Direct Adaptive Regulation of Discrete Time Nonlinear Systems with Arbitrary Nonlinearities by Backstepping

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Abstract

This paper presents a direct adaptive regulation result for a class of strict feedback discrete time nonlinear systems with arbitrary nonlinearities. This results is an extension of the current state of the art in discrete time adaptive backstepping, where until now results are restricted to the case where the plant’s nonlinearities can be expressed as a linear combination of known functions and restrictive linear growth conditions are imposed. Here, a more general approach is followed by using function approximation and on-line update of the approximators. The regulation theorem guarantees convergence of the state norm to a neighborhood of the origin, and the size of the neighborhood depends exclusively on the chosen approximator structure and ideal approximation errors. The stability is semi-global, and no high-gain bounding terms are required, since explicit bounds for the state are found from the stability analysis. The proposed method is a direct adaptive scheme where a stabilizing control law, rather than the plant nonlinearities, is directly approximated. An example is provided to illustrate the method.

1 Introduction

The problem of control of nonlinear systems presents fundamental difficulties due to the structural complexity arbitrary nonlinear systems may possess. In order to perform stability analysis, it is of basic importance to first determine a class of systems that is representative of real problems, and yet amenable to analysis. One such class is that of strict feedback systems, which have been studied in continuous time under the backstepping methodology [1] (see also the references therein, as well as [2], where tracking is addressed).

Backstepping currently appears to be the most systematic method for nonlinear control design through step-by-step construction of quadratic Lyapunov functions. The stability analysis is constructive, and it generates stabilizing control laws. In addition, backstepping provides some guidelines on the effects of design parameters on transient performance of the closed loop system. However, most results are restricted to continuous time analysis, largely due to the technical difficulties that discrete time stability analysis presents. The few existing results in adaptive control of discrete time systems concentrate on system in strict feedback and output feedback forms [3, 4, 5, 6, 7, 8, 9]. Most of these papers use ideas from the linear adaptive control literature, and they require the system nonlinearities to be linear combinations of known nonlinearities, and these nonlinearities must satisfy linear growth conditions. Recently, the papers [10, 11] introduce adaptive control methods for discrete time systems in output feedback and strict feedback forms by using a two-stage approach, where linearly parameterized nonlinearities are first identified and then a deadbeat controller is used to achieve tracking.

In this paper regulation of discrete time strict feedback systems is addressed, where the system nonlinearities are not required to be linear combinations of known functions. Rather, arbitrary nonlinearities are allowed and no growth restrictions are imposed, and function approximators are used and updated on-line to achieve regulation of the state to a neighborhood of the origin. The size of this neighborhood is only determined by how well the function approximators are chosen, that is, how small the approximation error would be if one used ideal parameters. The proposed method is a direct adaptive scheme where a stabilizing control law, rather than the plant nonlinearities, is directly approximated. The result is semi-global stability, and the scheme is able to guarantee stability without the need for bounding control terms, since the stability analysis yields explicit bounds for the state.

2 The Plant, and General Assumptions

We are interested in regulation to the origin of discrete time systems of the form

\[ x_i(k+1) = \phi_i(x_i(k)) + x_{i+1}(k), \quad i = 1, \ldots, n-1 \]
\[ x_n(k+1) = \phi_n(x_n(k)) + \psi_n(x_n(k))u(k) \] (1)

We use the notation \( \bar{x}_i = [x_1, \ldots, x_i]^T \) for convenience, and \( k \) is used to denote the discrete time index. This class of systems is a useful and general representation of systems in discrete time that may arise, for instance,
when converting from a continuous time dynamic description [12].

The nonlinearities $\phi_i(\cdot), i = 1, \ldots, n$ and $\psi_n(\cdot)$ are assumed to be piecewise continuous within some domain of interest, but otherwise unknown. However, as shown in Section 4, in practice the more prior knowledge available to the designer, the more successful the implementation of this method will be. The function $\psi_n(\bar{x}_n(k))$ is assumed bounded away from zero (for convenience, we assume it to be positive, although the negative case is straightforward) so that

$$0 < \bar{\psi} \leq \psi_n(k) \leq \bar{\psi} < \infty$$

(2)

Knowledge of the upper bound $\bar{\psi}$ is required. Note that we will freely omit the explicit list of arguments of functions when the dependencies are easily understood, and will only keep the time index for clarity.

