Robust $\mathcal{H}_\infty$ Filtering for Uncertain Discrete-Time State-Delayed Systems

Reinaldo Martinez Palhares, Member, IEEE, Carlos E. de Souza, Senior Member, IEEE, and Pedro Luis Dias Peres, Member, IEEE

Abstract—This paper addresses the problem of robust $\mathcal{H}_\infty$ filtering for linear discrete-time systems subject to parameter uncertainties in the system state-space model and with multiple time delays in the state variables. The uncertain parameters are supposed to belong to a given convex bounded polyhedral domain. A methodology is developed to design a stable linear filter that assures asymptotic stability and a prescribed $\mathcal{H}_\infty$ performance for the filtering error, irrespective of the uncertainty and the time delays. The proposed design is given in terms of linear matrix inequalities, which has the advantage in that it can be implemented numerically very efficiently.

Index Terms—Discrete-time systems, $\mathcal{H}_\infty$ filtering, linear matrix inequalities, robust filtering, time delays.

I. INTRODUCTION

RECENTLY, there has been a lot of interest on the problem of robust $\mathcal{H}_\infty$ filtering for discrete-time linear systems with parametric uncertainty. In robust $\mathcal{H}_\infty$ filtering, the noise signals are assumed to be arbitrary deterministic signals of bounded energy (or average power), and the problem is to design a filter that ensures a prescribed bound for the induced $\ell_2$-norm of the operator from the noise signals to the filtering error, irrespective of the uncertainty. Several robust $\mathcal{H}_\infty$ filtering approaches have been developed over the past few years. When norm-bounded uncertainties are considered, one may cite, among others, the Riccati equation approaches in [1]–[3] and the linear matrix inequality (LMI)-based technique in [4]. On the other hand, convex bounded uncertainties have been treated very recently in [5] and [6] by means of an LMI methodology.

The existing methods of robust $\mathcal{H}_\infty$ filter design, however, do not directly apply in the context of state-space models with time delays in the state variable. Since time delays usually result in unsatisfactory performance and are frequently a source of instability, their presence must be considered in realistic filter designs [7]–[10]. It turns out that the noise attenuation level guaranteed by a robust $\mathcal{H}_\infty$ filter design without considering time delays may collapse if the system actually exhibits non-negligible time delays. On the other hand, state-space models for linear filter analysis and design have proved to be quite useful in many situations in the context of signal processing [4], [11]–[15], and there is a large number of applications in this field where state delays are unavoidable and must be taken into account in a realistic filter design as, for instance, echo cancellation, local loop equalization, multipath propagation in mobile communication, array signal processing, and congestion analysis and control in high-speed communication networks [11], [16]–[21]. Furthermore, it should be remarked that state delays appear in any filtering application where the sensors are subject to time delays, which can arise due to transport phenomena (e.g., of information, material or energy) or required numerical processing of the measurements. In such situations, state delays will appear in the measurement equation of the state-space model.

The filtering problem for linear discrete-time systems with time delays has been attracting increasing attention over the past few years. Specifically, the design of Kalman-type filters has been addressed in [7] and [22] without considering the presence of parameter uncertainty in the system model. Very recently, the problem of robust $\mathcal{H}_\infty$ filtering for discrete-time systems with a single time-delay and norm-bounded parameter uncertainty has been addressed by means of coupled Riccati equations/inequalities with scaling parameters [23]. To the authors’ knowledge, the problem of $\mathcal{H}_\infty$ filtering in the context of discrete-time state-space models with multiple state delays has not been fully investigated in the literature, and, in particular, the design of robust $\mathcal{H}_\infty$ filters for this class of models in the presence of polytopic-type parameter uncertainty.

This paper deals with the robust $\mathcal{H}_\infty$ filtering problem for linear discrete-time systems subject to parameter uncertainties in all the matrices of the system state-space model and with multiple time delays in the state variables. The uncertain parameters are assumed to belong to a given convex bounded polytope. A methodology for the design of a full-order stable linear filter that assures asymptotic stability and a prescribed $\mathcal{H}_\infty$ performance for the filtering error system, irrespective of the uncertainty and the time delays, is developed. The suitable robust filter is obtained from the solution of a convex optimization problem in terms of LMIs, which can be solved numerically very efficiently using recently developed LMI algorithms [24]. The proposed filter design methodology has the advantage that it can easily handle additional filter design constraints that keep the problem...
convex, for instance, when the filter is required to satisfy certain structure constraints.

The notation used in this paper is quite standard. The bold-face characters $\mathbf{I}$ and $\mathbf{0}$ stand for the identity and the zero matrices of appropriate dimensions, respectively. $\mathbb{Z}^n$ denotes the space of all sequences mapping $\mathbb{Z}^+ \rightarrow \mathbb{R}^n$, and $\ell_2$ denotes the subset of all sequences $\xi \in \mathbb{Z}^n$, which satisfy $\|\xi\|_2 \triangleq \left( \sum_{n=1}^{\infty} |\xi(n)|^2 \right)^{1/2} < \infty$, where $\| \cdot \|$ stands for the Euclidean vector norm. For a symmetric block matrix, the symbol $\ast$ denotes the submatrices that lie below the main block-diagonal, whereas $\text{diag}\{\cdots\}$ stands for a block-diagonal matrix.

