The Oscillation of Perturbed Functional Differential Equations

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Abstract—We provide new oscillation criteria for the perturbed functional differential equations. This solves some open problems of [1]. An application to an equation arising in nonlinear neural networks is illustrated. © 2000 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

In this paper, we shall study the oscillatory behavior of perturbed functional differential equations of the form

\[ \delta x^{(n)}(t) + H(t, x[g(t)]) = P(t, x[g(t)]), \]

where \( n \geq 1, \delta = \pm 1, g : [t_0, \infty) \to \mathbb{R} = (-\infty, \infty), H, P : [t_0, \infty) \times \mathbb{R} \to \mathbb{R} \) are continuous, \( t_0 \geq 0 \), and \( \lim_{t \to \infty} g(t) = \infty \).

We shall assume that there exist continuous functions \( a, p, q : [t_0, \infty) \to [0, \infty) \) and positive constants \( \lambda \) and \( \mu, \gamma := \mu - \lambda > 1 \) such that

\[ H(t, x) \operatorname{sgn} x \leq a(t)|x|^\lambda + 1, \quad \text{for } x \neq 0, \quad t \geq t_0, \]

\[ P(t, x) \operatorname{sgn} x \geq p(t)|x|^\mu + q(t)|x|^\lambda, \quad \text{for } x \neq 0, \quad t \geq t_0, \]

and

\[ Q(t) := q(t) - \beta a^\gamma/(\gamma - 1)(t)p^{1/(1-\gamma)}(t) \geq 0, \quad \text{for } t \geq t_0, \]

and \( Q(t) \neq 0 \) on any ray of the form \([t^*, \infty)\) for some \( t^* \geq t_0 \), where \( \beta = (\gamma - 1)\gamma^{(1-\gamma)} \).

As usual, a nontrivial solution of equation \((\text{E}, \delta)\) is called oscillatory if it has arbitrarily large zeros, otherwise it is said to be nonoscillatory. Equation \((\text{E}, \delta)\) is called oscillatory if all of its solutions are oscillatory.
The oscillatory behavior of equation (E,l) with \( n \) even and \( g(t) = t \) has been introduced and discussed by Kartsatos [1–3]. In [1], Kartsatos raised some open problems regarding the oscillation of equation (E,\( \delta \)) without assuming either

\[
\lim_{t \to \infty} \frac{H(t, u[g(t)])}{P(t, u[g(t)])} = 0 \quad (*)
\]

or

\[
\lim_{t \to \infty} \sup_{|u| \leq K} \frac{H(t, u)}{P(t, u)} = 0 \quad (**)
\]

(see Problems IX and X).

The purpose of this paper is to provide sufficient conditions for the oscillation of equation (E,\( \delta \)) without necessarily requiring assumptions (*) or (**).

2. MAIN RESULTS

We need the following lemma.

**Lemma 1.** (See [4].) If \( A \) and \( B \) are nonnegative, then

\[
A^\lambda - \lambda AB^{\lambda-1} + (\lambda - 1)B^\lambda \geq 0, \quad \lambda > 1.
\]

and equality holds if and only if \( A = B \).

**Theorem 1.** Let \( n \) be even, and conditions (1)–(3) hold. If the equation

\[
x^{(n)}(t) + Q(t)\sigma(t)\sigma'(t) + Q(t)\sigma(t)[\sigma(t)] \geq 0 \quad (4)
\]

is oscillatory, where \( \sigma(t) = \min\{t, g(t)\} \) and \( \lim_{t \to \infty} \sigma(t) = \infty \), then equation (E,\( -1 \)) is oscillatory.

**Proof.** Let \( x(t) \) be a nonoscillatory solution of equation (E,\( -1 \)), say \( x(t) > 0 \) and \( x[g(t)] > 0 \) for \( t > t_0 \). Using conditions (1) and (3) in equation (E,\( -1 \)), we have

\[
0 \geq x^{(n)}(t) - a(t)x^{\lambda-1}\sigma(t)\sigma'(t) + p(t)x^{\lambda}[\sigma(t)] + q(t)x^\lambda \quad (5)
\]

Now we set

\[
A = p^{1/\gamma(t)}x[g(t)] \quad \text{and} \quad B = \left(\frac{a(t)}{\gamma}p^{-1/\gamma(t)}\right)^{1/(\gamma-1)}
\]

and apply Lemma 1, to get

\[
0 \geq x^{(n)}(t) + Q(t)x^{\lambda}[\sigma(t)] - \beta a^{\gamma/(\gamma-1)}(t)p^{1/(1-\gamma)}(t)x^{\lambda}[\sigma(t)]
\]

or

\[
x^{(n)}(t) + Q(t)x^{\lambda}[\sigma(t)] \leq 0, \quad t \geq T_1 > t_0 \quad (6)
\]

Since \( n \) is even, we see that \( x(t) \) is an increasing function for \( t \geq T_1 \) and (6) reduces to

\[
x^{(n)}(t) + Q(t)x^{\lambda}[\sigma(t)] \leq 0, \quad t \geq T \geq T_1.
\]

But, this in view of a result of [5] leads to a contradiction.

**Remark 1.** For \( a(t) = p(t) = q(t) \), condition (3) holds for all \( \lambda > 0 \) and \( \mu > \lambda + 1 \). However, for this case conditions of type (*) and (***) are not valid.