Whereas we allow the control signal $u(k)$ to be multiplied by a function of the state, $\psi_n(\bar{x}_n(k))$, we assume that the corresponding functions $\psi_i(\cdot)$ in the dynamics of $x_i(k+1)$ (that is, rewriting the first equation in (1) as $x_i(k+1) = \phi_i(x_i(k)) + \psi_i(x_i(k))x_{i+1}(k), i = 1, \ldots, n-1$) are all known constants. For convenience, and without loss of generality, we assume $\psi_i = 1, i = 1, \ldots, n-1$. Moreover, the state $\bar{x}_n(k)$ is assumed available for measurement in order to construct the control signal $u(k)$.

Systems in the form (1) may be best analyzed in the context of backstepping. As shown in Section 3, one needs to formulate $n$ auxiliary functions that are based on the nonlinearities of (1) and such that stability is achieved. Since we assume poor knowledge of the dynamics of (1), we resort to general function approximators that are updated on-line to achieve closed-loop stability. We do not specify the kind nor the structure of the approximators, so that neural networks, fuzzy systems, wavelets, splines or any other structure that possesses the universal approximation property may be used. We do restrict our attention to linearly parameterized approximators, though, for ease of analysis. To establish the notation for the discussion to follow, consider a piecewise continuous function $f^*(x)$ that we wish to approximate within some compact domain $S$. We will assume that the approximator is expressed as

$$f(x, \theta) = \zeta^\top(x)\theta$$

(3)

where $\theta \in \mathbb{R}^p$ is a column vector of parameters, and $\zeta(x) \in \mathbb{R}^p$ is a column vector of basis functions, and $p$ is the size of the approximator. Then, for a universal approximator of a given size there exists a parameter set $\theta^*$ such that

$$f^*(x) = f(x, \theta^*) + w(x)$$

(4)

where $w(x)$ is the representation error, bounded by $|w(x)| \leq W, W \geq 0$ when $x \in S$. The vector $\theta^*$ is “optimal” in the sense that the bound $W$ is the smallest possible given the size $p$. That such a $\theta^*$ exists is a result of the universal approximation property (conversely, this property is usually stated so that, for a given approximation error bound $W$, a large enough $p$ exists so that the desired approximation level can be achieved). The only restriction we place on the approximator structure is that

$$|\zeta(x)| \leq \bar{\zeta}$$

(5)

for all $x$, where $\bar{\zeta} > 0$ is some known constant. This can be easily achieved by using properly defined basis functions, such as radial basis functions.

### 3 Direct Adaptive Regulation

The main result in this paper is the regulation theorem that follows. The theorem is first stated, and then a proof is provided.

**Theorem 1:** Consider nonlinear discrete time systems in the form (1) satisfying the assumptions stated in Section 2. The diffeomorphism

$$z_1(k) = x_1(k)$$

$$z_i(k) = x_i(k) - \alpha_{i-1}(\bar{x}_{i-1}(k), \theta_{i-1}(k)), \ i = 2, \ldots, n$$

is used to perform an invertible coordinate transformation, where the approximators

$$\alpha_i(\bar{x}_i(k), \theta_i(k)) = \zeta^\top(\bar{x}_i(k))\theta_i(k), \ i = 1, \ldots, n-1$$

(7)

are used and are such that $|\zeta_i(\cdot)| \leq \bar{\zeta}_i$ for some known constants $\bar{\zeta}_i > 0$. The approximation errors $w_i(k)$ between $\alpha_i(\bar{x}_i(k), \theta_i^*)$ and the auxiliary functions $\alpha_i^*(\bar{x}_i(k))$ to be defined below are assumed bounded by known constants, so that

$$\alpha_i^*(\bar{x}_i(k)) = \alpha_i(\bar{x}_i(k), \theta_i^*) + w_i(k)$$

(8)

and $|w_i(k)| \leq W_i$ when $\bar{x}_n \in S_x$ and $S_x$ is a compact set. Each approximator is of size $p_i$ so that $\theta_i(k) \in \mathbb{R}^{p_i}$. The control signal is defined as

$$u(k, \theta_n) = u(\bar{x}_n(k), \theta_n(k)) = \zeta^\top(\bar{x}_n(k))\theta_n(k)$$