II. Problem Formulation

Consider the following uncertain discrete-time system model with multiple time-delayed states

\[ x(t+1) = A_0 x(t) + \sum_{j=1}^{q} A_j x(t-\tau_j) + B w(t) \]  \hspace{1cm} (1)

\[ y(t) = C_0 x(t) + \sum_{j=1}^{q} C_j x(t-\tau_j) + D w(t) \]  \hspace{1cm} (2)

\[ s(t) = L_0 x(t) + \sum_{j=1}^{q} L_j x(t-\tau_j) + T w(t) \]  \hspace{1cm} (3)

\[ x(t) = \phi(t); \ t = -\tau, -\tau + 1, \ldots, 0, \ \tau = \max_j \tau_j \]  \hspace{1cm} (4)

where $x(t) \in \mathbb{Z}^n$ is the state vector, $y(t) \in \mathbb{Z}^p$ is the measurements vector, $w(t) \in \mathbb{Z}^m$ is the noise signal vector (including process and measurement noises), $s(t) \in \mathbb{Z}^p$ is the signal to be estimated. \{\phi(t), \ t = -\tau, -\tau + 1, \ldots, 0\} is a known given initial condition sequence, and $\tau_j \geq 0, j = 1, \ldots, q$ are known constant time delays.

The noise signal $w$ is assumed to be an arbitrary signal in $\ell_2$, and the system matrices are supposed to be unknown (uncertain) but belonging to a known convex compact set of polytopic type, namely

\[ (A_0, A_1, \ldots, A_q, B, C_0, \ldots, C_q, D, L_0, \ldots, L_q, T) \in \mathcal{D} \]  \hspace{1cm} (5)

where $\mathcal{D}$ is a given convex bounded polyhedral domain described by $\kappa$ vertices as follows:

\[ \mathcal{D} \triangleq \left\{ (A_0, \ldots, A_q, B, C_0, \ldots, C_q, D, L_0, \ldots, L_q, T) \right\} \]

\[ \left( A_0, \ldots, A_q, B, C_0, \ldots, C_q, D, L_0, \ldots, L_q, T \right) \]

\[ = \sum_{i=1}^{\kappa} \alpha_i (A_{0i}, \ldots, A_{qi}, B, C_{0i}, \ldots, C_{qi}, D, L_{0i}, \ldots, L_{qi}, T_i), \ \alpha_i \geq 0, \ \sum_{i=1}^{\kappa} \alpha_i = 1 \}. \]  \hspace{1cm} (6)

This kind of convex bounded parameter uncertainty has been widely used in the context of robust control (see, for instance, [25] and references therein).

This paper is concerned with obtaining an estimate $\hat{s}(t)$ of the signal $s(t)$ that provides small estimation error $\|\hat{s}(t) - s(t)\|_2$, over the horizon $\{t = 1, 2, \ldots, \infty\}$, for all $w \in \ell_2$ and over the entire uncertainty domain $\mathcal{D}$ in a sense to be defined in the sequel. Attention is focused on the design of a stationary linear time-invariant, asymptotically stable, filter of order $n$ with state-space realization of the form

\[ \dot{x}(t+1) = A_f \dot{x}(t) + B f(t), \ \ \dot{x}(0) = 0 \]  \hspace{1cm} (7)

\[ \dot{s}(t) = C_f \dot{x}(t) + D f(t) \]  \hspace{1cm} (8)

where $A_f, B_f, C_f,$ and $D_f$ are matrices to be determined.

Defining the augmented state vector $\hat{x}(t) \triangleq [x'(t), \dot{x}'(t)]'$, the filtering error dynamics are described by

\[ \dot{x}(t+1) = \hat{A} \hat{x}(t) + \sum_{j=1}^{q} \hat{A}_j E \hat{x}(t-\tau_j) + \hat{B} w(t) \]  \hspace{1cm} (9)

\[ \dot{s}(t) = \hat{C} \hat{x}(t) + \sum_{j=1}^{q} \hat{C}_j E \hat{x}(t-\tau_j) + \hat{D} w(t) \]  \hspace{1cm} (10)

\[ \hat{x}(t) = [\phi'(t) \ 0]' \]  \hspace{1cm} (11)

where

\[ \hat{A} = \begin{bmatrix} A_0 & 0 \\ B_f C_0 & A_f \end{bmatrix}, \ \ \hat{C} = [L_0 - D_f C_0 - C_f] \]  \hspace{1cm} (12)

\[ \hat{A}_j = \begin{bmatrix} A_j \\ B_f C_j \end{bmatrix}, \ \ \hat{C}_j = L_j - D_f C_j, \ j = 1, \ldots, q \]  \hspace{1cm} (13)

\[ \hat{B} = \begin{bmatrix} B \\ B_f D \end{bmatrix}, \ \ \hat{D} = T - D_f D, \ E = [\mathbf{I} \ 0]. \]  \hspace{1cm} (14)

Throughout the paper, $\hat{A}_0, \hat{A}_1, \ldots, \hat{A}_q, \hat{C}_0, \ldots, \hat{C}_q, \hat{B},$ and $\hat{D}_i$, $i = 1, \ldots, \kappa$ denote the matrices $\hat{A}_0, \hat{A}_1, \ldots, \hat{A}_q, \hat{C}_0, \ldots, \hat{C}_q, \hat{B},$ and $\hat{D}$ evaluated at each of the vertices of the polytope $\mathcal{D}$.

