A novel approach for $H_2/H_\infty$ robust PID synthesis for uncertain systems

Eduardo N. Goncalves a, Reinaldo M. Palhares b,*, Ricardo H.C. Takahashi c

a Department of Electrical Engineering, Federal Center of Technological Education of Minas Gerais, Av. Amazonas 7675, 31510-470 Belo Horizonte, MG, Brazil
b Department of Electronics Engineering, Federal University of Minas Gerais, Av. Antonio Carlos 6627, 31270-010 Belo Horizonte, MG, Brazil
c Department of Mathematics, Federal University of Minas Gerais, Av. Antonio Carlos 6627, 31270-010 Belo Horizonte, MG, Brazil

Received 3 April 2007; received in revised form 27 June 2007; accepted 27 June 2007

Abstract

This paper presents a procedure for robust two-degrees-of-freedom PID synthesis based on a non-convex optimization problem considering the $H_2$ and/or $H_\infty$ guaranteed costs of the closed-loop transfer matrix as objective functions and constraints together with robust regional pole placement. The uncertain system can be represented by multiple, polytopic or affine parameter-dependent models. The efficiency of the proposed synthesis procedure is illustrated by examples.

Keywords: PID controller; Uncertain systems; Robust control

1. Introduction

The proportional-integral-derivative (PID) controllers still represent, nowadays, the most applied controller configuration in industry. More than 90% of all control loops are PID, with a wide range of applications: process control, motor drives, magnetic and optical memories, automotive, flight control, instrumentation, etc. [1]. This predominance of PID control is probably due to its simplicity (only three parameters to adjust), to its effectiveness for the majority of the industrial plants, and to the existence of a simple tuning method developed by John G. Ziegler and Nathaniel B. Nichols in 1942. In the Ziegler–Nichols step response method, the PID parameters are computed from simple features of the step response. However, it is known that such procedure leads to poor results in many cases [1]. Since 1942, several published works have tried to meet the need for specific synthesis procedures according to the application (auto-tuning, model predictive control, etc) considering quite different objectives and tuning algorithm complexity (see the references in [1]). The work [2] presents an overview of PID control and different tuning methods. The Ziegler–Nichols step response method is revisited and new tuning rules are developed in [3,4]. The design of PID controllers based on constrained optimization is developed in [5], where the objective is to maximize the integral gain subject to constraints over the Nyquist curve. In [6], a PID tuning procedure is presented based on the fitting of the process frequency response to a particular second-order plus dead time structure. In [7], a PID tuning method is presented, based on frequency loop-shaping, considering the minimization of the difference between the actual and a target loop transfer function, in an $L_\infty$ sense. The work [8] presents a single PID tuning rule for a first-order or second-order time delay model, and develops simple analytic rules for model reduction to obtain a model in these forms.

Despite the several PID tuning techniques available for the different PID configurations and applications (refer to [9] for an extensive list), it is still interesting to research...
new approaches that can lead to better performance and that can be applied to a broader class of problems, in special, the multiobjective robust synthesis to tackle uncertain systems represented by polytopic, or affine parameter-dependent models. The development of $\mathcal{H}_2/\mathcal{H}_\infty$ guaranteed cost characterizations in terms of linear matrix inequalities (LMIs) could be an alternative to develop PID tuning approaches for uncertain systems. Unfortunately, the PID controller is a structured reduced-order dynamic output-feedback controller and, to the best of the authors’ knowledge, there is no LMI-based formulation to tackle this problem: characterizing the problem as a static output-feedback control leads to bilinear matrix inequalities (BMIs) that are non-convex optimization problems. In [10], the PID tuning is characterized as a robust multiobjective static output-feedback control problem with an augmented state equation, and LMI-based formulations are applied to assess the resulting guaranteed costs. In [11], the Kharitonov theorem for interval plants is exploited for the purpose of characterizing all PID controllers that stabilize an uncertain plant. In [12], an integrated system identification and robust PID tuning strategy is presented, based on frequency loop-shaping, considering the minimization of the difference between the actual and a target loop transfer function, in an $\mathcal{L}_2/\mathcal{L}_\infty$ sense. In [13], a robust PID controller design method for multiple-models is developed via the LQR-LMI static output-feedback control approach.

