ℋ∞ and ℋ2 guaranteed costs computation for uncertain linear systems

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\textbf{H}_\infty and \textbf{H}_2 guaranteed costs computation for uncertain linear systems

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This paper proposes an LMIs characterization of guaranteed \textbf{H}_\infty and \textbf{H}_2 norms costs for linear systems with convex bounded parameter uncertainties. Both continuous-time and discrete-time systems are addressed.

1. Introduction

The \textbf{H}_\infty norm of a transfer function has recently been employed as an important parameter in robust analysis and synthesis control literature, as for instance by Doyle et al. (1989). A classical interpretation for the \textbf{H}_\infty norm of a monovariable system is established in terms of the Nyquist and Bode diagrams. In this case, the \textbf{H}_\infty norm is equivalent to the distance from the origin to the farthest point in the complex plane in the Nyquist diagram, or to the magnitude peak in the Bode diagram. The \textbf{H}_\infty norm is a measure of the system robustness, indicating the amount of uncertainty that is allowable without leading the system into instability.

The \textbf{H}_2 norm is a new fashion of the traditional Linear Quadratic criterion in the optimal control design (Doyle et al. 1989). It may be interpreted as a weighted energy measure of the system states and inputs.

These two criteria are very popular ones, establishing measures of system performance. In particular, the joint optimization of such criteria is increasingly being recognized as 'good' design practice, leading to robust closed-loop systems with optimal nominal performances (Khargonekar and Rotea 1991).

Several controller design techniques are available for optimizing either the \textbf{H}_2 or the \textbf{H}_\infty or both norms of the closed-loop system, for systems that have uncertain models (Geromel et al. 1992, 1993, 1994). The explicit account for system uncertainties and the design of controllers which are robust under these uncertainties is one of the main reasons for the increasing acceptance of the robust \textbf{H}_2/\textbf{H}_\infty control. The standard procedure in such techniques is to perform the controller synthesis with algorithms based on closed-loop system \textbf{H}_2 and/or \textbf{H}_\infty norms expressions. This leads to the controller parameters and, as a by-product, also to the closed-loop \textbf{H}_2/\textbf{H}_\infty norms (or \textbf{H}_2/\textbf{H}_\infty guaranteed costs, in the case of uncertain systems).

If one considers an open-loop system as precisely known, its closed-loop \textbf{H}_2 or \textbf{H}_\infty norm may be calculated from standard procedures, given the controller (which may have been synthesized by any means). There are several methods available in the literature for the computation of such norms, in the case of precisely known systems, see, for instance, Robel (1989), Hinrichsen et al. (1989), Doyle et al. (1989), Chang et al. (1990), Bruinsman and Steinbuch (1990), Boyd and Balakrishnan (1990), Kavranoglu (1994) for the continuous-time \textbf{H}_\infty norm, and Doyle et al. (1989) for the continuous-time \textbf{H}_2 norm. The discrete-time corresponding norms can be computed through similar procedures. However, given a system model with parameter uncertainties, there is presently no way for determining such norms of the closed-loop system, unless in the case of the controller being synthesized through some of the above referred \textbf{H}_2/\textbf{H}_\infty techniques.

This paper presents Linear Matrix Inequalities (LMIs) formulations for the computation of \textbf{H}_2 and \textbf{H}_\infty guaranteed costs for continuous-time and discrete-time uncertain systems with convex-bounded (polytope type) model uncertainties. These results allow the determination of guaranteed \textbf{H}_2 and \textbf{H}_\infty performance levels for open-loop systems, or for closed-loop systems with a controller synthesized by any design algorithm.

The notation used in this paper is fairly standard.
The space $\mathbb{R}^n$ is equipped with the euclidean norm $\| \|$, $\mathbb{R}^{m \times n}, m, n \in \mathbb{N}$, with the spectral norm, i.e. $\| X \| = \sigma_{\text{max}}(X)$, where $\sigma_{\text{max}}(X)$ is the maximum singular value of the matrix $X$. $\text{Tr}(X)$ is the trace of $X$, $Y^*$ stands for the transpose of $Y$, and $Y^{*\text{c}}$ for the conjugate transpose of $Y$.