Observe that as the filtering problem treated in this paper is over infinite horizon, one is required to ensure the asymptotic stability of the filtering error dynamics over the entire uncertainty domain $\mathcal{D}$.

The filtering problem addressed in this paper is as follows. Given a scalar $\gamma > 0$, determine an asymptotically stable linear filter of the form of (7) and (8) such that the resulting filtering error system (9) and (10) is globally asymptotically stable over the entire uncertainty domain $\mathcal{D}$.

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III. Robust $\mathcal{H}_\infty$ Analysis

In the sequel, a sufficient condition for the filtering error system (9)–(11) resulting from a given filter (7) and (8) to have a guaranteed $\gamma$ level of noise attenuation is established in terms
Lemma 1: Consider the filtering error system of (9) and (11) for a given filter (7) and (8), and let \( \gamma > 0 \) be a given scalar. If there exist symmetric matrices \( P \in \mathbb{R}^{2n \times 2n} \) and \( P_j \in \mathbb{R}^{n \times n}, j = 1, \ldots, q \) such that
\[
\begin{bmatrix}
Y_1 & Y_2 \\
Y_2 & Y_3
\end{bmatrix} > 0, \quad \forall i = 1, \ldots, \kappa
\]  
(16)
where
\[
Y_1 = \text{diag}(P, P)
\]  
(17)
\[
Y_2 = \begin{bmatrix}
P \hat{A}_0 & P \hat{A}_1 & \cdots & P \hat{A}_q & P \hat{B}_1 \\
\hat{C}_0 & \hat{C}_1 & \cdots & \hat{C}_q & \hat{D}_1
\end{bmatrix}
\]  
(18)
\[
Y_3 = \text{diag} \left\{ P - \sum_{j=1}^{q} E' P_j E, P_1, \ldots, P_q, \gamma^2 I \right\}
\]  
(19)
then the uncertain filtering error system has a guaranteed level of noise attenuation.

Proof: First, the global asymptotic stability of the estimation error system of (9) with \( u(t) \equiv 0 \) is established. For that, the Lyapunov functional
\[
V(\hat{e}_r(t)) = \hat{e}'(t) P \hat{e}(t) + \sum_{k=1}^{\tau_1} \hat{e}'(k) E' P_1 E \hat{e}(k)
\]  
+ \cdots + \sum_{k=1}^{\tau_q} \hat{e}'(k) E' P_q E \hat{e}(k)
\]  
(20)
where \( P \in \mathbb{R}^{2n \times 2n} \) and \( P_j \in \mathbb{R}^{n \times n}, j = 1, \ldots, q \) are symmetric positive definite matrices to be determined, \( \hat{e}_r(t) \) denotes the vector \( [\hat{e}(t)'(t+\tau_1) \cdots \hat{e}(t+\tau_q)]' \), and \( E \) is as in (14).

Along the solutions of (9) with \( u(t) \equiv 0 \) and for any arbitrary initial condition, one has
\[
\Delta V(\hat{e}_r(t)) = V(\hat{e}_r(t+1)) - V(\hat{e}_r(t))
\]  
= \( \hat{e}'(t) \left( \tilde{A}_0 P \tilde{A}_0 - P + \sum_{j=1}^{q} E' P_j E \right) \hat{e}(t) \)
+ \( 2 \sum_{j=1}^{q} \hat{e}'(t) \tilde{A}_j P \tilde{A}_j x(t - \tau_j) \)
- \( \sum_{j=1}^{q} \hat{e}'(t - \tau_j) P_j x(t - \tau_j) \)
+ \( \sum_{j=1}^{q} \sum_{k=1}^{q} \hat{e}'(t - \tau_j) \tilde{A}_j P \tilde{A}_k x(t - \tau_k) \).
\]  
(21)
Defining \( \xi(t) = [\hat{e}'(t) x'(t - \tau_1) \cdots x'(t - \tau_q)]' \), (21) can be rewritten in the following quadratic form:
\[
\Delta V(\hat{e}_r(t)) = \xi'(t) Q \xi(t)
\]  
(22)
where
\[
Q = \begin{bmatrix}
Q_1 & Q_2 \\
Q_2 & Q_3
\end{bmatrix}
\]  
(23)
\[
Q_1 = P - \tilde{A}_0 P \tilde{A}_0 - \sum_{j=1}^{q} E' P_j E
\]  
(24)
\[
Q_2 = \begin{bmatrix}
-\tilde{A}_0 P \tilde{A}_1 & \cdots & -\tilde{A}_0 P \tilde{A}_q \\
\tilde{A}_1 P \tilde{A}_0 & \cdots & \tilde{A}_1 \tilde{A}_q \\
\vdots & \ddots & \vdots \\
\tilde{A}_q P \tilde{A}_0 & \cdots & -\tilde{A}_q \tilde{A}_q
\end{bmatrix}
\]  
(25)
\[
Q_3 = \begin{bmatrix}
P - \tilde{A}_0 \tilde{A}_1 & \cdots & -\tilde{A}_0 \tilde{A}_q \\
\vdots & \ddots & \vdots \\
* & \cdots & * \\
-\tilde{A}_1 \tilde{A}_0 & \cdots & -\tilde{A}_1 \tilde{A}_q
\end{bmatrix}
\]  
(26)
Consider that the LMIs of (16) are affine in the system matrices, using Schur’s complements, it can be easily verified that (16) ensures that \( Q > 0 \) over the entire uncertainty domain \( \mathcal{D} \). Note that the LMIs of (16) also imply that \( P \) and \( P_j \), \( j = 1, \ldots, q \) are positive definite matrices. Therefore, from standard Lyapunov–Krasovskii stability results, it can be concluded that the filtering error system is globally asymptotically stable over the entire uncertainty domain \( \mathcal{D} \).