(9)

Consider the adaptation laws

$$\theta_i(k) = \theta_i(k-1) - \eta_i \frac{\zeta_i(\bar{x}_i(k-1))q_i(k)}{1 + \gamma_i|\zeta_i(\bar{x}_i(k-1))|^2}, \ i = 1, \ldots, n$$

(10)

where $\gamma_i > 0$ and $0 < \eta_i < 2\gamma_i$ are design constants, and we let

$$\bar{z}_i(k) = z_i(k) - z_{i+1}(k-1), \ i = 1, \ldots, n-1$$

$$q_i(k) = D_c(\bar{z}_i(k), W_i), \ i = 1, \ldots, n-1$$

$$q_n(k) = D_c(z_n(k), W_n)$$

(11)
where we used (8), and with maximum bound

$$|z(k)|^2 \leq \max\left(V(0), \frac{b_n + V_0(0)}{b_0}\right) = B_{max} \quad (12)$$

for all $k$, and that it will have ultimate bound

$$\lim_{k \to \infty} |z(k)|^2 \leq \frac{b_n}{b_0} \quad (13)$$

where the function $V(k)$ and the constants $b_0$ and $b_n$ are defined in the proof below.

Continuous dead-zones are used in the proof to provide for stability. For some $y, b \in \mathbb{R}$, we define the continuous dead-zone as

$$D_c(y, b) = \begin{cases} 
    y - b & \text{if } y \geq b \\
    0 & \text{if } -b < y < b \\
    y + b & \text{if } y \leq -b 
\end{cases} \quad (14)$$

The following lemmas from [12] are used in the analysis:

**Lemma 1:** Let $x, y, w \in \mathbb{R}$. If $y = x + w$ and $|w| \leq b$, $b > 0$, then $D_c(y, b) = \delta x$, where $0 \leq \delta < 1$.

**Lemma 2:** Given some non-negative sequences $V_1(k)$ and $V_2(k)$, if $V(k) = V_1(k) + V_2(k)$ with $V_2(k + 1) - V_2(k) \leq 0$ and $V(k + 1) - V(k) \leq -k_1V_1(k) + k_2$, then $V_2(k)$ is bounded and $V_1(k)$ is uniformly ultimately bounded, with maximum bound $V_1(k) \leq \max\left(V_1(0), \frac{k_2 + V_2(0)}{k_1}\right)$ for all $k$, and ultimate bound

$$\lim_{k \to \infty} V_1(k) = \frac{k_2}{k_1}. \quad (15)$$

Now we are ready to state the proof of Theorem 1

**Proof:** The proof is performed inductively in $n$ steps. With $z_1(k) = x_1(k)$ and $z_2(k) = x_2(k) - \alpha_1(k, \theta_1)$ from (6) (we will use the notations $\alpha_1(k, \theta_1) = \alpha_1(\bar{x}_1(k), \theta_1(k))$ interchangeably), we have that

$$z_1(k + 1) = \phi_1(k) + z_2(k) + \alpha_1(k, \theta_1) + \alpha_1(k, \theta_1^*) - \alpha_1(k, \theta_1^*) = z_2(k) + \alpha_1(k, \theta_1) - w_1(k) \quad (16)$$

where we used (8), and

$$\hat{\theta}_i(k) = \theta_i(k) - \theta_i^* \quad (16)$$

is the parameter error vector, so that $\alpha_1(k, \hat{\theta}_1) = \zeta_{1i}^\top(k)\hat{\theta}_1(k)$. In order to measure the parameter errors, we define the $n$ functions

$$V_{\hat{\theta}_i}(k) = \zeta_{1i}^\top(k)\hat{\theta}_i(k) \quad (17)$$

for $i = 1, \ldots, n$, and we let

$$V_i(k) = z_i^2(k) + c_{1i}V_{\hat{\theta}_i}(k) \quad (18)$$

where $c_{1i} > 0$ is a constant to be defined below. We now need to find an upper bound for the difference

$$V(k + 1) - V(k) = -\zeta_i^2(k) + z_i^2(k + 1) + c_{1i}(V_{\hat{\theta}_i}(k + 1) - V_{\hat{\theta}_i}(k)) \quad (19)$$