This paper presents a two-step robust PID tuning procedure based on a non-convex multiobjective optimization phase and on an LMI-based analysis phase that can be applied to uncertain linear time-invariant systems represented by multiple, polytopic or affine parameter-dependent models. In the optimization phase, a non-convex optimization problem is solved to compute the PID parameters, considering a finite set of points of the uncertainty domain. In the analysis phase, a branch-and-bound algorithm that employs LMI-based sufficient conditions as upper bound functions is applied to verify the objective function and constraints for all uncertainty domain. If any constraint is violated or there is the possibility to improve the objective function, new points can be included in the finite set for a new iteration of the iterative synthesis procedure. The motivation to consider this procedure is its previous successfully applications for robust state-feedback controller synthesis [14], robust static or dynamic output-feedback controller synthesis [15], and filter synthesis [16]. Despite the difficulty to tackle non-convex optimization problems with only three decision variables (the PID parameters), it will be illustrated by examples that the proposed robust PID tuning procedure can provide better performance than previous published approaches considered here. Preliminary results have been published in an earlier conference work [17].

In this paper, capital letters are used to indicate the Laplace transform of a signal, e.g., $H(s)$ is the Laplace transform of $\eta(t)$.

### 2. Problem statement

This paper consider the two-degrees-of-freedom ISA PID configuration\(^1\) presented in Fig. 1, where $R(s)$ is the set point, $D(s)$ is the load disturbance, $U(s)$ is the control signal, $C(s)$ is the system output, and $H(s)$ is the measurement noise. The ISA PID control law is given by

$$U(s) = -kp\left(\frac{T_d s + 1}{T_i s} + \frac{T_d s}{\rho T_d s + 1}\right)(C(s) + H(s)) + kp\frac{T_d s + 1}{T_i s}R(s)$$

(1)

in which $k_p$ is the proportional gain, $T_i$ is the integral action time or reset time, $T_d$ is the derivative action time or rate time, and $\rho \geq 1/\rho$ is the noise filtering constant that varies between 3 and 10 [3] with $\rho = 10$ the typical value.

In the PID controller tuning, the resulting system must present the following features: good set point tracking, satisfactory load disturbance rejection, minimal influence of the measurement noise, limited range and rate of variation of the control signal, and robustness to model uncertainties. There are different ways to enforce these requirements in the PID controller synthesis strategy. For example, the load disturbance rejection can be achieved by the minimization of appropriate closed-loop transfer-function norms, and requirements over time responses can be attained by regional pole placement constraints. To achieve a robust PID controller, the tuning approach can consider systems represented by polytopic or affine parameter-dependent models.

Since the main procedure to be employed here deals with the control of uncertain systems described by state space models, the block diagram in Fig. (1) is represented in a modified form as in Fig. 2.

Considering Fig. 2, the linear time-invariant system to be controlled is represented in state-space as

$$\dot{x}(t) = Ax(t) + B_u w(t) + B_u u(t)$$

$$z_1(t) = C_1 x(t) + D_{w2} w(t) + D_{u2} u(t)$$

$$z_2(t) = C_2 x(t) + D_{w2} w(t) + D_{u2} u(t)$$

$$y(t) = C_y x(t) + D_{w} w(t) + D_{u} u(t)$$

(2)

in which $x \in \mathbb{R}^n$ is the state vector, $w(t) = [r(t), d(t), \eta(t)]^T$ is the exogenous input vector, $u(t) \in \mathbb{R}$ is the control variable, $z_1(t) \in \mathbb{R}^{n_1}$, and $z_2(t) \in \mathbb{R}^{n_2}$ are the controlled output vectors associated with the $\mathcal{H}_\infty$ and $\mathcal{H}_2$ performance, respectively, and $y(t) = [c(t) + \eta(t), r(t)]^T$ is the output vector.

This work is concerned with PID tuning for uncertain systems in which the matrices of the state-space model in (2) are not precisely known. Define the system matrix as

---

\(^1\) Instrument Society of America.
In the case of polytopic models, the system matrix belongs to a bounded polyhedral convex domain, or polytope, described by the convex combination of its $N$ vertices

$$P(\theta) = \sum_{i=1}^{N} \theta_i P_i$$

in which

$$P_i = \begin{bmatrix} A_i & B_{u,i} & B_{e,i} \\ C_{v,i} & D_{v0,i} & D_{v1,i} \\ C_{x,i} & D_{x0,i} & D_{x1,i} \end{bmatrix}$$

are the set of known vertices and $\theta = [\theta_1 \cdots \theta_N]^T \in \Omega_M$ is the polytopic coordinate vector, with $\Omega_M$ defined as

$$\Omega_M \triangleq \left\{ \theta \in \mathbb{R}^N : \theta_i > 0, \ i = 1, \ldots, N, \ \sum_{i=1}^{N} \theta_i = 1 \right\}$$

