Stability of Delaunay-type Structures for Manifolds

[Extended Abstract]

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ABSTRACT

We introduce a parametrized notion of genericity for Delaunay triangulations which, in particular, implies that the Delaunay simplices of \( \delta \)-generic point sets are thick. Equipped with this notion, we study the stability of Delaunay triangulations under perturbations of the metric and of the vertex positions. We then show that, for any sufficiently regular submanifold of Euclidean space, and appropriate \( \epsilon \) and \( \delta \), any sample set which meets a localized \( \delta \)-generic \( \epsilon \)-dense sampling criteria yields a manifold intrinsic Delaunay complex which is equal to the restricted Delaunay complex.

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1. INTRODUCTION

One of the central properties of Delaunay complexes, which was demonstrated when they were introduced [Del34], is that under very mild assumptions they triangulate Euclidean space. This paper addresses issues that arise when the Delaunay paradigm is employed for triangulating non-Euclidean manifolds whose dimension may exceed two.

For a submanifold of Euclidean space, the restricted Delaunay complex [ES97], which is defined by the ambient metric restricted to the submanifold, was employed by Cheng et al. [CDR05] as the basis for a triangulation. However, it was found that sampling density alone was insufficient to ensure a triangulation, and manipulations of the complex were employed.

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In an earlier work, Leibon and Letscher [LL00] announced sampling density conditions which would ensure that the Delaunay complex defined by the intrinsic metric of the manifold was a triangulation. In fact, the stated result is incorrect [BDG12b]: sampling density alone is insufficient to guarantee an intrinsic Delaunay triangulation. Topological defects can arise when the vertices lie too close to a degenerate or “quasi-cospherical” configuration.

We address this problem with the introduction of a parametrized notion of genericity for Delaunay complexes. Our interest in the intrinsic Delaunay complex stems from its close relationship with other Delaunay-like structures that have been proposed in the context of non-homogeneous metrics. For example, anisotropic Voronoi diagrams [LS03] and anisotropic Delaunay triangulations emerge as natural structures when we want to mesh a domain of \( \mathbb{R}^m \) while respecting a given metric tensor field.

This paper builds over preliminary results on anisotropic Delaunay meshes [BWy11] and manifold reconstruction using the tangential Delaunay complex [BG11]. The central idea in both cases is to define Euclidean Delaunay triangulations locally and to glue these local triangulations together by removing inconsistencies between them. We view the inconsistencies as arising from instability in the Delaunay triangulations, and provide explicit bounds on the stability with respect to the genericity parameter.

The stability of Delaunay triangulations has not previously been studied in this way. Related work can be found in the context of kinetic data structures [AGG10] or in the context of robust computation [BS04], and in particular, the concept of protection we introduce in Section 3 is embodied in the guarded insphere predicate which has been employed in a 2D Delaunay triangulation algorithm [FKMS05].

After presenting background material in Section 2, we introduce the concept of \( \delta \)-generic point sets for Euclidean Delaunay triangulations in Section 3. We show that Delaunay simplices of \( \delta \)-generic point sets are thick; they satisfy a quality bound. Then in Section 4 we quantify how \( \delta \) leads to robustness in the Delaunay triangulation when either the points or the metric are perturbed. The primary challenge is bounding the displacement of simplex circumcentres. Finally, we use these results in Section 5 to demonstrate conditions under which the intrinsic Delaunay complex and the restricted Delaunay complex coincide and are manifold. In this extended abstract, we have omitted proofs for some of the lemmas. In all cases, the missing proofs are routine exercises, and they can be found in the full version of the paper [BDG12b].
2. BACKGROUND

Within the context of the standard m-dimensional Euclidean space $\mathbb{R}^m$, when distances are determined by the standard norm, $\| \cdot \|$, we use the following conventions. The distance between a point $p$ and a set $X \subset \mathbb{R}^m$, is the infimum of the distances between $p$ and the points of $X$, and is denoted $d_{\infty}(p, X)$. We refer to the distance between two points $a$ and $b$ as $\| b - a \|$ or $d_{\infty}(a, b)$ as convenient. A ball $B_\rho(c; r) = \{ x \mid \| x - c \| < r \}$ is open, and $\overline{B}_\rho(c; r)$ is its topological closure. Generally, we denote the topological closure of a set $X$ by $\overline{X}$, the interior by $Int X$, and the boundary by $\partial X$. The convex hull is denoted $ch(X)$, and the affine hull is $aff(X)$.

