t+\tau} f(s) \, ds = \int_{t-\tau}^{t+\tau} f(t) \, ds \pm \int_{t-\tau}^{t-\tau} f(s) \, ds$ and on Jensen’s inequality [Gu et al., 2003, p. 323], the previous equation can be bound as

\begin{equation}
\dot{V}^\mu(\xi) \leq \dot{\overline{V}}^\mu(\alpha, \delta) = \tau^2 e^T(t)Z(\xi(t)) - [e(t - d(t)) - e(t - d(t))] - [e(t - d(t)) - e(t - d(t))] - \delta e(t - d(t)) - e(t - \tau)]Z[e(t - d(t)) - e(t - d(t))] - e(t - \tau)] - (1 - \delta)[e(t - d(t)) - e(t - d(t))]Z[e(t - d(t)) - e(t - d(t))] \times Z[e(t - d(t)) - e(t - d(t))],
\end{equation}

with $\delta = d(t)/\tau$. Then the estimation of $\dot{V}^\mu(\xi)$ induces a convex domain of matrices, and the resulting stability criterion can be written as a convex combination of two LMIs. The main idea of this paper is then to consider a new Lyapunov–Krasovskii functional composed by (5) and (7), namely

\begin{equation}
V(\xi) = V^\mu(\xi) + V^\nu(\xi),
\end{equation}

with $V^\mu(\xi)$ given in (5) and $V^\nu(\xi)$ given in (7).

Since the upper bound for $\dot{V}^\mu(\xi)$ in (9) is expressed in terms of a convex combination defining a two-vertices polytope for which the domain is related to the time-delay interval the proposed method is given as a convex set of linear matrix inequalities defined at the vertices of the polytope. A feasibility test should then be performed at these vertices yielding sufficient conditions for the stability of the entire domain.

3. Asymptotic Stability Analysis

In this section, an asymptotic stability condition for error system (3) of the master–slave synchronization for Lur’e systems using a time-varying delay feedback control based on LMIs is presented. The issues presented in the previous section will be used in the proof of the main theorem stated below.

**Theorem 1.** Let $\Delta_{\text{max}}$, a known maximum number of data packet dropouts, $h$, the sampling period, and $N$, the desired number of partitions, be given. Calculate $\tau = (\Delta_{\text{max}} + 1)h$, the upper bound of $d(t)$ in (2) and suppose that assumption (4) holds. The master–slave synchronization scheme (1) synchronizes, with error system (3) globally asymptotically stable, if there exist $n \times n$ matrices $Y_1, Y_2, \lambda = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n) > 0$, $P = P^T$, $Q_i$, $R_{ij}, S_i = S_i^T$, ($i, j = 0, 1, \ldots, N - 1$), $S_N > 0$, and $Z = Z^T$, such that the following LMIs for $\delta = 0$ and $\delta = 1$ hold

\begin{equation}
\begin{bmatrix}
P & Q_s \\
* & R_s + S_s
\end{bmatrix} > 0,
\end{equation}

and

\begin{equation}
\begin{bmatrix}
\Xi(\delta) & D_s \\
* & -R_s - S_s & 0 \\
* & * & -3S_s
\end{bmatrix} < 0,
\end{equation}

where

\begin{equation}
\Xi(\delta) = D_s^T(D_s - \delta D_s).
\end{equation}
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where

\[ Q_S = [Q_0 \ Q_1 \ \ldots \ Q_N]; \]

\[ R_S = \begin{bmatrix} R_{0,0} & R_{0,1} & \ldots & R_{0,N} \\ & \ast & \ldots & \ast \\ & & \ddots & \vdots \\ & & & R_{N,N} \end{bmatrix}, \]

\[ S_S = \text{diag}\left\{ \frac{1}{\tau} S_0, \frac{1}{\tau} S_1, \ldots, \frac{1}{\tau} S_N \right\}. \]

with \( r = \tau/N \)

\[ \Xi(\delta) = \begin{bmatrix} Y_A + A^T Y_T^2 + Q_0 + \frac{Q_T^2}{r} + S_0 - (2 - \delta)Z & \ast & \ast & \ast \\ P - Y_T^2 + Y_A & \tau^2 Z - Y_2 - Y_T^2 & \ast & \ast \\ F^T Y_T^2 + (2 - \delta)Z & F^T Y_T^2 & -3Z & \ast \\ -Q_T^2 & 0 & (1 + \delta)Z - S_N - (1 + \delta)Z & \ast \\ B^T Y_T^2 + \kappa \Lambda C & B^T Y_T^2 & 0 & 0 & -2\Lambda \end{bmatrix}. \]