To establish the \( \mathcal{H}_\infty \) performance for the filtering error system, first notice that as (9) is globally asymptotically stable over the entire uncertainty domain \( \mathcal{D} \) and \( w \in \mathcal{L}_2 \), then \( \delta \in \mathcal{L}_2 \), and \( \hat{e}_r(t) \) tends to zero as \( t \to \infty \). Next, assuming zero initial conditions for the filtering error system, the performance index
\[
J = \int_{t_0}^{\infty} \left[ \hat{e}'(t) \hat{e}(t) - \gamma^2 u'(t) u(t) \right] dt
\]  
(27)
where \( V(\hat{e}_r(t)) = 0 \) and \( V(\hat{e}_r(t))_{t=\infty} \to 0 \) and using (9) and (10), one can easily obtain that
\[
J = \sum_{t_0}^{\infty} \left[ \hat{e}'(t) \left( \tilde{A}_0 P \tilde{A}_0 - P + \sum_{j=1}^{q} E' P_j E + \tilde{C}_0 \tilde{C}_0 \right) \hat{e}(t) \right.
+ 2 \sum_{j=1}^{q} \hat{e}'(t) \tilde{A}_j P \tilde{A}_j x(t - \tau_j)
- \sum_{j=1}^{q} \hat{e}'(t - \tau_j) P_j x(t - \tau_j)
+ \sum_{j=1}^{q} \sum_{k=1}^{q} \hat{e}'(t - \tau_j) \tilde{A}_j P \tilde{A}_k x(t - \tau_k).
\]  
(28)
Defining $\eta(t) \triangleq [z'(t) \ x'(t-\tau_1) \ldots x'(t-\tau_q) \ w'(t)]'$, one can rewrite (24) in the following form:

$$J = -\sum_{t=0}^{\infty} \eta(t)W\eta(t)$$

(25)

where we have the first equation at the bottom of the page.

In the light of the above, it can be easily verified that

$$W = \gamma_3 - \gamma_2 \gamma_1^{-1} \gamma_2$$

where $\gamma_1$ and $\gamma_3$ are as defined in (17) and (19), respectively, and

$$\gamma_2 = \begin{bmatrix} P \hat{A}_0 & P \hat{A}_1 & \cdots & P \hat{A}_q & P \hat{B} \\ \hat{C}_0 & \hat{C}_1 & \cdots & \hat{C}_q & \hat{D} \end{bmatrix}.$$  

(26)

Next, using Schur’s complements, it can be readily verified that the LMIs of (16) ensure that $W > 0$ over the entire uncertainty domain $D$. This implies that $J < 0$ for any nonzero $w \in E$ and over the entire uncertainty domain $D$, i.e., the filtering error system has a guaranteed $\gamma$ level of noise attenuation, which concludes the proof.

**Remark 1:** It should be noted that as the conditions of Lemma 1 do not use any information about the size of the time delays $\tau_j$, $j = 1, \ldots, q$, this lemma ensures that the filtering error system of (9)–(11) is globally asymptotically stable with a prescribed $\gamma$ level of noise attenuation over the entire uncertainty domain $D$ and for any finite and constant time delays $\tau_j \geq 0$, $j = 1, \ldots, q$.

**IV. ROBUST $H_\infty$ FILTER DESIGN**

The following theorem provides a sufficient condition for the existence of a linear stable filter assuring a robust $H_\infty$ performance for linear systems described by discrete-time state-space models subject to polytopic-type parameter uncertainty and multiple delayed states of the form (1)–(4).