2. Problem statement

Consider the following linear time-invariant (LTI) continuous-time system

$$
\begin{align*}
\dot{x}(t) &= Ax(t) + Bw(t) \\
z(t) &= Cx(t) + Dw(t)
\end{align*}
$$

and also the LTI discrete-time system:

$$
\begin{align*}
x(k+1) &= Ax(k) + Bw(k) \\
z(k) &= Cx(k) + Dw(k)
\end{align*}
$$

In both above systems, $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times r}$, $C \in \mathbb{R}^{q \times n}$ and $D \in \mathbb{R}^{q \times r}$. The matrix $A$ is assumed to be asymptotically stable, which means in the continuous-time case that its eigenvalues have strictly negative real part, and in the discrete-time case that its eigenvalues are strictly inside the unit disc with its centre at the origin. The system parameters are considered to be unknown but belonging to a known convex compact set of polytope type. Define, for systems that are not necessarily strictly proper:

$$
(A, B, C, D) \in \mathcal{U}
$$

$$
\mathcal{U} \triangleq \left\{ (A, B, C, D) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times r} \times \mathbb{R}^{q \times n} \times \mathbb{R}^{q \times r} \right\}
$$

$$
(A, B, C, D) = \sum_{i=1}^{n} \alpha_i(A_i, B_i, C_i, D_i); \alpha_i \geq 0; \sum \alpha_i = 1.
$$

Denote by $\mathcal{U}_c$ the uncertainties polytope in the case of continuous-time systems, and by $\mathcal{U}_d$ the uncertainties polytope for discrete-time systems. Define also an uncertainty set for strictly proper systems:

$$
D = 0 => (A, B, C) \in \mathcal{P}
$$

$$
\mathcal{P} \triangleq \left\{ (A, B, C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times r} \times \mathbb{R}^{q \times n} \right\}
$$

$$
(A, B, C) = \sum_{i=1}^{n} \alpha_i(A_i, B_i, C_i); \alpha_i \geq 0; \sum \alpha_i = 1.
$$

Denote also by $\mathcal{P}_c$ the continuous-time polytope, and by $\mathcal{P}_d$ the discrete-time polytope.

In any case, let $\mathcal{V}(\cdot)$ denote the set of vertices of the argument polytope.

The transfer matrix from the input vector $w$ to the output vector $z$ is given, in the continuous-time case, by:

$$
H(s) = C(sI - A)^{-1}B + D.
$$

The same expression is obtained for the discrete-time system, replacing the frequency variable $s$ by the time-shift operator $z$. The $\mathcal{H}_\infty$ norm of this transfer matrix is defined, for the continuous-time case, as (Doyle et al. 1989):

$$
\| H \|_\infty = \sup_{s \in \mathbb{C}_+} \{ \sigma_{\text{max}}(H(s)) \} = \text{ess sup}_{\omega \in \mathbb{R}} \{ \sigma_{\text{max}}(H(j\omega)) \}
$$

and for the discrete-time case as:

$$
\| H \|_\infty = \text{ess sup}_{\omega \in [-\pi, \pi]} \{ \sigma_{\text{max}}(H(e^{j\omega})) \}.
$$

The $\mathcal{H}_2$ norm is defined only for $D = 0$, for the continuous-time case as:

$$
\| H \|_2 = \left( \frac{1}{2\pi} \int_{-\infty}^{\infty} \text{Tr} (H(j\omega)H^*(j\omega)) \, d\omega \right)^{1/2}
$$

and for the discrete-time case as:

$$
\| H \|_2 = \left( \frac{1}{2\pi} \int_{-\pi}^{\pi} \text{Tr} (H(e^{j\omega})H^{*}(e^{j\omega})) \, d\omega \right)^{1/2}.
$$

There is not, up to now, any methodology for dealing with such norm evaluation in the case of uncertain systems. This note develops such a methodology, leading to guaranteed costs (i.e. upper bounds) in the following sense:

$$
\gamma_c \geq \| H \|_\infty \quad \forall (A, B, C, D) \in \mathcal{U}_c,
$$

$$
\gamma_d \geq \| H \|_\infty \quad \forall (A, B, C, D) \in \mathcal{U}_d,
$$

$$
\delta_c \geq \| H \|_2 \quad \forall (A, B, C) \in \mathcal{P}_c,
$$

$$
\delta_d \geq \| H \|_2 \quad \forall (A, B, C) \in \mathcal{P}_d.
$$

The computation of $\gamma_c$, $\gamma_d$, $\delta_c$, $\delta_d$ is here formulated as a Linear Matrix Inequalities (LMIs) optimization problem. In the case of precisely known systems, these guaranteed costs become equal to the corresponding norms.