\[ D_S = [D_{S,1} \ D_{S,2} \ \ldots \ D_{S,N}], \]

\[ D_N = [D_{N,1} \ D_{N,2} \ \ldots \ D_{N,N}], \]

\[ D_{n,i} = \begin{bmatrix} -[Q_{i-1} - Q_i] + \frac{r}{2}(R_{0,i-1} + R_{0,i}) \\ \frac{r}{2}[Q_{i-1} + Q_i] \\ 0 \\ -\frac{r}{2}[R_{N-1,N} + R_{0,N}^T] \\ 0 \end{bmatrix}, \]

\[ D_{b,i} = \begin{bmatrix} -\frac{r}{2}(R_{0,i-1} - R_{0,i}) \\ \frac{r}{2}[Q_{i-1} - Q_i] \\ 0 \\ \frac{r}{2}[R_{N-1,N} - R_{0,N}^T] \\ 0 \end{bmatrix}, \]

\[ R_d = \begin{bmatrix} R_{d(1,1)} & R_{d(1,2)} & \ldots & R_{d(1,N)} \\ & \ast & \ldots & \ast \\ & & \ddots & \vdots \\ & & & \ast \\ & & & \ast \end{bmatrix}, \]

Proof. Since the term \( V^2(e(t)) \) in (7) is the proposed functional \( V(e(t)) \) given in (10) with \( Z > 0 \) is positive definite (see block matrix at position (3,3) in the LMI in (16)), then to show that the condition \( V(e(t)) \geq |e(t)|^2 (\epsilon > 0) \) is satisfied, it suffices to show that \( V^2(e(t)) > 0 \). Following similar arguments used in [Gu et al., 2003, Proposition 5.20], one can verify that if (11) is satisfied, and due to \( S_N > 0 \), then \( V^2(e(t)) > 0 \).
In order to obtain (12) the structures chosen for the matrix functions $Q(\xi)$, $R(\xi, s)$ and $S(\xi)$ in (6) are considered and the interval of integration $[-\tau, 0]$ in (21) is divided into $N$ segments $[\theta_i, \theta_{i+1}]$ for $i = 1, \ldots, N$, where $\theta_i = -ir$ and $r = \tau/N$. For example, choose one term from (21) as shown below

$$
\begin{align*}
\tilde{V}(\epsilon_1) &= 2e^T(t) \int_{-\tau}^{0} Q(\xi)e(t + \xi) \, d\xi
\end{align*}
$$

results, after tedious calculation, in

$$
\begin{align*}
\tilde{V}(\epsilon_1) &\leq \zeta^T \Sigma \zeta + 2\zeta^T \sum_{i=1}^{N} \int_{0}^{1} [(1 - \alpha)(D_i + D_i')
+ \alpha(D_i' - D_i')\Phi_i(\alpha)\, d\alpha
- \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{0}^{1} \int_{0}^{1} \phi_i(\alpha) \Phi_{ij}(\beta) \, d\beta \, d\alpha
- \sum_{i=1}^{N} \int_{0}^{1} \phi_i(\alpha) \Phi_i(\alpha) \, d\alpha,
\end{align*}
$$

or

$$
\begin{align*}
\tilde{V}(\epsilon_1) &\leq \tilde{V}(\epsilon_1)
\end{align*}
$$
The terms $D_a$ and $D_b$ are both given in (17), $S_i$, $R_i$ are defined in (19) and (18), respectively, and $\phi_i'(\alpha) = [\epsilon_i'(t - \tau + \alpha \tau) \epsilon_i'\epsilon(t - 2\tau + \alpha \tau) \ldots \epsilon_i'(t - N\tau + \alpha \tau)]$ (for details see [Gu et al., 2003, p. 188]).

Moreover, using assumption (4) that deals with the nonlinearities, one can write that

$$
\begin{align*}
0 & \leq -2 \sum_{i=1}^{n} \lambda_i n_i(c_i^T \epsilon, y)[\gamma_i(c_i^T \epsilon, y) - \kappa c_i^T \epsilon] \\
& = -2\Lambda \gamma^T[C_i(t), y(t)][\gamma_i[C_i(t), y(t)] - \kappa C_i(t)], \\
& = -2\Lambda \gamma^T>C_i(t), y(t)] + B_d[C_i(t), y(t)].
\end{align*}
$$

(25)

where $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} > 0$.