**Theorem 1:** Consider the system (1)–(4), and let $\gamma > 0$ be a given scalar. If there exist symmetric matrices $R \in \mathbb{R}^{n \times n}$, $X \in \mathbb{R}^{n \times n}$, and $P_j \in \mathbb{R}^{n \times n}$, $j = 1, \ldots, q$ and matrices $M \in \mathbb{R}^{n \times n}$, $N \in \mathbb{R}^{n \times r}$, $Z \in \mathbb{R}^{n \times r}$, and $D_f \in \mathbb{R}^{p \times r}$ satisfying

$$
\begin{bmatrix}
\Phi_1 & \Phi_2 \\
\Phi_2^T & \Phi_3
\end{bmatrix} > 0, \quad \forall i = 1, \ldots, \kappa
$$

(27)

where we have (28)–(33), shown at the bottom of the page, then the robust $H_\infty$ filtering problem for system (1)–(4) is solvable. Moreover, under the above conditions, the transfer function matrix $H_{by}(z)$ of a suitable filter is given by

$$H_{by}(z) = C_f (zI - A_f)^{-1} B_f + D_f$$

(34)

---

$$W \triangleq \begin{bmatrix} W_1 & W_2 \\ W_2 & W_3 \end{bmatrix}$$

$$W_1 = P - \hat{A}_0 P \hat{A}_0 - \sum_{j=1}^{q} E^T P_j E - \hat{C}_0 \hat{C}_0$$

$$W_2 = -\hat{A}_0 P \begin{bmatrix} \hat{A}_1 & \cdots & \hat{A}_q & \hat{B} \\ \hat{C}_1 & \cdots & \hat{C}_q & \hat{D} \end{bmatrix} - \hat{C}_0 \begin{bmatrix} \hat{C}_1 & \cdots & \hat{C}_q \end{bmatrix}$$

$$W_3 = \begin{bmatrix}
P_1 - Y_{11} - \hat{C}_1 \hat{C}_1 - Y_{12} - \hat{C}_2 \hat{C}_2 & \cdots & -Y_{1q} - \hat{C}_q \hat{C}_q - \hat{A}_0 P \hat{B} - \hat{C}_1 \hat{D} \\
* & P_2 - Y_{22} - \hat{C}_2 \hat{C}_2 & \cdots & -Y_{2q} - \hat{C}_q \hat{C}_q - \hat{A}_0 P \hat{B} - \hat{C}_2 \hat{D} \\
* & * & \ddots & \cdots & \cdots & \cdots \\
* & * & * & P_q - Y_{qq} - \hat{C}_q \hat{C}_q - \hat{A}_0 P \hat{B} - \hat{C}_q \hat{D} \\
* & * & * & * & \hat{C}_f \hat{D} - \hat{D} \hat{D}^T + \hat{D} \hat{B} \hat{P}_f
\end{bmatrix}$$

$$Y_{jk} = \hat{A}_j P \hat{A}_k.$$
where \( D_f \) satisfies (27), and
\[
A_f = (R - X)^{-1} M, \quad B_f = (R - X)^{-1} Z, \quad C_f = N. \tag{35}
\]

**Proof:** Assume there exist matrices \( R, X, P_j, j = 1, \ldots, q, M, N, Z, \) and \( D_f \) satisfying (27). The positive definiteness of \( \Phi_2 \) and \( \Phi_3 \) imply that \( R, X, P_j, j = 1, \ldots, q, \) and \( X - R \) are symmetric positive definite matrices. Therefore, it follows that the matrix \( I - XR^{-1} \) is nonsingular. This implies that it is always possible to find square and nonsingular matrices \( U \) and \( V \) such that \( I - XR^{-1} = UV \).

Now, introduce the nonsingular matrices \( J \) and \( \bar{J} \) as follows:
\[
J \triangleq \begin{bmatrix} R^{-1} & \mathbf{0} \\ V & \mathbf{0} \end{bmatrix}, \quad \bar{J} \triangleq \begin{bmatrix} \mathbf{0} & X \\ \mathbf{0} & U \end{bmatrix} \tag{36}
\]
and define the following symmetric matrix:
\[
P \triangleq \bar{J} J^{-1} \tag{37}
\]
where the matrices \( U \) and \( V \) are such that \( P \) can be partitioned as
\[
P \triangleq \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^T & P_{22} \end{bmatrix}, \quad \text{with } P_{22} \triangleq \mathbf{I}.
\]
The above implies that
\[
P_{11} = X, \quad P_{12} = U, \quad U = -RV. \tag{38}
\]
Moreover, since \( UV^T = I - XR^{-1} \), it can be easily established using (38) that \( X - UU^T = R > 0 \). This implies, by using Schur’s complements, that \( P > 0 \).