The uncertainty domain $\Omega_M$ can be also represented by a simplex in the dimension $N-1$.

$$\Omega_M \triangleq \left\{ \theta \in \mathbb{R}^{N-1} : \theta_i > 0, \ i = 1, \ldots, N-1, \ \sum_{i=1}^{N-1} \theta_i \leq 1 \right\}$$

with $\theta_N = 1 - \sum_{i=1}^{N-1} \theta_i$.

In the case of uncertain models depending affinely on an uncertain parameter vector $p = [p_1 \cdots p_d]^T$, in which $p_i$ has lower and upper bounds $\bar{p}_i$ and $\underline{p}_i$, respectively, the system matrix is computed as

$$P(p) = P_0 + \sum_{i=1}^{d} p_i P_i$$

with $P_i, i = 0, \ldots, d$, as in (4). In this case, $p \in \Omega_p$, in which $\Omega_p$ is a polytope with the shape of a hyper-rectangle, with the vertices generated by the combination of the uncertain parameter bounds,

$$\Omega_p \triangleq \left\{ p \in \mathbb{R}^d : \bar{p}_i \leq p_i \leq \underline{p}_i, \ i = 1, \ldots, N-1 \right\}$$

or any other shape if there are additional constraints over the uncertain parameters.

Let $\pi$ denote $\theta$ or $p$, and $\Omega$ denote $\Omega_M$ or $\Omega_p$.

Notice that the ISA PID controller, $K(s)$, can be described by the following realization considering the state variables $x_c(t)$ and $x_{e2}(t)$ in the modified block diagram presented in Fig. 2

$$K(s) = \begin{bmatrix} A_i & B_{u,i} & B_{e,i} \\ C_{v,i} & D_{v0,i} & D_{v1,i} \\ C_{x,i} & D_{x0,i} & D_{x1,i} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\frac{1}{T_1} \\ 0 & -\frac{1}{\rho T_2} & -\frac{1}{\rho T_2} \\ k_p & k_p & -k_p \left(1 + \frac{1}{\rho} \right) \frac{1}{\rho} \end{bmatrix}$$

Considering the modified diagram presented in Fig. 2, define the following closed-loop transfer functions: $T_{z_{1d}}(s, x) \triangleq Z_{1}(s)/D(s)$ and $T_{z_{2s}}(s, x) \triangleq Z_{2}(s)/\eta(s)$ which are related to the $H_{\infty}$ and $H_2$ performance criteria, respectively, with $Z_{1}(s) \triangleq W_1(s)C(s)$ and $Z_{2}(s) \triangleq W_2(s)U(s)$. The weighting function $W_2(s)$ is introduced to guarantee that the transfer function $T_{z_{2s}}(s, x)$ will be strictly proper. The weighting functions $W_1$ and $W_2$ are also selected such that the $H_{\infty}$ and $H_2$ norms have similar magnitudes.

![Fig. 1. Block diagram of the ISA PID configuration.](image1)

![Fig. 2. Modified ISA PID block diagram and the controlled output definitions.](image2)
Particularly, $W_2(s)$ is designed as a low-pass filter with high cutting frequency.

The closed-loop system dynamic matrix is denoted by $\mathcal{A}(z)$. Consider $\mathcal{D} \subseteq \mathbb{C}$, the desired closed-loop pole placement region (half-plane, disk, sector, or the intersection of them) with $\mathbb{C}$ denoting the complex left half-plane. Let $\gamma_{w.c.}$ and $\delta_{w.c.}$ be the worst case of the $H_{\infty}$ and $H_2$ norms in the uncertainty domain

$$
\gamma_{w.c.} \triangleq \max_{\zeta \in \Omega} \|T_{z_1z}(s, \zeta)\|_{\infty}
$$

(10)

$$
\delta_{w.c.} \triangleq \max_{\zeta \in \Omega} \|T_{z_2z}(s, \zeta)\|_2
$$

(11)

The robust PID tuning problem can be stated as: find the values of $k_p$, $T_i$, and $T_d$ that minimize the maximum norm values $\gamma_{w.c.}$ and $\delta_{w.c.}$ subject to $k_p > 0$, $T_i \geq 0$, $T_d \geq 0$, and $\sigma(\mathcal{A}(z)) \subseteq \mathcal{D}$, $\forall \zeta \in \Omega$. In this expression, $\sigma()$ denotes the spectrum of the argument matrix. In the sequel, $\sigma(k)$ will denote the $k$th eigenvalue in such set.