3. Continuous-time $\mathcal{H}_2$ norm

The following relation holds for the $\mathcal{H}_2$ norm of a strictly proper stable transfer function $H(s)$ and the system controllability gramian $L_c$:

$$
\left\{ \begin{array}{l}
\| H \|_2^2 = \text{Tr} (CL_cC') \\
L_c: \{ AL_c + L_cA' + BB' = 0 \}
\end{array} \right\}
$$

Take a symmetric definite positive matrix $W$ such that:

$$
AW + WA' + BB' \leq 0,
$$

which implies, from the negativity of matrix $A$ eigenvalues:

$$
\text{Tr}(CW'C') \geq \text{Tr}(CL_cC),
$$
From Schur's complement, (12) is equivalent to:
\[ \Phi_e(A, B, W) \geq 0 \] (14)
with the matrix function \( \Phi_e \) defined as:
\[ \Phi_e(A, B, W) \triangleq \begin{bmatrix}
-(AW + WA') & B \\
B' & I
\end{bmatrix}. \] (15)

For any symmetric semi-definite positive matrix \( \Delta \) holds:
\[ CW' \leq \Delta \Rightarrow \text{Tr} \{CW'\} \leq \text{Tr} \{\Delta\} \]
\[ CWC' \leq \Delta \Leftrightarrow \begin{bmatrix}
W & WC' \\
CW & \Delta
\end{bmatrix} \succeq 0. \] (16)

The last relation is obtained by Schur's complement.

Define the matrix function \( r \) by:
\[ r(W, A, B, W) \triangleq \begin{bmatrix}
W & AW & B \\
B' & 0 & I
\end{bmatrix}. \]

From Schur's complement, (23) is equivalent to:
\[ \text{Tr} \{CWC'\} \geq \text{Tr} \{CL_eC\}. \] (24)

For any symmetric semi-definite positive matrix \( \Delta \), (16) and (18) still hold. Let \( \Omega(C, W, \Delta) \) be defined as in (17).

Define now the sets \( \Theta \) and \( \Theta_v \) as:
\[ \Theta \triangleq \{(\Delta, W)|\Delta = \Delta' \geq 0; W = W' > 0; \Phi_e(A, B, W) \geq 0; \Gamma(C, W, \Delta) \geq 0; \forall (A, B, C) \in \mathcal{R}_d\} \]
\[ \Theta_v \triangleq \{(\Delta, W)|\Delta = \Delta' \geq 0; W = W' > 0; \Phi_e(A_i, B_i, W) \geq 0; \Gamma(C_i, W, \Delta) \geq 0; \forall (A_i, B_i, C_i) \in \Psi_v(\mathcal{R}_d)\}. \] (19)

From the linearity of the above matrix inequalities, and from the convexity of \( \mathcal{R}_d \), these sets are equal (\( \Theta = \Theta_v \)), and the constraints on the whole set \( \mathcal{R}_d \) may be verified by inspection over the set of its vertices only. The following LMI optimization problem is then defined:
\[ \delta_e^2 = \min_{(\Delta, W)} \text{Tr} \{\Delta\} \]
subject to: (\( \Delta, W \)) \( \in \Theta_v \). (20)

\[ \text{4. Discrete-time } \mathcal{H}_2 \text{ norm} \]

A similar development is followed for the \( \mathcal{H}_2 \) norm of a strictly proper and stable discrete transfer function \( H(z) \):
\[ \|H\|^2_2 = \text{Tr} \{CL_eC'\} \]
\[ L_e: \{AL_eA' + BB' - L_e = 0\}. \] (22)

Take a symmetric positive definite matrix \( W \) such that:
\[ AWA' + BB' - W \leq 0, \] (23)
which implies, from the fact that the matrix \( A \) eigenvalues are inside the unit disc:
\[ \text{Tr} \{CWC'\} \geq \text{Tr} \{CL_eC\}. \] (24)

From Schur's complement, (23) is equivalent to:
\[ \Phi_d(A, B, W) \geq 0 \] (25)
with the matrix function \( \Phi_d \) defined as:
\[ \Phi_d(A, B, W) \triangleq \begin{bmatrix}
W & AW & B \\
B' & 0 & I
\end{bmatrix}. \] (26)

For any symmetric semi-definite positive matrix \( \Delta \), (16) and (18) still hold. Let \( \Omega(C, W, \Delta) \) be defined as in (17).