Now considering the error system in (3), the following null term can be obtained

$$
0 = 2[\epsilon_i'(t)Y_i + \epsilon_i'(t)Y_i][\gamma_i'[C_i(t), y(t)] + Fr(t - d(t)) + B_d[C_i(t), y(t)].
$$

(26)

Taking into account the upper bound for $\bar{V}(e_i)$ and $\bar{V}(e_i)$ in (9), the inequality in (25), and the null term in (26), one has that

$$
\begin{align*}
\dot{V}(e_i) & \leq \bar{V}(e_i) + \dot{V}(e_i, \delta) - 2\Lambda \gamma^T>C_i(t), y(t)] \\
& \times [\gamma_i[C_i(t), y(t)] - \kappa C_i(t)] \\
& + 2[\epsilon_i'(t)Y_i + \epsilon_i'(t)Y_i][\gamma_i[C_i(t), y(t)] + Fr(t - d(t)) + B_d[C_i(t), y(t)].
\end{align*}
$$

(27)

and consequently

$$
\begin{align*}
\dot{V}(e_i) & \leq \epsilon_i^T \Xi(\delta) \epsilon_i \\
& + 2\epsilon_i^T \int_0^t [D_a + (1 - 2\alpha)D_b] \phi(\epsilon) \alpha d\alpha.
\end{align*}
$$

(28)

where

$$
\Xi(\delta) = \begin{bmatrix}
Y_iA + AT_iY_i^T + Q_i + Q_i^T + S_i - (2 - \delta)Z \\
\gamma^2 Z - \gamma(Y_i + Y_i^T)Z \\
B_i^T K_i^T + (2 - \delta)Z \\
-(1 + \delta)Z \end{bmatrix}
$$

(29)

4. Control Synthesis for Synchronization by Sampled Data

In this section, the control synthesis for synchronization is considered. The strategy is to extend the analysis result of Theorem 1 to the time-delay feedback control design.

Theorem 2. Let $\Delta_{\text{max}}$, a known maximum number of data packet dropouts, $h$, the sampling period, $N$, the desired number of partitions, and $\gamma$, a tuning scalar parameter; be given. Calculate $\tau = (\Delta_{\text{max}} + 1)h$, the upper bound of $d(t)$ in (2) and suppose that assumption (4) holds. The master–slave synchronization scheme (1) synchronizes if there exist matrices of appropriated dimensions: $K$, $Y_i$, $\Lambda = \text{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_n\} > 0$, $P = P_i^T$, $Q_i$, $R_i = R_j^T$, $S_i = S_i^T$, $(i, j = 0, 1, \ldots, N - 1)$, $S_N > 0$, and $Z = Z_i^T$, such that the LMI in (11) and the following LMI for $\delta = 0$ and $\delta = 1$ hold

$$
\begin{bmatrix}
\Xi(\delta) & D_a & D_b \\
* & -R_d & 0 \\
* & -3S_d & 0
\end{bmatrix} < 0,
$$

(30)

where

$$
\Xi(\delta) = \begin{bmatrix}
Y_iA + AT_iY_i^T + Q_i + Q_i^T + S_i - (2 - \delta)Z \\
\gamma^2 Z - \gamma(Y_i + Y_i^T)Z \\
B_i^T K_i^T + (2 - \delta)Z \\
-(1 + \delta)Z \end{bmatrix}
$$

(31)
with $D_s$ and $D_h$ defined in (17), $R_d$ and $S_d$ defined in (18) and (19), respectively.

Furthermore the error system in (3) is globally asymptotically stable with control gain given by $K = Y_1^{-1} \bar{K}$ in the affirmative case.