The regional pole-placement constraint is useful because the optimal $H_{\infty}$ controllers may lead to slow set point responses [10].

### 3. PID tuning procedure

Let $\mathcal{O}_i \subseteq \mathcal{O}$ be a finite set of points that belong to the polytope of uncertain parameters, initialized as the set of polytope vertices $\mathcal{O}_0 = \nu(\mathcal{O})$, in which $\nu(\cdot)$ denotes the set of vertices of its argument. Consider the “worst-case” points in the finite set $\mathcal{O}_i$, given a PID controller $K(s)$

$$
\gamma_{w.c.} = \max_{\zeta \in \mathcal{O}_i} \|T_{z_1z}(s, \zeta)\|_{\infty}
$$

$$
\delta_{w.c.} = \max_{\zeta \in \mathcal{O}_i} \|T_{z_2z}(s, \zeta)\|_2
$$

(12)

Let $\gamma_{g.c.}$ and $\delta_{g.c.}$ be $\varepsilon$-guaranteed costs such as

$$
\gamma_{w.c.} \leq \gamma_{g.c.} \leq (1 + \varepsilon)\gamma_{w.c.}
$$

(13)

$$
\delta_{w.c.} \leq \delta_{g.c.} \leq (1 + \varepsilon)\delta_{w.c.}
$$

(14)

and $\mathcal{O}_{(\infty)} \subseteq \mathcal{O}$ and $\mathcal{O}_{(2)} \subseteq \mathcal{O}$ be some polytope coordinates such as

$$
\|T_{z_1z}(s, \mathcal{O}_{(\infty)})\|_{\infty} \geq (1 - \varepsilon)\gamma_{g.c.}
$$

(15)

$$
\|T_{z_2z}(s, \mathcal{O}_{(2)})\|_2 \geq (1 - \varepsilon)\delta_{g.c.}
$$

(16)

Define the set of PID controllers with the realization presented in (9)

$$
\mathcal{K}(s) : \left\{\begin{array}{l}
k_p > 0, \ T_i \geq 0, \ T_d \geq 0, \ \rho = 0.1 \\
\sigma(\mathcal{A}(z)) \subseteq \mathcal{D}, \ \forall \zeta \in \mathcal{O}_i
\end{array}\right\}
$$

(17)

The following auxiliary problem is employed in the proposed PID tuning procedure:

**Auxiliary Problem:** Given the scalars $\lambda_1 > 0$, $\lambda_2 > 0$, $\lambda_1 + \lambda_2 = 1$, $\epsilon_1 > 0$, and $\epsilon_2 > 0$, find $k_p$, $T_i$, and $T_d$ to construct $K(s)$, such as

$$
K^*(s) = \arg \min_{K(s)} \max_{\zeta \in \mathcal{D}} \left(\lambda_1 \|T_{z_1z}(s, \zeta)\|_{\infty} + \lambda_2 \|T_{z_2z}(s, \zeta)\|_2\right)
$$

subject to:

$$
\begin{align*}
\mathcal{A}(s) &\in \mathcal{D} \\
\|T_{z_1z}(s, \zeta)\|_{\infty} &\leq \epsilon_1 \\
\|T_{z_2z}(s, \zeta)\|_2 &\leq \epsilon_2
\end{align*}
$$

This Auxiliary Problem combines two scalarization techniques to transform the original multiobjective problem into a scalar optimization problem: the weighting problem and $\varepsilon$-constraint problem. In the weighting problem the objective is to minimize a weighted sum of the objective functions. In the $\varepsilon$-constraint problem it is considered one objective function each time with the other objective function treated as a constraint. Different trade-offs can be achieved varying $\lambda_1$ and $\lambda_2$, without the norm constraints ($\epsilon_1 = \epsilon_2 = \infty$) or, e.g., keeping $\lambda_1 = 0$, $\lambda_2 = 1$, $\epsilon_2 = \infty$, and varying $\epsilon_1$ in a desired range.