We will make use of other metrics besides the Euclidean one. A generic metric is denoted $d$, and the associated open and closed balls are $B(c; r)$, and $\overline{B}(c; r)$. If a specific metric is intended, it will be indicated by a subscript, for example in Section 5 we introduce $d_M$, the intrinsic metric on a manifold $M$, which has associated balls $B_M(c; r)$.

If $A$ is a $k \times j$ matrix, we denote its $i^{th}$ singular value by $s_i(A)$. We use the operator norm $\| A \| = s_1(A)$, and employ the following standard observation:

**Lemma 1.** If $\eta > 0$ is an upper bound on the norms of the columns of $A$, then $\| A \| \leq \sqrt{j}\eta$.

We will also be interested in obtaining a lower bound on the smallest singular value, for which the following observation is useful:

**Lemma 2.** If $A$ is a $k \times j$ matrix of rank $j \leq k$, then the pseudo inverse $A^+ = (A^T A)^{-1} A^T$ is the unique left inverse of $A$ whose kernel is the orthogonal complement of the column space of $A$. Furthermore, $s_i(A^+) = s_j-s_{j+1}(A)^{-1}$.

If $U$ and $V$ are vector subspaces of $\mathbb{R}^m$, with $\dim U \leq \dim V$, the angle between them is defined by

$$\cos \angle(U, V) = \inf_{u \in U} \sup_{v \in V} \frac{u^T v}{\| u \| \| v \|}.$$  

This is the largest principal angle between $U$ and $V$. The angle between affine subspaces $K$ and $H$ is defined as the angle between the corresponding parallel vector subspaces.

2.1 Sampling parameters and perturbations

The structures of interest will be built from a finite set $P \subset \mathbb{R}^m$, which we consider to be a set of sample points. If $D \subset \mathbb{R}^m$ is a bounded set, then $P$ is an $\epsilon$-sample set for $D$ if $d_m(x, P) < \epsilon$ for all $x \in D$. We say that $\epsilon$ is a sampling radius for $D$ satisfied by $P$. If no domain $D$ is specified, we say $P$ is an $\epsilon$-sample set if $d_m(p, q) > \epsilon$ for all $p, q \in P$. We usually assume that the sparsity of a $\epsilon$-sample set is proportional to $\epsilon$, thus: $\lambda = \rho \epsilon$.

We consider a perturbation of the points $P \subset \mathbb{R}^m$ given by a function $\zeta: P \rightarrow \mathbb{R}^m$. If $\zeta$ is such that $d_m(p, \zeta(p)) \leq \rho$, we say that $\zeta$ is a $\rho$-perturbation. As a notational convenience, we frequently define $P = \zeta(P)$, and let $\tilde{p}$ represent $\zeta(p) \in P$. We will only be considering $\rho$-perturbations where $\rho$ is less than half the sparsity of $P$, so $\zeta: P \rightarrow P$ is a bijection.

Points in $P$ which are not on the boundary of $ch(P)$ are interior points of $P$.

**Lemma 3.** Suppose $P$ is an $\epsilon$-sample set, and $\zeta: P \rightarrow \tilde{P}$ is a $\rho$-perturbation with $2\rho < \epsilon$. If point $p \in P$ satisfies $d_{\infty}(p, \partial ch(P)) \geq 3\epsilon$, then $\tilde{p} = \zeta(p)$ is an interior point of $\tilde{P}$.