In fact, this solves the open problems IX and X in [1] for equation (E,\( -1 \)) when conditions (1)–(3) hold.

**Remark 2.** From the proof of Theorem 1, we see that equation (E,\( -1 \)) under assumptions (1)–(3) is reduced to an inequality of type (6). Now for any \( n \geq 1 \) and any \( \lambda > 0 \), one can apply the results of [6] to this inequality and obtain complete oscillation criteria for equation (E,\( -1 \)), or make use of comparison results of [7] and compare the oscillatory and asymptotic behavior of equations of type (4) to that of (E,\( -1 \)).

Next, we present the following oscillation criterion for equation (E,1), \( \lambda = 1 \) and \( n \) is odd. The other cases for any \( n \geq 1 \) and \( \lambda > 0 \) can be obtained similarly.
THEOREM 2. Let $\lambda = 1$, $n$ is odd, and conditions (1)-(3) hold. If

$$\int_{t_0}^{\infty} s\sigma^{n-2}(s)g^{-\epsilon}(s)Q(s)\,ds = \infty, \quad \text{for some } \epsilon > 0, \quad (7)$$

$$\int_{t_0}^{\infty} g^{n-1}(s)Q(s)\,ds = \infty, \quad (8)$$

and

$$\limsup_{t \to \infty} \int_t^{\rho(t)} \frac{(g(s) - \rho(t))^{j}}{j!} \frac{(\rho(t) - s)^{n-j-1}}{(n-j-1)!} Q(s)\,ds > 1 \quad (9)$$

for some $j = 0, 1, \ldots, n-1$, where $\sigma(t) = \min\{t, g(t)\} \to \infty$ as $t \to \infty$ and $\rho(t) = \min\{\max\{s, g(s)\} : s \geq t\}$, then equation (E,1) is oscillatory.

PROOF. Let $x(t)$ be a nonoscillatory solution of equation (E,1), say $x(t) > 0$ and $x[g(t)] > 0$ for $t > t_0$. Using conditions (1) and (2) in equation (E,1), we obtain

$$0 \leq x^{(n)}(t) + a(t)x^{\lambda+1}[g(t)] - p(t)x^{\mu}[g(t)] - q(t)x^{\lambda}[g(t)]
= x^{(n)}(t) - q(t)x^{\lambda}[g(t)] + x^{\lambda}[g(t)] [a(t)x[g(t)] - q(t)x^{\gamma}[g(t)]], \quad \text{for } t \geq t_0.$$

Now we let $A$ and $B$ be as in the proof of Theorem 1 and apply Lemma 1, to get

$$x^{(n)}(t) - Q(t)x^{\lambda}[g(t)] \geq 0, \quad \text{for } t \geq T \geq t_0. \quad (10)$$

The rest of the proof follows by applying a result of [6].

As an application, we consider the following equation which arises in the study of nonlinear neural networks:

$$\delta \frac{dx(t)}{dt} = -q(t)|x[g(t)]|^{\lambda} \text{sgn} \, x[g(t)] + a(t)|x[g(t)]|^{\lambda} \tanh x[g(t)]
- p(t)|x[g(t)]|^{\mu} \text{sgn} \, x[g(t)], \quad \text{in (1)},$$

where $\delta = \pm 1$, $\lambda$ and $\mu$ are real constants, $\lambda > 0$ and $\mu > \lambda + 1$, the functions $a$, $g$, $p$, and $q$ are as in equation (E,1) and conditions (1) and (2) hold. As in the proof of Theorems 1 and 2, one can easily see that equations (N,1) and (N,-1) are reduced respectively to the following inequalities:

$$\left\{ \frac{dx(t)}{dt} + Q(t)|x[g(t)]|^{\lambda} \text{sgn} \, x[g(t)] \right\} \leq 0 \quad \text{in (1)},$$

$$\left\{ \frac{dx(t)}{dt} - Q(t)|x[g(t)]|^{\lambda} \text{sgn} \, x[g(t)] \right\} \geq 0, \quad \text{in (2)}$$

where $Q(t)$ is defined in (3).

Now, it is easy to see that (11) is oscillatory if one of the following conditions holds:

I) $\lambda = 1$, $g(t) \leq t$ and $g'(t) \geq 0$ for $t \geq t_0$, and

$$\liminf_{t \to \infty} \int_{g(t)}^{t} Q(s)\, ds > \frac{1}{e},$$

II) $0 < \lambda < 1$ and

$$\int_{R_{\alpha}} Q(s)\, ds = \infty, \quad \text{where } R_{\alpha} = \{ t \in [t_0, \infty) : t_0 \leq g(t) \leq t \}.$$
Also, we see that (12) is oscillatory if one of the following conditions holds:

(III) $\lambda = 1$, $g(t) \geq t$, and $g'(t) \geq 0$ for $t \geq t_0$, and

$$\lim_{t \to \infty} \inf_{t} \int_{t}^{g(t)} Q(s) \, ds > 1,$$

(IV) $\lambda > 1$ and

$$\int_{A_g} Q(s) \, ds = \infty,$$

where $A_g = \{ t \in [t_0, \infty) : g(t) \geq t \}$.

Thus, the oscillation of the equation $(N, \delta)$, $\delta = \pm 1$ follows from those for inequalities (11) and (12).

REFERENCES