Note that

$$z_i^2(k + 1) - 2z_i^2(k) + 2(\alpha_1(k, \hat{\theta}_1) - w_1(k))^2 \quad (20)$$

In order to find an upper bound for (19), define the signal $\bar{z}_i(k) = z_i(k) - z_i(k - 1)$, so that $\bar{z}_i(k + 1) = \alpha_1(k, \hat{\theta}_1) - w_1(k)$. Moreover, letting

$$q_i(k) = D_c(\bar{z}_i(k), W_i) \quad (21)$$

we find, applying Lemma 1, that

$$q_i(k + 1) = \delta_i(k)\alpha_1(k, \hat{\theta}_1) \quad (22)$$

where $0 \leq \delta_i(k) < 1$. Then,

$$\alpha_1(k, \hat{\theta}_1) - w_1(k) = q_i(k + 1) + \bar{q}(k + 1) \quad (23)$$

where $\bar{q}(k + 1) = \bar{z}_i(k + 1) - q_i(k + 1) + w_1(k)$. Notice that, due to the definition of the signals and the dead zone, we have $|\bar{q}_i(k + 1)| \leq |\bar{z}_i(k + 1) - q_i(k + 1)| + |w_1(k)| = 2W_1$. Moreover, it is easy to show that, for any $\epsilon > 0$ and scalar $x$ and $y$, $(x + y)^2 \leq (1 + \epsilon)x^2 + (1 + \frac{1}{\epsilon})y^2$. Then,

$$\begin{aligned}
(\alpha_1(k, \hat{\theta}_1) - w_1(k))^2 &\leq (|q_i(k + 1)| + 3W_1)^2 \\
&\leq (1 + \epsilon)9W_1^2 + \left(1 + \frac{1}{\epsilon}\right)q_i^2(k + 1) \\
&\leq (1 + \epsilon)9W_1^2 + \left(1 + \frac{1}{\epsilon}\right)(1 + \gamma_1\zeta_i^2)\frac{q_i^2(k + 1)}{1 + \gamma_1\zeta_i^2(k)} \quad (24)
\end{aligned}$$

We can now give an upper bound for $z_i^2(k + 1)$, and can investigate parameter behavior next. Considering the adaptation law (10) with $i = 1$, we have that

$$V_{\hat{\theta}_i}(k + 1) - V_{\hat{\theta}_i}(k) = \hat{\theta}_i^\top(k + 1)\hat{\theta}_i(k + 1) - \hat{\theta}_i^\top(k)\hat{\theta}_i(k)$$

$$= \left[\hat{\theta}_i(k) - \eta_i\zeta_i(x_1(k))q_i(k + 1))^2 \right]^2 - \hat{\theta}_i^\top(k)\hat{\theta}_i(k)$$

$$= -2\eta_i\zeta_i^\top(x_1(k))\hat{\theta}_i(k)q_i(k + 1) + \eta_i^2\frac{\zeta_i^2(x_1(k))q_i^2(k + 1)}{(1 + \gamma_1\zeta_i^2(k))^2} \quad (25)$$