**Proof.** Let $S = \partial B$ be the bounding sphere for $B = B_\rho(p, 3\epsilon/2)$. Then $d_m(p, S) \geq \epsilon$ and for any $x \in S$, $d_m(x, \partial ch(P)) \geq \epsilon$. Thus the sampling assumption ensures that for every point $x \in S$, there is a point $q \in P$ with $p \neq q$ and $d_m(x, q) < \epsilon$. It follows that $d_m(x, \xi(q)) < 3\epsilon/2$, and thus that $\tilde{p}$ is not the closest point in $\tilde{P}$ for any point on $S$.

Thus $\tilde{p}$ cannot belong to $\partial ch(P)$. Indeed if $\tilde{p} \in \partial ch(P)$, then take a unit vector $v$ in an outward direction orthogonal to a closed half-space supporting $ch(P)$ at $\tilde{p}$. The ray from $\tilde{p}$ defined by $v$ must intersect $S$ at some point $y$, and $\tilde{p}$ would be the closest point in $\tilde{P}$ to $y$, a contradiction. 

2.2 Simplices

Given a set of $j + 1$ points $\{p_0, \ldots, p_j\} \subset \mathbb{R}^m$, a (geometric) $j$-simplex $\sigma = [p_0, \ldots, p_j]$ is defined by the convex hull: $\sigma = ch([p_0, \ldots, p_j])$. The points $p_i$ are the vertices of $\sigma$. Any subset $\{p_0, \ldots, p_j\}$ of $\{p_0, \ldots, p_j\}$ defines a $k$-simplex $\tau$ which we call a face of $\sigma$. We write $\tau \leq \sigma$ if $\tau$ is a face of $\sigma$, and $\sigma \leq \tau$ if $\tau$ is a proper face of $\sigma$, i.e., if the vertices of $\tau$ are a proper subset of the vertices of $\sigma$.

The boundary of $\sigma$, is the union of its proper faces: $\partial\sigma = \bigcup_{\tau \subset \sigma} \tau$. In general this is distinct from the topological boundary defined above, but we denote it with the same symbol. The interior of $\sigma$ is $int\sigma = \sigma \setminus \partial\sigma$. Again this is generally different from the topological interior. Other geometric properties of $\sigma$ include its diameter (its longest edge), $\Delta(\sigma)$, and its shortest edge, $L(\sigma)$.

For any vertex $p \in \sigma$, the face opposite $p$ is the face determined by the other vertices of $\sigma$, and is denoted $\sigma_p$. If $\tau$ is a $j$-simplex, and $p$ is not a vertex of $\tau$, we may construct a $(j + 1)$-simplex $\sigma = p \cup \tau$, called the join of $p$ and $\tau$. It is the simplex defined by $p$ and the vertices of $\tau$, i.e., $\tau = \sigma_p$.

Our definition of a simplex has made an important departure from standard convention: we do not demand that the vertices of a simplex be affinely independent. A $j$-simplex $\sigma$ is a degenerate simplex if $dim(aff(\sigma)) < j$. If we wish to emphasise that a simplex is a $j$-simplex, we write $j$ as a superscript: $\sigma^j$; but this always refers to the combinatorial dimension of the simplex.

If $\sigma$ is non-degenerate, then it has a circumcentre, $C(\sigma)$, which is the centre of the smallest circumscribing ball for $\sigma$. The radius of this ball is the circumradius of $\sigma$, denoted $R(\sigma)$. In the context of the Euclidean Delaunay complexes we will work with, the degenerate simplices we may encounter also have these properties. The ratio of the circumradius to the shortest edge is denoted $\Phi(\sigma) = R(\sigma)/L(\sigma)$. We will make use of the affine space $N(\sigma)$ composed of the centres of the balls that circumscribe $\sigma$. This space is orthogonal to $aff(\sigma)$ and intersects it at the circumcentre of $\sigma$. Its dimension is $m - dim(aff(\sigma))$.