Proof. Consider the LMI in (12), substitute $F$ by $-KH$ and perform a change with the particular choice $Y_2 = \gamma Y_1$, where $\gamma$ is a tuning scalar parameter. Then, introducing the linearization variable: $\hat{K} = Y_1 K$, the LMI in (28) is obtained. Moreover, from Theorem 1, it is easy to see that the control gain assures the asymptotic stability for (4), which concludes the proof. ■

5. Numerical Illustrative Examples

Consider the circuit of Chua governed by the following equations

\[
\begin{align*}
\dot{x} &= a(y - f(x)), \\
y &= x - y - z, \\
\dot{z} &= by,
\end{align*}
\]

where $f(x)$, the nonlinear characteristic of the system, is given by

\[
f(x) = m_1 x + \frac{1}{2}(m_0 - m_1)(|x + c| - |x - c|)
\]

and the parameters are $a = 9$, $b = 14.28$, $c = 1$, $m_0 = -1/7$, $m_1 = 2/7$. With this set of parameters, the system settles on the well-known double scroll attractor.

System (30) can be represented in the Lur'e form as follows

\[
A = \begin{bmatrix}
-a m_1 & a & 0 \\
1 & -1 & 1 \\
0 & b & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
-a (m_0 - m_1) \\
0 \\
0
\end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}
\]

and $\sigma(\nu) = 1/2(|\nu + c| - |\nu - c|)$ is a nondecreasing Lipschitz continuous function, such that (4) is satisfied with Lipschitz constant $\varepsilon = 1$.

As considered in [Yakicon et al., 2001], the output matrix in (1a) and (1b) is selected as $H = [1 \ 0 \ 0]$, which means that the master system is connected to the slave system by the first state variable. Besides, it is assumed that the sampling period is given by $h = 10$ ms (100 Hz).

5.1. Synchronizing Chua’s circuits over networks with packet dropouts

The objective in this example is to design a state-feedback controller as in (1c) such that the closed-loop system is asymptotically stable for the maximum number of data packet dropouts $\Delta_{\text{max}}$. In addition, the influence of the parameters $\gamma$ and $N$ in Theorem 2 is investigated. The results are presented in Table 1.

The results shown in Table 1 indicate that there is no improvement by increasing the parameter $N$. However, an improvement is obtained for a good choice of the parameter $\gamma$. Motivated by the results presented in Table 1, $\gamma = 0.5$ and $N = 1$ are chosen in the application of Theorem 2, with $h = 10$ ms (100 Hz). In this case, a controller is obtained that synchronizes the scheme in (1) for the maximum number of data packet dropouts $\Delta_{\text{max}} = 18$, with corresponding controller gains $K^T = \begin{bmatrix} 3.7752 & 0.7573 & 3.4921 \end{bmatrix}$.

To further illustrate the results, the coupled Chua’s circuits were implemented on the MATLAB/Simulink numerical simulation environment, along with a random zero-order sample and hold function. Figure 2 shows the result of the simulation of coupled Chua’s circuits for the maximum number of data packet dropouts $\Delta_{\text{max}} = 18$. It can be readily seen that the synchronization is achieved after approximately five seconds. Both the continuous-time synchronization error and the resulting signal after the action of the random zero-order sample and hold function are shown in Fig. 3. The detailed view of both signals can be seen in Fig. 3(b). Notice that the sampling time changes at random between the minimum value ($h = 10$ ms) and maximum number of data packet dropouts $\Delta_{\text{max}} \times 10$ ms. The error distribution is close to an uniform distribution as it can be seen in Fig. 4.

In the simulation tests for the same values of the gain $K$, the maximum number of data packet dropouts can be increased up to $\Delta_{\text{max}} = 84$ before losing synchronization. Figures 5 and 6 show, for $\Delta_{\text{max}} = 83$, the comparison between the trajectories of the $x$ variable for the master and slave and the comparison between the continuous-time error and the result of the random zero-order sample and hold function. Note that the control takes around 15 sec to almost synchronize. The synchronization is achieved for short period of times before the control loses it [see Fig. 6(b) for an example].
Table 1. Results obtained by applying Theorem 2 for $h = 10$ ms and different values for $\gamma$ and $N$. The objective is to design a state-feedback controller as in (1e) such that the closed loop system is asymptotically stable for the maximum number of data packet dropouts $\Delta_{\text{max}}$.

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Fig. 2. Comparison between the trajectories of the $x(t)$ variable for Master and Slave in coupled Chua’s circuits: (a) whole trajectory and (b) detailed view of the first five seconds. The maximum number of data packet dropouts $\Delta_{\text{max}} = 18$.  

Synchronizing Continuous Time Chaotic Systems Over Nondeterministic Networks
5.2. Chaos based information transmission over the internet

From the previous example, a practical application of synchronizing master–slave Chua’s circuits to transmit encrypted information over the Internet can be investigated.