The proposed two phase iterative robust PID tuning procedure, to be presented in the sequel, transforms the problem of dealing with an infinite number of points of the uncertainty domain in a problem that considers only a finite set of points. At each iteration, in the synthesis phase, the PID parameters are computed by means of an optimization algorithm that considers the finite set of polytope points $\mathcal{O}_i$. In the analysis phase, considering the whole polytope $\mathcal{O}$, if the maximum value of the objective function occurs in a point outside the finite set $\mathcal{O}_i$ or if any constraint is not attained in a point belonging to the polytope $\mathcal{O}$, then such points are included in the finite set and another optimization is processed. The procedure ends when all constraints are attained for all points in the polytope $\mathcal{O}$ and there is no possibility to minimize the objective function by means of introducing new points of $\mathcal{O}$ in the finite set $\mathcal{O}_i$, considering a relative accuracy index $\epsilon_d$.

**PID tuning procedure:**

**Step 1.** Initialize $i \leftarrow 0$, $\mathcal{O}_0 \leftarrow \nu(\mathcal{O})$.

**Step 2.** $i \leftarrow i + 1$, $\mathcal{O}_i \leftarrow \mathcal{O}_{i-1}$.

**Step 3.** Solve the auxiliary problem to compute $K^*(s)$ and $\gamma_{w.c.}$ and/or $\delta_{w.c.}$.

**Step 4.** If $\lambda_1 > 0$ or $\epsilon_1 < \infty$ then compute $\gamma_{g.c.}$ and $\mathcal{O}_{(\infty)}$ for $K^*(s)$.

If $\mathcal{O}_{(\infty)} \not\in \mathcal{O}_i$, then if $\{\lambda_1 > 0 \text{ and } (\gamma_{g.c.} - \gamma_{w.c.})/\delta_{w.c.} > \epsilon_3\}$ or $\gamma_{g.c.} > \epsilon_1$ then $\mathcal{O}_i \leftarrow \mathcal{O}_i \cup \{\mathcal{O}_{(\infty)}\}$.

**Step 5.** If $\lambda_2 > 0$ or $\epsilon_2 < \infty$ then compute $\delta_{g.c.}$ and $\mathcal{O}_{(2)}$ for $K^*(s)$.

If $\mathcal{O}_{(2)} \not\in \mathcal{O}_i$, then if $\{\lambda_2 > 0 \text{ and } (\delta_{g.c.} - \delta_{w.c.})/\epsilon_2 > \epsilon_3\}$ or $\delta_{g.c.} > \epsilon_2$ then $\mathcal{O}_i \leftarrow \mathcal{O}_i \cup \{\mathcal{O}_{(2)}\}$.

**Step 6.** Verify if $\exists j \sigma(\mathcal{A}(z_j)) \not\in \mathcal{D}$ for any $z_j \in \mathcal{O}$. If true, then $\mathcal{O}_i \leftarrow \mathcal{O}_i \cup \{z_j\}$.

**Step 7.** If $\mathcal{O}_i \neq \mathcal{O}_{i-1}$, then goto step 2, else stop.

#### 3.1. Optimization phase

The auxiliary scalar optimization problem can be solved by means of the cone-ellipsoidal algorithm [18]. Consider
the ellipsoid in the iteration $k$ described as $\delta_k = \{x \in \mathbb{R}^d : (x - x_k)^T Q_k^{-1} (x - x_k) \leq 1\}$, where $x_k$ is the ellipsoid center and $Q_k = Q_k^T > 0$ is the matrix that determines the direction and the dimension of the ellipsoid axes. Given the initial values $x_0$ and $Q_0$, the ellipsoidal algorithm is described by the following recursive equations:

$$
\begin{align*}
  x_{k+1} &= x_k - \frac{1}{d+1} Q_k \hat{m} \\
  Q_{k+1} &= \frac{d^2}{d-1} \left( Q_k - \frac{2}{d+1} Q_k \hat{m} \hat{m}^T Q_k \right)
\end{align*}
$$