The altitude of $p$ in $\sigma$ is $D(p, \sigma) = d_m(p, aff(\sigma))$. A poorly-shaped simplex can be characterized by the existence of a relatively small altitude. The thickness of a $j$-simplex $\sigma$ is the dimensionless quantity given by $T(\sigma) = 1$ if $j = 0$, and $T(\sigma) = min_{p \in \sigma} \Phi(\sigma)$ otherwise. We say that $\sigma$ is $\psilon_T$-thick, if $T(\sigma) \geq \epsilon_T$. If $\sigma$ is $\epsilon_T$-thick, then so are all of its faces.

Indeed if $\tau \leq \sigma$, then the smallest altitude in $\tau$ cannot be smaller than that of $\sigma$, and also $\Delta(\tau) \leq \Delta(\sigma)$.

Other parameters such as the volume [Whi57], or the radius of the largest contained ball centred at the barycentre [Mun68], can be used to characterize simplex quality. We find a direct bound on the altitudes to be more convenient, in part due to the following consequence of Lemma 2:
Lemma 4 (Thickness and singular value). Let \( \sigma = [p_0, \ldots, p_j] \) be a non-degenerate \( j \)-simplex in \( \mathbb{R}^m \), with \( j > 0 \), and let \( P \) be the \( m \times j \) matrix whose \( i \)-th column is \( p_i - p_0 \). Then \( s_i(P) \geq \sqrt{T(\sigma)}(\Delta(\sigma)) \).

Proof. Let \( w_i^T \) be the \( i \)-th row of \( P^T \). Then \( w_i \) belongs to the column space of \( P \), and it is orthogonal to all \( (p_i - p_0) \) for \( i \neq i \). Let \( u_i = w_i/\|w_i\| \). It follows from the definition that \( u_i^T(p_i - p_0) = D(p_i, \sigma) \). Thus \( w_i = D(p_i, \sigma)^{-1}u_i \). Since \( s_i(A^T) = s_i(A) \) for any matrix \( A \), we have \( s_i(P^T) \leq \sqrt{T(\sigma)(\Delta(\tau_{\sigma}))} \), by Lemma 1. Thus Lemma 2 yields \( s_i(P) \geq \frac{1}{\sqrt{m}} \min_{i \leq j \leq \sigma} \|D(p_i, \sigma)\|^{-1}\). In other words, \( \Delta(\tau_{\sigma})^{-1} \) provides a convenient upper bound on the condition number of \( P \).

Whitney [Whi57, p. 127] proved that the affine hull of a thick simplex makes a small angle with any hyperplane which lies near all the vertices of the simplex. Employing Lemma 4 in the proof of Whitney’s Lemma yields a simpler proof [BDGO11, Lemma 2.6] and a sharper result:

**Lemma 5** (Whitney angle bound). Suppose \( \sigma \) is a \( j \)-simplex whose vertices all lie within a distance \( \eta \) from a hyperplane, \( H \subset \mathbb{R}^m \). Then \( \sin(\angle(\sigma, H)) \leq \frac{2\eta}{\Delta(\sigma)} \).

### 2.3 Complexes

The Delaunay complexes we study are abstract simplicial complexes, but their simplices carry a canonical geometry induced from the inclusion map \( \iota : P \hookrightarrow \mathbb{R}^m \). (We assume \( \iota \) is injective on \( P \), and so do not distinguish between \( P \) and \( \iota(P) \).

To each abstract simplex \( \sigma \in K \), we have an associated geometric simplex \( \text{ch}(\iota(\sigma)) \), and normally when we write \( \sigma \in K \), we are referring to this geometric object. Occasionally, when it is convenient to emphasise a distinction, we will write \( \iota(\sigma) \) instead of \( \sigma \). Thus we view such a \( K \) as a set of simplices in \( \mathbb{R}^m \), and we refer to it as a complex, but it is not generally a (geometric) simplicial complex.