Note that $\bar{z}_i(k + 1)$ is either inside or outside the dead zone. If $\bar{z}(k + 1)$ is within the dead zone, then $q_i(k + 1) = 0$ by definition, and in this case $V_{\hat{\theta}_i}(k + 1) -
In this manner, we can write
\[ \alpha_1(k, \hat{\theta}_1) = \xi_1^T(k)\hat{\theta}_1(k) = \frac{q_1(k+1)}{\delta_1(k)} (26) \]
This yields \[ \frac{V_{\hat{\theta}_1}(k + 1)}{\eta_1} - V_{\hat{\theta}_1}(k) = \frac{q_1^2(k+1)}{1+\gamma_1|\xi_1(k)|^2}, \]
Since \[ \frac{2}{\delta_1(k)} - \eta_1 \frac{|\xi_1(k)|^2}{1+\gamma_1|\xi_1(k)|^2} \geq 2 - \frac{\eta_1}{\gamma_1}, \] the choices \( \gamma_1 > 0 \) and \( 0 < \eta_1 < 2\gamma_1 \) yield (letting \(\alpha_{1.2} = 2 - \frac{\eta_1}{\gamma_1} \))
\[ V_{\hat{\theta}_1}(k + 1) - V_{\hat{\theta}_1}(k) = -\eta_1 c_{1.2} \frac{q_1^2(k+1)}{1+\gamma_1|\xi_1(k)|^2} \leq 0 (27) \]
where the inequality is valid for all \( k \). Therefore, \( V_{\hat{\theta}_1}(k) \) is a bounded sequence. Now we can return to bounding the difference of the sequence \( V_1(k) \), so that
\[ V_1(k + 1) - V_1(k) \leq -z_1^2(k) + 2z_2^2(k) + 2(1 + \gamma_1)q_1^2(k+1) + 2\gamma_1|\xi_1(k)|^2 \]
\[ - \frac{q_1^2(k+1)}{1+\gamma_1|\xi_1(k)|^2} \leq -\gamma_1 c_{1.1}\]
with \( b_1 = 2(1 + \epsilon)9W_1^2 \). If \( c_{1.1} > 2(1 + \frac{1}{\gamma_1})|\xi_1(k)|^2 \), then
\[ V_1(k + 1) - V_1(k) \leq -z_1^2(k) + 2z_2^2(k) + b_1 (29) \]
This completes the first step of the proof.
Next, we proceed inductively for the \( r^{th} \) step, with \( 2 \leq r \leq n - 1 \). From (6) we have that \( z_{r+1}(k) = x_r(k+1) - \alpha_r(x_r(k), \theta_r(k)) \), and the unknown auxiliary function we wish to approximate is given by
\[ \alpha^*_r(x_r(k)) = \phi_r(x_r(k)) + \alpha_{r-1}(x_{r-1}(k+1), \theta_{r-1}(k+1)) \]
In this manner, we can write
\[ z_r(k + 1) = \phi_r(k) + z_{r+1}(k) - \alpha_r(x_r(k), \theta_r(k)) \]
\[ + \alpha_r(x_r(k), \theta_r^*) - \alpha_r(k, \theta_r^*) \]
\[ = z_{r+1}(k) + \alpha_r(k, \hat{\theta}_r) - w_r(k) (31) \]
Let \( V_r(k) = V_{r-1}(k) + (2^r-1)z_1^2(k) + c_{r.1}V_{\hat{\theta}_r}(k) \). Noting that \( V_{r-1}(k + 1) - V_{r-1}(k) \leq -\sum_{i=1}^{r-1} z_i^2(k) + b_{r-1} + (2^r-2)z_r^2(k) \) with \( b_{r-1} = b_{r-2} + 9(1 + \epsilon)(2^r-2)W_{r-1}^2 = 9(1 + \epsilon)\sum_{i=1}^{r-1} (2^i+1 - 2)W_i^2 \), and \( b_1 \) is given above,
\[ V_{r}(k + 1) - V_{r}(k) \leq -\sum_{i=1}^{r-1} z_i^2(k) + b_{r-1} + (2^r-2)z_r^2(k) \]
\[ + (2^r-1)(z_r^2(k+1) - z_r^2(k)) + c_{r.1}(V_{\hat{\theta}_r}(k+1) - V_{\hat{\theta}_r}(k)) \]
\[ = \sum_{i=1}^{r} z_i^2(k) + b_{r-1} + (2^r-1)z_r^2(k+1) + c_{r.1}(V_{\hat{\theta}_r}(k+1) - V_{\hat{\theta}_r}(k)) \]
\[ = \sum_{i=1}^{r} z_i^2(k) + b_{r-1} + (2^r-1)z_r^2(k+1) + c_{r.1}(V_{\hat{\theta}_r}(k+1) - V_{\hat{\theta}_r}(k)) \]
Finally, we have that \( V_{r}(k + 1) - V_{r}(k) \leq -\sum_{i=1}^{r} z_i^2(k) + (2^r-1)z_{r+1}^2(k) + 2(2^r-1)\]
\[ - (1 + \epsilon)9W_r^2 (1 + \frac{1}{\gamma_1})1 + \gamma_1|\xi_1(k)|^2 \]
\[ \leq -\hat{\gamma}_r c_{r.2} \frac{q_r^2(k+1)}{1+\gamma_r|\xi_r(k)|^2} \]
Letting \( b_r = b_{r-1} + 9(1 + \epsilon)(2^r-2)W_r^2 \), and with \( c_{r.1} > \frac{1(1 + \gamma_r|\xi_r(k)|^2)}{\gamma_r|\xi_r(k)|^2} \)
we obtain
\[ V_{r}(k + 1) - V_{r}(k) \leq -\sum_{i=1}^{r} z_i^2(k) + (2^r-1)z_{r+1}^2(k) + b_r \]
At the \( n^{th} \) step of the proof, we have that \( z_n(k+1) = \phi_n(k) + \psi_n(k)u(k) - \alpha_{n-1}(k+1, \theta_{n-1}) \). Due to the dynamic uncertainty, the control function
\[ u^*(\bar{x}_n(k)) = \frac{1}{\psi_n(k)}(-\phi_n(k) + \alpha_{n-1}(k+1, \theta_{n-1})) (36) \]
is unkown, so using an approximator with optimal parameter vector we can write \( u^*(\bar{x}_n(k)) = u(\bar{x}_n(k), \theta_n^*) + w_n(k) \). It follows that
\[ z_n(k+1) = \phi_n(k) + \psi_n(k)(u(k, \theta_n) + u(k, \theta_n^*)) \]
\[ - u(k, \theta_n^*) - \alpha_{n-1}(k+1, \theta_{n-1}) \]
\[ = \psi_n(k)(u(k, \hat{\theta}_n) - w_n(k)) (37) \]
Consider the Lyapunov function candidate $V(k) = V_{n-1}(k) + (2^n - 1)z_n^2(k) + c_{n,1}V_{\theta_n}(k)$. Then, if $r = n - 1$ above,