Define now the sets \( \Theta \) and \( \Theta_v \) as:
\[ \Omega \triangleq \{(\Delta, W)|\Delta = \Delta' \geq 0; W = W' > 0; \Phi_d(A, B, W) \geq 0; \Gamma(C, W, \Delta) \geq 0; \forall (A, B, C) \in \mathcal{R}_d\} \]
\[ \Omega_v \triangleq \{(\Delta, W)|\Delta = \Delta' > 0; W = W' > 0; \Phi_d(A_i, B_i, W) \geq 0; \Gamma(C_i, W, \Delta) \geq 0; \forall (A_i, B_i, C_i) \in \Psi_v(\mathcal{R}_d)\}. \] (28)

Clearly, these sets are equal (\( \Omega = \Omega_v \)). The following LMI optimization problem is then defined:
\[ \delta_d^2 = \min_{(\Delta, W)} \text{Tr} \{\Delta\} \]
subject to: (\( \Delta, W \)) \( \in \Omega_v \). (29)

\[ \text{5. Continuous-time } \mathcal{H}_\infty \text{ norm} \]

The Bounded Real Lemma (see for instance Boyd et al. 1994) states that:
\[ \|H\|_\infty < 1 \Leftrightarrow \exists P = P^T > 0: \]
\[ \begin{bmatrix}
A'P + PA + C'C & PB + C'D \\
B'P + D'C & D'D - I
\end{bmatrix} \leq 0. \] (30)

It is straightforward to rewrite it as:
\[ \|H\|_\infty < \gamma_c \Rightarrow \exists P = P^T > 0: \]
\[ \begin{bmatrix}
A'P + PA + \gamma_c^{-2}C'C & PB + \gamma_c^{-2}C'D \\
B'P + \gamma_c^{-2}D'C & \gamma_c^{-2}D'D - I
\end{bmatrix} \leq 0. \] (31)

Define \( \mu_c \triangleq \gamma_c^2 \). From Schur's complements, (31) is equivalent to:
\[ \Psi_e(A, B, C, D, \mu_c, P) \geq 0 \] (32)
with \( \Psi_e \) defined as:
\[ \Psi_e(A, B, C, D, \mu_c, P) \triangleq \begin{bmatrix}
-(A'P + PA) & PB & C' \\
B'P & I & D' \\
C & D & \mu_c I
\end{bmatrix}. \] (33)
As before, these sets are equal \( \Gamma = \Gamma_v \). The guaranteed cost norm computation may be stated as:

\[
\gamma_v^2 = \min_{(\mu_d, \gamma_v)} \mu_d
\]

subject to: \((\mu_d, \gamma_v) \in \Sigma_v\).

7. Numerical examples

7.1. Continuous time model: DC motor with position control

The first example is concerned with a DC motor with a proportional-integral (PI) position control. An open-loop augmented state-space model is given by

\[
\dot{x} = Ax + Bu + B_1w
\]

\[
z = Cy + D_1u + D_2w
\]

where

\[
A = \begin{bmatrix}
-(R_a + \rho)/(L_a + \tau) & -k_i/(L_a + \tau) & 0 & 0 \\
k_i/J & -k_i/J & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & -1
\end{bmatrix},
\]

\[
B_1 = \begin{bmatrix}
0 & 0 \\
-1/J & 0 \\
0 & 0 \\
0 & 1
\end{bmatrix},
\]

\[
B_2 = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix},
\]

\[
C = \begin{bmatrix}
0 & 0 & -1 + \zeta & 0 \\
0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
D_1 = \begin{bmatrix}
0 & 1 \\
0 & 0
\end{bmatrix},
\]

\[
D_2 = \begin{bmatrix}
0 \\
0
\end{bmatrix}.
\]

In this model the state variables in vector \( x \) are, respectively, the armature current, the angular velocity, the angular position and the integral of the position error. The control input \( u \) is the armature voltage, and the disturbances input vector \( w \) components are the load torque and the angle position reference. The output variables in vector \( y \) are the position angle error and the integral of the position angle error. The PI control law is simply a proportional feedback of the model output variables. The control gain is chosen to be \( K = [K_p, K_i] \) with \( K_p = 1 \) and \( K_i = 0.5 \). The system closed loop model is described by

\[
A_c = A + B_2K_C
\]

\[
B_c = B_1 + B_2K_D_1
\]

\[
C_c = \begin{bmatrix}
0 & 0 & 0 & 1 + \zeta & 0 \\
0 & 0 & 0 & 0 & 1
\end{bmatrix},
\]

\[
D_c = \begin{bmatrix}
0 & 0 \\
0 & 0
\end{bmatrix}.
\]