with

$$
\hat{m} = \frac{m_k}{\sqrt{m_k^T Q_k m_k}}
$$

where $x \in \mathbb{R}^d$ is the vector of optimization parameters (in this case the PID parameters: $x = [k_p, 1/T_i, 1/T_d]^T$) and $m_k$ is the gradient (or sub-gradient) of the most violated constraint of $g(x) : \mathbb{R}^d \rightarrow \mathbb{R}^1$, when $x_k$ is not a feasible solution, or the gradient (or sub-gradient) of the objective function, $f(x) : \mathbb{R}^d \rightarrow \mathbb{R}$, when $x_k$ is a feasible solution. The finite difference method is applied to compute the gradient. The reader can also see [14, Section 3.1] for details of how to compute matrix $Q_k$ and the gradient $m_k$. The optimization algorithm ends when $(f_{\text{max}} - f_{\text{min}})/f_{\text{min}} \leq \epsilon$, where $f_{\text{max}}$ and $f_{\text{min}}$ are the minimal and maximal values of the objective function at the last $N_i$ iterations and $\epsilon$ is the prescribed relative accuracy.

The optimization algorithm requires the choice of the initial entries of $x_0$ and $Q_0$. The vector $x_0$, with the initial values of the PID parameters, can be generated randomly or one can use the results of a simple PID tuning technique, e.g., the Ziegler–Nichols method. The matrix $Q_0$ defines the size and orientation of the initial ellipsoid where the optimal solution will be searched. The convergence of the algorithm depends on the size of the initial ellipsoid and the prescribed relative accuracy. Obviously, there is a tradeoff between the computational effort and the level of objective function minimization that defines the best choice of $Q_0$, $N_i$, and $\epsilon$.

3.2. Analysis phase

The computation of the $\varepsilon$-guaranteed costs ($\gamma_{\varepsilon,c}$ or $\delta_{\varepsilon,c}$), maximum value of the closed-loop transfer function norms ($\gamma_{\infty,c}$ or $\delta_{\infty,c}$) for all uncertain domain, and the corresponding coordinates ($x_{\gamma_{\infty,c}}$ or $x_{\delta_{\infty,c}}$) is performed by means of an analysis procedure based on the branch-and-bound algorithm [19]. The basic strategy of this procedure is to partition the uncertainty domain such as a lower bound and an upper bound functions converge to the global maximum value of the norm. This algorithm ends when the difference between the bound functions is smaller than the prescribed relative accuracy $\epsilon$. The maximum value of the $H_\infty$ (or $H_2$) norm computed in the vertices of the polytope and its subdivisions is employed by the algorithm as the lower bound function. The upper bound function is the maximum value of the $H_\infty$ (or $H_2$) guaranteed cost computed by means of LMI-based formulations: Lemma 1 presented in [20] in the case of $H_\infty$ guaranteed cost and combination of Lemmas 1 and 2 presented in [21] in the case of $H_2$ guaranteed cost.

A partition technique based on simplicial meshes [22] is applied for branching the search region. This technique is particularly interesting because: (i) it can tackle uncertainty domains not restricted to the hyper-rectangle case; (ii) it leads to an efficient branching sequence (which shrinks the search region very fast); and (iii) it allows efficient upper bound function evaluations (requiring the minimal number of LMI constraints). Besides the fact that the uncertainty domain for a polytopic system is already a simplex, the simplex is the polytope with the minimal number of vertices in a specific space dimension. Lower number of vertices means lower complexity of the LMI-based analysis formulations (lower number of scalar decision variables and LMI rows). This partition technique allows the application of this analysis procedure to both affine parameter-dependent and polytopic models with high efficiency. In the case of affine parameter-dependent models, the first partition is performed by the Delaunay triangulation that decomposes any polytope in a set of simplices.

The same partition strategy is also applied to the regional pole placement analysis procedure. In this case, the analysis procedure must subdivide any subpolytope whose robust $D$-stability is not established until one of the following conditions is achieved: all subpolytopes are identified as robustly $D$-stable by an LMI-based sufficient condition, proving that the uncertain system is robustly $D$-stable, or it is found at least one vertex of a subpolytope that is not $D$-stable, proving that the uncertain system is not robustly $D$-stable [23]. When the system is not robustly $D$-stable, the analysis procedure returns the coordinates $x_{(u)}$ of one instance of a not $D$-stable model $T_{z_{(u)}}(s, x_{(u)})$ such as $\exists! (\sigma_i(\mathcal{A}(x_{(u)})) \notin \mathcal{D})$. The capability to find an instance of a not $D$-stable system is due to the fast search space reduction by the elimination of the simplices that are robustly $D$-stable.

The reader can access the Matlab Code for the guaranteed cost computation and the stability analysis at the website [24].