$$V(k + 1) - V(k) \leq -\sum_{i=1}^n z_i^2(k) + (2^n - 1)z_n^2(k) + 1 + b_{n-1} + c_{n,1}(V_{\theta_n}(k + 1) - V_{\theta_n}(k))$$

(38)

Let $q_n(k) = D_z(z_n(k), W_n \tilde{\psi})$, so that $q_n(k + 1) = \delta_n(k + 1)q_n(k + 1) + \psi_n(k)u(k, \theta_n) \geq 0 \leq \delta_n(k) < 1$. Also, we have that $\psi_n(k)u(k, \theta_n) = z_n(k + 1) + \psi_n(k)w_n(k) = q_n(k + 1) + \tilde{q}_n(k + 1)$ with $\tilde{q}_n(k + 1) = z_n(k + 1) - q_n(k + 1) + \psi_n(k)w_n(k)$. It is easy to verify that $|\tilde{q}_n(k + 1)| \leq 2W_n \tilde{\psi}$. We can now write the bound

$$|u(k, \tilde{\theta}_n)| \leq \frac{1}{\psi_n(k)}(q_n(k + 1) + 2W_n \tilde{\psi} \psi_n(k))$$

(39)

Therefore,

$$z_n^2(k + 1) \leq \psi_n^2(k) \left( \frac{1}{\psi_n(k)}(q_n(k + 1) + 2W_n \tilde{\psi} \psi_n(k)) \right)^2 \leq \psi_n^2(k) \left( (1 + \epsilon)W_n \tilde{\psi} \frac{\tilde{\eta} + \sum_{i=1}^\epsilon \tilde{\psi}_n(k)}{\psi_n(k)} \right)^2 \leq (1 + \epsilon)9\tilde{\psi}^2W_n^2 + (1 + \frac{1}{\epsilon})(1 + 2\gamma_n^2)\tilde{\eta} \tilde{\psi} \tilde{\eta} \frac{q_n^2(k + 1)}{1 + 2\gamma_n^2}(38)$$