The nominal parameter values in the model are \( R_a = 7.56 \Omega, \quad L_a = 0.0555 \, \text{H}, \quad k_i = 4.23 \, \text{N m A}^{-1}, \quad J = 0.068 \, \text{Kg m}^2, \quad k_v = 0.1 \, \text{N m s rad}^{-1} \).
The armature resistance \( R_a \) and inductance \( L_a \) and the angle measurement device gain are considered to be uncertain:
\[
-0.2 \times R_a \leq \rho \leq 0.5 \times R_a
\]
\[
-0.1 \times L_a \leq \tau \leq 0.1 \times L_a
\]
\[
-0.1 \leq \zeta \leq 0.1.
\]

The above considered uncertainties lead to an eight-vertices polytope. Computing the guaranteed \( \mathcal{H}_\infty \) and \( \mathcal{H}_2 \) cost norms \( \forall (A_{\text{el}}, B_{\text{el}}, C_{\text{el}}) \in \mathcal{V}(\mathcal{P}) \) for this system, one obtains \( \gamma_c = 3.3472 \) and \( \delta_c = 1.2663 \), respectively.

7.2. Discrete time model: a stock market

As a second example, consider the discrete-time model of a stock market with imperfect information, discussed by Consiglio (1994). A first sub-system \( P \) has as inputs the professional trader's fundamental value \( h \) and the specialist's fundamental value \( k \), and as output the price \( p \). Another subsystem \( H \) is feedback-connected (see the Figure), with \( p \) as the input and \( h \) as the output. In Consiglio (1994) both these sub-systems are considered to be linear, with models given by:
\[
P(z) = \frac{z}{z + \eta \beta} (K(z) - \eta H(z)) \tag{44}
\]
\[
H(z) = \frac{\Omega \beta}{z^2 + (\Omega \beta - 1)z} P(z) \tag{45}
\]
in which \( \Omega, \beta, \eta \) are positive parameters. The system behaviour is punctually analysed in Consiglio (1994) for variations in these system parameters.

With the procedure proposed here, a different kind of analysis may be performed in the \( \mathcal{H}_\infty \)-norm setting. Consider, for instance, that a parameter in \( P(z) \) is uncertain: \( 0.1 \leq \beta \leq 0.2 \). Take also \( \eta = 0.2 \) and \( \Omega = 0.3 \). The \( \mathcal{H}_\infty \) norm guaranteed cost of \( P(z) \) (considering only the input \( h \) for this two-vertices polytope is calculated as: \( \gamma_d(P) = 0.25 \).

The small-gain theorem (Boyd et al. 1994) provides information about the maximum \( \mathcal{H}_\infty \) norm of the transfer function \( H(z) \), which still leads to a stable closed-loop in the feedback configuration of the Figure. From that theorem, \( H(z) \) may have \( \mathcal{H}_\infty \) norm up to \( \gamma_d(H) = 1/\gamma_d(P) = 4.0 \), still keeping the system stability. Note that the system closed-loop stability does not depend on the particular model assumed for \( H(z) \), so that both model parameter errors and model structure errors are allowed, provided that this norm bound is fulfilled for every allowed uncertainty. By simple system transformations, model structure uncertainties may appear even in subsystem \( P(z) \).

As this analysis takes the whole uncertainties set at once, it becomes a robustness analysis, which is stronger than a punctual stability analysis. Additionally, not only parameter uncertainties, but also model structure uncertainties may be dealt with, as a by-product of the \( \mathcal{H}_\infty \)-analysis.

8. Conclusions

This paper has presented LMIs formulations for the computation of guaranteed costs of \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) norms for uncertain systems with convex-bounded model uncertainties of polytope type. The resulting optimization problems may be solved via interior point algorithms (Boyd et al. 1994), which are very efficient and reliable even for high-dimensional problems.

The resulting bounds on the \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) norms may be, in this way, incorporated as an analysis tool for the evaluation of closed-loop performance under different controller design methodologies. The \( \mathcal{H}_2 \) and \( \mathcal{H}_\infty \) norms' information may be also extended to pure analysis problems.

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