Pre and postmultiplying the LMIs of (27) by
\[
\text{diag} \{ R^{-1}, I, R^{-1}, I \} \tag{39}
\]
followed by the change of variables
\[
\hat{Z} \triangleq MR^{-1}, \quad \tilde{Z} \triangleq NR^{-1} \tag{40}
\]
and introducing the matrices \( A_f, B_f, \) and \( C_f \) given by
\[
A_f \triangleq U^{-1}\hat{Z}(V^T)^{-1}, \quad B_f \triangleq U^{-1}Z, \quad C_f \triangleq \tilde{Z}(V^T)^{-1} \tag{41}
\]
one can readily obtain that (27) is equivalent to
\[
\begin{bmatrix} \Lambda_1 & \Lambda_2 \\ \Lambda_2 & \Lambda_3 \end{bmatrix} > \mathbf{0}, \quad \forall i = 1, \ldots, \kappa \tag{42}
\]
where
\[
\Lambda_1 = \text{diag} \{ J'\hat{J}, I \} \tag{43}
\]
\[
\Lambda_{2i} = \begin{bmatrix} J' \hat{A}_{qi} J & \cdots & J' \hat{A}_{qi} J' \hat{B}_i \\ C_{qi} J & \cdots & C_{qi} J' \hat{B}_i \\ \hat{C}_{qi} J & \cdots & \hat{C}_{qi} J' \hat{B}_i \end{bmatrix} \tag{44}
\]
\[
\Lambda_3 = \text{diag} \begin{bmatrix} J'J - \sum_{j=1}^{q} J'P_j E_j P_j, J, \ldots, J'P_q E_q P_q, J' \end{bmatrix}. \tag{45}
\]

Next, pre and postmultiplying the inequalities of (42) by \( \text{diag} \{ (J'J)^{-1}, I, (J'J)^{-1}, I \} \) and \( \text{diag} \{ J^{-1}, I, J^{-1}, I \} \), respectively, and considering (37), it can be readily established that (42) is equivalent to the LMIs of (16). Hence, one can conclude from Lemma 1 that the filter with a state-space realization \( (A_f, B_f, C_f, D_f) \), where \( A_f, B_f, \) and \( C_f \) are as in (41) and \( D_f \) satisfies (27), assures that the resulting filtering error system has a guaranteed \( \gamma \) level of noise attenuation.

Now, using (41) and taking into account (40), the transfer function matrix \( H_{sy}(z) \), from \( y \) to \( \hat{s} \), of the above filter can be expressed as
\[
H_{sy}(z) = N \left[ z \mathbf{I} - (UV^T)^{-1} M \right]^{-1} (UV^T)^{-1} Z + D_f. \tag{46}
\]
Finally, considering that \( UV^T R = R - X \), the matrices of the filter in (35) are readily obtained.

**Remark 2:** In view of Theorem 1, the robust \( H_\infty \) filtering problem for system (1)–(4) can be solved in terms of the feasibility of the LMIs of (27). In fact, any feasible solution to (27) yields a suitable robust filter. Further, the robust filter with the smallest \( \gamma \) attenuation level obtainable from Theorem 1 can be easily determined by solving the following convex optimization problem:
\[
\min_{R, X, P_j, M, N, Z, D_f} \delta, \quad \text{subject to } \gamma^2 = \delta. \tag{47}
\]
The optimal filter transfer function matrix is as in (34) and (35), whereas the minimum value of \( \gamma \), namely, \( \gamma^* \), is given by \( \gamma^* = \sqrt{\delta^*} \), where \( \delta^* \) is the optimal value of \( \delta \).

**Remark 3:** It should be noted that the filter of Theorem 1 ensures global asymptotic stability and a guaranteed \( H_\infty \) performance for the filtering error system for all admissible uncertainties, irrespective of the size of the time delays. Thus, this filter provides a prescribed \( H_\infty \) performance that is robust with respect to parameter uncertainty in the matrices of the system state-space model as well as to uncertainty in the time delays in the state variables.

Further, observe that the filter provided by Theorem 1 is, in general, biproper, i.e., there is a directly feedforward from the noise signal \( w \) to the estimate \( \hat{s} \) of \( s \). However, if a strictly proper filter is desired, this can be easily achieved by imposing the constraint \( D_f = 0 \) to the LMIs of (27). Note that the smallest \( \gamma \) level of noise attenuation that can be achieved with a strictly proper filter, in general, is likely to be larger than that achieved with a biproper filter. The reason for this is that in the design of a strictly proper filter, setting \( D_f = 0 \) reduces the degree of freedom in the optimization problem of Remark 2, as compared with the case of the design of a proper filter. This fact is illustrated via an example in the next section.

**Remark 4:** It should be remarked that the robust filtering methodology of Theorem 1 finds application in a number of important problems in signal processing, where state delays are unavoidable and must be taken into account in a realistic filter design as, for instance, in echo cancellation, local loop equalization, multipath propagation in mobile communication, array signal processing, and congestion analysis and control in high-speed communication networks [11], [16]–[21].

A possible application of the proposed filter design is to the maximum likelihood sequence estimation problem, which is implemented through the algorithm of Viterbi and can be used in receivers for mobile communications [26]–[31]. The algorithm of Viterbi models the symbol sequence to be determined as a
sequence of states. The decision on the correct sequence depends on the state transition probabilities that are related to the channel characteristics. These characteristics can be resumed in a time-varying state vector that can be estimated from the sequence obtained as a delayed output of the algorithm of Viterbi. The state estimation, which is subject to symbol interference and to other kind of noises, must be performed in the presence of time delays. Although the aim of our paper is not to provide an accurate state-space model for this problem, all the capabilities of the proposed robust filtering methodology would apply, namely, noise attenuation (by means of an $\mathcal{H}_\infty$ attenuation level) and state estimation in the presence of modeling uncertainty and time delays.