By noting that (34) also holds when $r = n$, we can now write $V(k + 1) - V(k) \leq -\sum_{i=1}^n z_i^2(k) + (2^n - 1)(1 + \epsilon)9\tilde{\psi}^2W_n^2 + b_{n-1} + \frac{q_n^2(k + 1)}{1 + 2\gamma_n^2}(2^n - 1)(1 + \frac{1}{\epsilon})(1 + 2\gamma_n^2)\tilde{\eta} \tilde{\psi} \tilde{\eta} \frac{q_n^2(k + 1)}{1 + 2\gamma_n^2}$. Let

$$b_n = b_{n-1} + (1 + \epsilon)(2^n - 1)9\tilde{\psi}^2W_n^2$$

$$= 9(1 + \epsilon)\sum_{i=1}^n (2^{i+1} - 2)W_n^2 + (2^n - 1)\tilde{\psi}^2W_n^2$$

(40)

With $c_{n,1} > \frac{(2^n - 1)(1 + \frac{1}{\epsilon})(1 + 2\gamma_n^2)}{\tilde{\eta} \tilde{\psi} \tilde{\eta}}$ we obtain

$$V(k + 1) - V(k) \leq -\sum_{i=1}^n z_i^2(k) + b_n$$

(41)

In order to study the stability properties of the closed-loop system, define $V_{\theta_n}(k) = \sum_{i=1}^n c_{i,1}V_{\theta_i}(k)$ and $V_{\bar{\theta}_n}(k) = \sum_{i=1}^n (2^n - 1)z_i^2(k)$. Then $V(k) = V_{\bar{\theta}_n}(k) + V_{\theta_n}(k)$, and moreover $\sum_{i=1}^n z_i^2(k) \leq V_{\bar{s}}(k) \leq (2^n - 1)\sum_{i=1}^n z_i^2(k)$, which implies

$$V(k + 1) - V(k) \leq b_0V_{\bar{s}}(k) + b_n$$

(42)

where $b_0 = \frac{\tilde{\psi}^2W_n}{\tilde{\eta}}$. Moreover, $V_{\theta_n}(k + 1) - V_{\theta_n}(k) \leq 0$, so we conclude, by applying Lemma 2 that the maximum bound of the transformed state $\bar{z} = [z_1, \ldots, z_n]^T$ is given by (12), and that its ultimate bound is (13). Finally, boundedness of $\bar{z}$ implies boundedness of the plant’s state $\bar{x}_n$ since they are related by the diffeomorphism (6).

**Remark 1**: Note that the size of the final bound, $b_{n,0}$ is exclusively controlled by the choice of the approximator structure. Although continuous dead-zones are used to provide for closed-loop stability, no additional robustifying terms are required. The constant $b_{n,0}$ is fixed by the order of the plant, while $b_n \approx 9\sum_{i=1}^{n-1} (2^{i+1} - 2)W_i^2 + (2^n - 1)\tilde{\psi}^2W_n^2$ (43)

since the constant $\epsilon > 0$ can be arbitrarily small. The constant $b_n$ grows in direct proportion to the square of the approximation error bounds. Clearly, then, in order to obtain the best performance, any additional knowledge about the plant that may be available should be used in order to reduce, as much as possible, the approximation error bounds $W_i$.