4. Illustrative examples

The results presented in the following examples have been computed with the software MATLAB® in a computer with an 2.8 GHz processor and 1 Gb system RAM.

Example 1. Consider a continuous stirred tank reactor with a multiple-model description presented in [13] and also considered in [25]. Considering a stable operation range, three models are obtained for different operating points
\[ G_1(s) = \frac{0.04612}{s^2 + 9.251s + 22.19}, \quad G_2(s) = \frac{0.04107}{s^2 + 2.674s + 10.97}, \quad G_3(s) = \frac{0.03707}{s^2 + 0.01248s + 5.862} \]

These three models are the vertices of the polytope considering a polytopic model of the uncertain system. In [13], the classical PID controller is designed to minimize an LQR cost and to place the closed-loop poles in the intersection of the half plane \( \text{Real}(\sigma) < -1 \) and the sector region centered at the origin with inner angle \( 3\pi/4 \). In [13], the PID parameters are computed by means of an LMI approach that results in \( k_p = 516.6, \quad T_a = 0.2784, \quad \text{and} \quad T_i = 0.6749 \). In [25], the PID parameters are computed as \( k_p = 698.1, \quad T_a = 0.5259, \quad \text{and} \quad T_i = 0.6197 \). In the proposed approach, considering \( W_1(s) = w_1 = 40, \quad W_2(s) = w_2/(T_a s + 1), \quad w_2 = 5 \times 10^{-8}, \quad T_a = 0.10^{-5}, \quad \) a input disturbance with \( G_d(s, \theta) = 500G(s, \theta), \quad G(s) = B(s^2 + a_5 s + a_6), \) and a measurement noise, the three vertices are realized as

\[
\begin{align*}
\frac{dx}{dt} &= \begin{bmatrix} 0 & -a_0 & 0 \\ 1 & -a_1 & 0 \\ 0 & 0 & -1/T_a \end{bmatrix} x + \begin{bmatrix} r \\ 0 \\ 0 \end{bmatrix} u \\
&+ \begin{bmatrix} 0 & 500b_0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ d \eta \end{bmatrix} + \begin{bmatrix} b_0 \\ 0 \\ 1/T_a \end{bmatrix} \\
z_1 &= [0 \ w_1 \ 0] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \quad z_2 &= [0 \ 0 \ w_2] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\
y &= \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ d \eta \end{bmatrix}
\end{align*}
\]

The weight function \( W_2(s) \) is chosen to make the closed-loop transfer function strictly proper and the constants \( w_1 \) and \( w_2 \) are chosen to make the norm values near 1. As an initial design, the PID tuning procedure is applied to compute the PID controller considering only the \( H_\infty \) criterion \( (\lambda_1 = 1, \quad \lambda_2 = 0, \quad \epsilon_1 = \epsilon_2 = \infty) \). The Ziegler-Nichols step response method computes the PID parameters as \( k_p = 2.5209 \times 10^5, \quad T_i = 0.25, \quad \text{and} \quad T_d = 0.04 \). Starting from these parameters, the proposed approach computes the PID parameters as \( k_p = 7.2970 \times 10^5, \quad T_i = 0.0315, \quad \text{and} \quad T_d = 0.5772 \). The optimization improves the \( H_\infty \) guaranteed cost from 2144.1 to 1.1527 after 30.625 s of computational time. The time responses of the PID controllers achieved with the LMI approach in [13], the numerical optimization approach in [25], and the proposed procedure are presented in Fig. 3 for an unity step in \( r(t) \), a negative unit step in \( d \) starting from \( t = 5 \) s, and a noise signal varying randomly in the range \( |\eta| < 0.01 \).

The time response to the unity step in \( r(t) \) can be improved by means of a pre-filter or considering a multiobjective design. Considering the second strategy, the PID tuning procedure is applied to compute the controller that minimizes the \( H_\infty \) and \( H_2 \) guaranteed costs \( (\lambda_1 = \lambda_2 = 0.5) \) and places the poles in the intersection of the half plane \( \text{Real}(\sigma) < -4 \) and the sector region centered at the origin with inner angle \( 2\pi/3 \). The proposed approach computes the PID parameters as \( k_p = 1.2713 \times 10^4, \quad T_i = 0.4289, \quad \text{and} \quad T_d = 0.1149 \) after 13.422 s of computational time. The time response of this second PID controller is presented in Fig. 4. The multiobjective design provides a better command response and noise attenuation keeping a satisfactory disturbance rejection.