A. Block-Decoupled Design

The proposed robust $\mathcal{H}_\infty$ filter design method can be readily extended to the case where the filter transfer function matrix $H_{sy}(z)$ is required to satisfy additional structure constraints. A typical example is the case where a “block-decoupled” filter is required, i.e., when $H_{sy}(z)$ is required to be block diagonal.

The design of a robust $\mathcal{H}_\infty$ filter with constraints on the structure of its transfer function matrix can be easily achieved by imposing the desired structure on the matrices $N$, $Z$, and $D_f$ and a corresponding block diagonal structure on the matrices $R$, $X$, and $M$ (in some cases $M$ can have a less restrictive structure).

For example, suppose that the filter transfer function matrix $H_{sy}(z)$ is required to have the following block-triangular structure:

$$H_{sy}(z) = \begin{bmatrix} \mathbb{I} & 0 \\ \mathbb{0} & \mathbb{0} \end{bmatrix}$$

where $\mathbb{I}$ stands for a block of appropriate dimensions with no additional restriction on its entries. If one adds the convex constraints

$$M = \begin{bmatrix} \mathbb{I} & 0 \\ \mathbb{0} & \mathbb{0} \end{bmatrix}, \quad N = \begin{bmatrix} \mathbb{I} & 0 \\ \mathbb{0} & \mathbb{0} \end{bmatrix}, \quad Z = \begin{bmatrix} \mathbb{I} & 0 \\ \mathbb{0} & \mathbb{0} \end{bmatrix}$$

$$D_f = \begin{bmatrix} \mathbb{I} & 0 \\ \mathbb{0} & \mathbb{0} \end{bmatrix}, \quad R = \begin{bmatrix} \mathbb{I} & 0 \\ \mathbb{0} & \mathbb{0} \end{bmatrix}, \quad X = \begin{bmatrix} \mathbb{I} & 0 \\ \mathbb{0} & \mathbb{0} \end{bmatrix}$$

to the LMIs of Theorem 1, the filter transfer function matrix of the robust $\mathcal{H}_\infty$ filter will have the desired structure.

V. Numerical Example

To illustrate the potentials of the proposed robust $\mathcal{H}_\infty$ filter design, consider the uncertain linear discrete-time signal model given by (1)–(6) with $q = 2$ time delays, and

$$A_0 = \begin{bmatrix} 0 & 0.3 \\ -0.2 & \rho \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 \\ 0.1 & \phi \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 0.1 \\ 0 & \psi \end{bmatrix}$$

(47)

$$C_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 0.2 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 0.3 & 0 \end{bmatrix}$$

(48)

$$B = \begin{bmatrix} 0 & 1 \end{bmatrix}^T, \quad D = 1, \quad T = 0$$

(49)

$$L_0 = \begin{bmatrix} 1 \end{bmatrix}, \quad L_1 = 0, \quad L_2 = 0$$

(50)

where $\rho$, $\phi$, and $\psi$ are uncertain real parameters satisfying

$$-0.5 \leq \rho \leq 0.4, \quad -0.1 \leq \phi \leq 0.1, \quad 0.1 \leq \psi \leq 0.2 $$

(51)

The minimum $\gamma$ attenuation level obtained from Theorem 1 is $\gamma^* = 13.0107$, and the state-space realization (35) of the corresponding optimal filter is given by

$$A_f = \begin{bmatrix} 0.1184 & 0.3017 \\ -0.0327 & -0.0819 \end{bmatrix}, \quad B_f = \begin{bmatrix} -0.1962 \\ -0.8044 \end{bmatrix}$$

$$C_f = \begin{bmatrix} 0.7409 & 2.0050 \end{bmatrix}, \quad D_f = 0.2006. $$

(52)

To illustrate the performance of the filter (52), Fig. 1 shows the maximum singular value plots of the corresponding filtering error transfer function as computed at the eight vertices of the uncertainty domain $\mathcal{D}$ and for $\tau_1 = 1$ and $\tau_2 = 2$. On the other hand, Fig. 2 displays the maximum singular value plots of the filtering error transfer function for $\tau_1 = 3$ and $\tau_2 = 25$. The effectiveness of the $\gamma^*$ attenuation level is apparent.

Next, one restricts the filter to be strictly proper. Applying Theorem 1 with the constraint $D_f = 0$, it has been found that the minimum achievable $\gamma$ level of noise attenuation is $\gamma^* = \gamma^*$.

Fig. 1. Maximum singular value of the filtering error transfer function at the vertices of $\mathcal{D}$ for the filter (52) with $\tau_1 = 1$ and $\tau_2 = 2$.

Fig. 2. Maximum singular value of the filtering error transfer function at the vertices of $\mathcal{D}$ for the filter (52) with $\tau_1 = 3$ and $\tau_2 = 25$. 