**Remark 2**: The compact set $\mathcal{S}_x$ determines the region of interest where approximators are defined. The approximators are not necessarily valid outside this region, so that $|w_i(k)| > W_i$ may be possible if $\bar{x}_n$ is not within $\mathcal{S}_x$. This is usually a problem for adaptive control techniques that use on-line function approximation, where $\bar{x}_n \in \mathcal{S}_x$ can only be guaranteed through the use of a high-gain bounding control term. In this case, we see that bound (12) determines an invariant set, so that if the initial conditions are such that $|\bar{z}(0)| \leq B_{max}$, then $|\bar{z}(k)| \leq B_{max}$ for all time, and so $\bar{x}_n \in \mathcal{S}_x$ can be guaranteed with an appropriately defined $\mathcal{S}_x$ without the need of any further bounding term. In this way, bound (12) can be used to properly define the function approximators. Knowledge of $V_{\theta_n}(0)$ is necessary, but this quantity may be overbounded if the approximator parameters are known to belong to some bounded region.

### 4 Illustrative Example

We will illustrate the use of Theorem 1 via a simple second order example. Consider the open-loop unstable system given by

$$x_1(k + 1) = 1.1x_1(k) + x_2(k)$$

$$x_2(k + 1) = -\sin(x_1(k))x_2(k) + (1.1 + \tanh(x_2(k)))u(k)$$

(44)

We choose the initial conditions $x_1(0) = -1.3$ and $x_2(0) = 1.2$. Note that we can establish a bound $\psi = 2.1$ so that $\psi \geq \psi_2(k)$. In order to implement the stability result of Theorem 1 on (44) we need two approximators. We choose radial basis functions for the approximators, letting $\zeta_i(k) = \exp\left(-\frac{(x_i(k) - c_{1,i})^2}{\sigma_{1,i}^2}\right), \ldots, \exp\left(-\frac{(x_i(k) - c_{1,i})^2}{\sigma_{1,i}^2}\right)$ with the centers $c_{1,i}$ evenly spaced along the intervals $[-3, 3]$, and we set the spread $s_1 = 1.5$. 


For the second approximator we choose \( \zeta_2(k) = \left[ \exp\left( \frac{(x_1(k)-\sigma_{2,1})^2}{s_1^2} \right) \exp\left( \frac{(x_2(k)-\sigma_{2,2})^2}{s_2^2} \right) \ldots \right] \),
\[
\exp\left( \frac{(x_1(k)-\sigma_{2,1})^2}{s_1^2} \right) \exp\left( \frac{(x_2(k)-\sigma_{2,2})^2}{s_2^2} \right) \right]^{\top},
\]
with the centers \( \sigma_{2,i} \) and \( \sigma_{3,i} \) both evenly spaced along the interval \([-3,3] \) (i.e., \( \mathcal{S}_x = \{ x_2 \in \mathbb{R}^2 : -3 \leq x_1 \leq 3, -3 \leq x_2 \leq 3 \} \)) and \( s_2 = s_3 = 3/8 \). It is important to note that this choice for \( \mathcal{S}_x \) is a trial-and-error refinement of the bound given by Theorem 1. Indeed, the maximum and ultimate bounds (12) and (13) are very conservative estimates of the closed-loop performance, and while they are a good starting point, it is often possible to tighten the bounds by careful experimentation.

Again using trial and error, we set \( W_1 = 0.05 \) and \( W_2 = 0.2 \), and we choose the adaptation constants \( \eta_1 = 0.19, \gamma_1 = 0.1, \eta_2 = 0.2 \) and \( \gamma_2 = 0.12 \), which satisfy the conditions for the adaptation law (10). We initialize \( \theta_1(0) = 0 \) and \( \theta_2(0) = 0 \). Figure 1 contains the regulation results. The top plot shows the evolution of the states, where it is seen how they converge to a small neighborhood of the origin. The bottom plot shows the control action of the adaptive regulator.

5 Conclusions

In this paper we have presented a direct adaptive regulation scheme for a class of strict feedback discrete time nonlinear systems with arbitrary nonlinearities. Function approximation and on-line update of the approximators are used. The regulation theorem guarantees convergence of the state norm to a neighborhood of the origin, and the size of the neighborhood depends exclusively on the chosen approximator structure and ideal approximation errors. The nonlinearities are only assumed to be piecewise continuous, and no growth conditions are imposed. The stability is semiglobal, and the need for bounding terms is obviated by finding an explicit bound for the state from the stability analysis. A simple second order example illustrates the method.

References