The general idea is depicted in Fig. 7. The application scenario can be thought of as the attempt to exchange sensible data through the Internet by means of the numerical integration of differential equations describing master and slave chaotic systems, while employing a binary Chaos Shift Keying modulation technique [Dedieu et al., 1993]. In this case, a given binary message is used to select which signal \( s(t) = x(t) \), when the bit to be transmitted is “1”; or \( s(t) = h(x(t)) \), with \( h : \mathbb{R} \rightarrow \mathbb{R} \) an arbitrary nonlinear function, when the bit to be transmitted is “0”. The goal is to show the effectiveness of the previous theoretical results on establishing performance limits in a practical, and yet easy to understand, application. In this context, it is important to notice that the information transmission technique itself could be much improved.

An example of transmitted signal, considering a symbol duration \( t_{\text{symbol}} = 5 \text{ sec} \), is depicted in Fig. 8. Notice that the nonlinear function \( h(\cdot) \) should be chosen such that it introduces minimal detectable time variations on the statistics of the transmitted signal for each symbol, in order to enhance the system security against eavesdrops.

In Fig. 9, simulation results corresponding to the transmission of information using a pair of master–slave connected Chua’s circuits are shown for the case where a maximum number of data packet dropouts \( \Delta_{\text{max}} = 15 \). It is easily seen that, despite the random loss of data packets, the energy associated to the error signal at the receiver end is smaller when a bit “1” is transmitted, than when a bit “0” is sent down the Internet channel. This greatly facilitates the synthesis of some symbol detection algorithm based on variation of error energy.
Fig. 5. Comparison between the trajectories of the $x(t)$ variable for Master and Slave in the coupled Chua's circuit: (a) whole trajectory and (b) detailed view of the period between 20 sec and 30 sec. The maximum number of data packet dropouts $\Delta_{\text{max}} = 85$.

Fig. 6. Comparison between the continuous-time error and the result of the random zero-order sample and hold function: (a) whole trajectory and (b) detailed view of the period between 20 sec and 30 sec. The maximum number of data packet dropouts $\Delta_{\text{max}} = 85$. 
Fig. 7. Chaos based information transmission system over the Internet. The digital coupling between Master and Slave systems suffers from data packet dropouts and variable time delay associated to the nondeterministic nature of Internet communication.

Fig. 8. Transmitted signal encoding a given binary message.

Fig. 9. Information transmission results corresponding to the transmitted signal shown in Fig. 8, and using the controller gains obtained for $N = 4, \gamma = 0.1$, when the maximum number of data packet dropouts $\Delta_{\text{max}} = 15$ (see Table 1). Sampling period $h = 10\,\text{ms}$, and symbol duration $t_{\text{symbol}} = 5\,\text{sec}$. 

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When the Internet channel quality degrades from what it was originally expected, such that the maximum number of data packet dropouts becomes $\Delta_{\text{max}} = 75$, the corresponding simulation results are shown in Fig. 10. In this case there is no more clear separation between error energies associated to transmitted symbols as a result of many events of desynchronization. This shows that the controller gains, originally calculated for $\Delta_{\text{max}} = 15$, are unsuitable in this case.

6. Conclusions

The master–slave synchronization of continuous time nonlinear dynamical systems of Lur'e type, with nondecreasing and Lipschitz continuous nonlinearities, over nondeterministic networks with packet dropouts was investigated in the context of synchronization of time-delayed systems with time-varying delays.

By representing the hybrid system comprised by two continuous time dynamical systems coupled relying on an event driven sample-and-hold process as one single continuous time error system with nondifferential time-varying delay, analysis and design criteria to guarantee synchronization achievement were obtained. The theoretical framework was developed based on discretized Lyapunov functional for systems with pointwise delay, and the convex analysis for systems with time-varying delay.

Illustrative examples were provided to show the practical application of the proposed methodology, particularly in the context of transmitting secure information through the Internet based on synchronized chaotic systems.

In a future paper the authors plan to take into account the quantizer effects (see e.g., [Fridman & Dambrine, 2009]) and to explore possibilities of obtaining more precise methods using nonlinear matrix inequalities and discontinuous Lyapunov functionals (see e.g., [Fridman et al., 2008; Liu & Fridman, 2012a, 2012b]).

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References


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