Ref. [13] also tackles an unstable operation range, considering three models for different operating points

\[ G_1(s) = \frac{0.036}{s^2 - 0.405s + 4.996}, \quad G_2(s) = \frac{0.026}{s^2 - 2.647s - 0.879}, \quad G_3(s) = \frac{0.016}{s^2 - 2.251s - 2.252} \]

Fig. 3. Time responses of \( y(t) \) at the three polytope vertices for the stable open-loop systems with the proposed approach considering only the \( H_\infty \) criterion (Example 1).

Fig. 4. Time responses of \( y(t) \) at the three polytope vertices for the stable open-loop systems with the proposed approach considering the multiobjective optimization (Example 1).
lem present better tracking responses and disturbance rejections and they are less influenced by the uncertain parameter variations than the PID controllers stated in [13,25].

Example 2. Consider the uncertain system presented in [11]

\[ G(s) = \frac{5.2(s + 2)}{s(s^3 + b_2s^2 + b_1s + b_0)} \]

in which the coefficients \( b_0, b_1, \) and \( b_2 \) of the denominator lie within the following bounds: \( 9.5 \leq b_0 \leq 11.5, 12 \leq b_1 \leq 15, \) and \( 3.5 \leq b_2 \leq 4.8. \) In [11], a robust classical PID is designed to guarantee the gain and phase margins. The following realization is considered here with the definition of \( W_1(s) = 1, \) the addition of an input disturbance, \( G_d(s) = G(s), \) and measurement noise (see Fig. 1)

\[
\begin{align*}
\frac{d}{dt} & \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -b_0 \\
0 & 1 & 0 & -b_1 \\
0 & 0 & 1 & -b_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \\
& + \begin{bmatrix} 0 & 10.4 & 0 & 0 \\
0 & 5.2 & 0 & d \\
0 & 0 & 0 & \eta \end{bmatrix} \begin{bmatrix} r \\ d \\ \eta \end{bmatrix} \\
& + \begin{bmatrix} 10.4 \\ 5.2 \\ 0 \end{bmatrix} \begin{bmatrix} u \\ \eta \end{bmatrix} \\
\end{align*}
\]

\[
z_1 = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} , \ y = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} r \\ d \end{bmatrix}
\]

The proposed PID tuning procedure is applied to compute the controller that minimizes the \( H_\infty \) guaranteed cost of \( T_{zd}(s, z) \) and places the poles at the intersection of the half-plane \( \text{Real}(\sigma_i) < -0.5, \forall p \in \Omega_p, \) where \( p = [b_0 \ b_1 \ b_2]^T \) and \( \Omega_p \) is a hyper-rectangle. In this problem, the \( H_2 \) performance criterion is not considered since the noise signal has low influence over the output signal. The
The proposed procedure computes the PID parameters as \( k_p = 1.6909 \), \( T_i = 2.9200 \), and \( T_d = 0.4629 \). The PID parameters \( k_p = 3.1950 \), \( T_i = 1.3975 \), and \( T_d = 0.2236 \), calculated with the Ziegler–Nichols frequency response method have been used as the initial condition to the optimization algorithm. The \( H_\infty \) guaranteed cost is improved from 1.834 to 0.777 after 58 s of computational time. Fig. 7 presents the time responses of the PID controllers presented in [11] and computed here, for an unity step in \( r(t) \), a negative unit step in \( d \) starting from \( t = 15 \) s, and a noise signal varying randomly in the range \(|\eta| \leq 0.01\). The proposed PID controller presents better disturbance rejection and it is less influenced by the uncertain parameters variations.

5. Conclusion

In this paper, an efficient two-step (synthesis/analysis) iterative procedure for robust two-degrees-of-freedom PID tuning was presented, for uncertain linear time-invariant systems with multiple, polytopic or affine parameter-dependent models. The PID parameters are computed by means of a non-convex optimization approach to minimize the \( H_\infty \) and/or \( H_2 \) guaranteed costs of closed-loop transfer-functions between the exogenous inputs and the controllable variables, and to robustly place the closed-loop poles at LMI regions. Examples are presented to show the better performance of the proposed procedure in relation to previously published robust tuning methods, when comparing the set point tracking, the load disturbance rejection, and the robustness to model uncertainties.

Acknowledgement

This work has been supported in part by the Brazilian agencies CNPq and FAPEMIG.

References