The minimum $\gamma$ attenuation level obtained from Theorem 1 is $\gamma^* = 13.0107$, and the state-space realization (35) of the corresponding optimal filter is given by

$$A_f = \begin{bmatrix} 0.1184 & 0.3017 \\ -0.0327 & -0.0819 \end{bmatrix}, \quad B_f = \begin{bmatrix} -0.1962 \\ -0.8044 \end{bmatrix}$$

$$C_f = \begin{bmatrix} 0.7409 & 2.0050 \end{bmatrix}, \quad D_f = 0.2006. $$

(52)
discrete-time systems with multiple time delays in the state variables and uncertain parameters belonging to a convex bounded domain have been considered. A methodology for the design of a stable linear filter that assures asymptotic stability and a prescribed $\mathcal{H}_\infty$ performance for the filtering error system, irrespective of the uncertainty and the time delays, is developed. The suitable robust filter is obtained from the solution of a convex optimization problem described in terms of LMIs, which can be solved numerically very efficiently.

**ACKNOWLEDGMENT**

The authors would like to thank Dr. A. Lopes, from the School of Electrical and Computer Engineering, University of Campinas, Campinas, Brazil, for his helpful comments.

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**Fig. 3.** Maximum singular value of the filtering error transfer function at the vertices of $\mathcal{D}$ for the filter (53) with $\tau_1 = 1$ and $\tau_2 = 2$.

**Fig. 4.** Maximum singular value of the filtering error transfer function at the vertices of $\mathcal{D}$ for the filter (53) with $\tau_1 = 3$ and $\tau_2 = 25$.

13,0838. The state-space realization (35) of the corresponding optimal filter is given by

$$A_f = \begin{bmatrix} -0.78838 & 0.31765 \\ -0.23762 & -0.09374 \end{bmatrix}, \quad B_f = \begin{bmatrix} -0.5304 \\ -0.9805 \end{bmatrix}$$

$$C_f = \begin{bmatrix} 0.99995 \\ 1.09997 \end{bmatrix}, \quad D_f = 0$$

(53)

The performance of the filter (53) is illustrated in Fig. 3, which shows the maximum singular value diagram of the resulting filtering error transfer function as computed at the vertices of $\mathcal{D}$ and for $\tau_1 = 1$ and $\tau_2 = 2$. On the other hand, Fig. 4 displays the maximum singular value diagram for $\tau_1 = 3$ and $\tau_2 = 25$.

**VI. CONCLUSIONS**

This paper investigated the design of robust $\mathcal{H}_\infty$ filters for linear discrete-time state-delayed systems subject to parameter uncertainties in the system state-space model. Linear


Carlos E. de Souza (M’81–SM’94) was born in João Pessoa, Brazil. He received the B.E. degree in electrical engineering (with highest honors) from the Universidade Federal de Pernambuco, Recife, Brazil, in 1976 and the Dr.-Ing. degree from the Université de Paris VI, Paris, France, in 1980.

From 1980 to 1984, he was a Lecturer with the Department of Electrical Engineering, Universidade Federal de Uberlândia, Uberlândia, Brazil. In 1985, he joined the Department of Electrical and Computer Engineering, University of Newcastle, Callaghan, Australia, as a Lecturer and became an Associate Professor in 1997. Since 1998, he has been a Professor with the Department of Systems and Control, Laboratório Nacional de Computação Científica (LNCC), Petrópolis, Brazil, and is the Director of the Research Center for Control of Dynamic Systems. During a sabbatical from 1992 to 1993, he was a Visiting Professor with the Laboratoire d’Automatique de Grenoble, Grenoble, France. He has also held numerous short-term visiting appointments at universities in several countries, including the United States, France, Switzerland, Israel, Australia, and Brazil. His research interests include robust signal estimation, optimal $H_\infty$ and $H_2$ estimation, robust control, optimal $H_2$ and $H_\infty$ control, and time-delay systems. He has published over 170 research papers. He was Subject Editor for the *International Journal of Robust and Nonlinear Control (IJRNC)* and Guest Editor for the IJRNC Special Issue on $H_\infty$ and Robust Filtering. He is currently a member of the Editorial Board of the IJRNC and Co-Chairman of the IFAC Technical Committee on Linear Systems.

Dr. de Souza is a Fellow of the Brazilian Academy of Sciences.

Pedro Luis Dias Peres (M’92) was born in Sorocaba, Brazil, in 1960. He received the B.Sc. and M.Sc. degrees in electrical engineering from the University of Campinas (UNICAMP), Campinas, Brazil, in 1982 and 1985, respectively, and the Doctorat en Automatique degree from the University Paul Sabatier, Toulouse, France, in 1989.

In 1990, he joined the School of Electrical and Computer Engineering, UNICAMP, where he is currently an Associate Professor. His main interests are control theory, convex analysis and optimization, and circuit theory.

Dr. Peres is a member of the Brazilian Society of Automatic Control.

Reinaldo Martinez Palhares (M’97) was born in Anápolis, Brazil, in 1969. He received the B.Sc. degree in electrical engineering from the Federal University of Goiás, Goiás, Brazil, in 1992 and the M.Sc. and Ph.D. degrees in electrical engineering from the University of Campinas, Campinas, Brazil, in 1995 and 1998, respectively.

In 1998, he joined the Graduate Program in Electrical Engineering, Pontifical Catholic University of Minas Gerais, Belo Horizonte, Brazil, as an Associate Professor. His main interests include robust control/filter theory and time-delay systems.