The *carrier* of an abstract complex \( K \) is the underlying topological space \( |K| \), associated with a geometric realization of \( K \). For our complexes, the inclusion map \( \iota \) induces a continuous map \( \iota : |K| \to \mathbb{R}^m \), defined by barycentric interpolation on each simplex. If this map is injective, we say that \( K \) is *embedded*. In this case \( \iota \) also defines a geometric realization of \( K \), and we may identify the carrier of \( K \) with the image of \( \iota \).

A subset \( K' \subset K \) is a *subcomplex* of \( K \) if it is also a complex. The *star* of a subcomplex \( K' \subset K \) is the subcomplex generated by the simplices incident to \( K' \). I.e., it is all the simplices that share a face with a simplex of \( K' \), plus all the faces of such simplices. This is a departure from a common usage of this same term in the topology literature. The star of \( K' \) is denoted \( \text{star}(K') \) when there is no risk of ambiguity, otherwise we also specify the parent complex, as in \( \text{star}(K'; K) \).

A *triangulation* of \( P \subset \mathbb{R}^m \) is an embedded complex \( K \) with vertices \( P \) such that \( |K| = \text{ch}(P) \).

**Definition 6** (Triangulation at a point). A complex \( K \) is a triangulation at \( p \in \mathbb{R}^m \) if:

- \( p \) is a vertex of \( K \).
- \( \text{star}(p) \) is embedded.
- \( p \) lies in \( \text{int}(|\text{star}(p)|) \).
- For all \( \tau \in K', \sigma \in \text{star}(p) \), if \( \text{int} \tau \cap \sigma \neq \emptyset \), then \( \tau \in \text{star}(p) \).

A complex \( K \) is a \( j \)-manifold complex if the star of every vertex is isomorphic to the star of a triangulation of \( \mathbb{R}^j \).

If \( \sigma \) is a simplex with vertices in \( P \), then any map \( \varsigma: P \to P \subset \mathbb{R}^m \) defines a simplex \( \varsigma(\sigma) \) whose vertices in \( P \) are the images of vertices of \( \sigma \). If \( K \) is a complex on \( P \), and \( K \) is a complex on \( P \), then \( \varsigma \) induces a simplicial map \( \varsigma(K) \to K \). If \( \varsigma(\sigma) \in K \) for every \( \sigma \in K \). We denote this map by the same symbol, \( \varsigma \). We are interested in the case when \( \varsigma \) is an isomorphism, which means it establishes a bijection between \( K \) and \( K \). We then say that \( K \) and \( K \) are isomorphic, and write \( K \cong K \) if \( K \cong K \) if we wish to emphasise that the correspondence is given by \( \varsigma \). We observe [BDG12b] the following local version of a standard result:

**Lemma 7**. Suppose \( K \) is a complex with vertices \( P \subset \mathbb{R}^m \), and \( K \) a complex with vertices \( P \subset \mathbb{R}^m \). Suppose also that \( K \) is a triangulation at \( p \in P \), and that \( \varsigma : P \to P \) induces an injective simplicial map \( \varsigma(\sigma) \to \text{star}(\varsigma(\sigma)) \). If \( K \) is a triangulation at \( \varsigma(p) \), then \( \varsigma(\text{star}(p)) = \text{star}(\varsigma(p)) \).

### 3. PARAMETERIZED GENERICITY

In this section we examine the Delaunay complex of \( P \subset \mathbb{R}^m \), taking the view that poorly-shaped simplices arise from almost degenerate configurations of points. We introduce the concept of a protected Delaunay ball, which leads to a parameterized definition of genericity. We then show that a lower bound on the protection of the maximal simplices yields a lower bound on their thickness.

#### 3.1 The Delaunay complex

An *empty ball* is one that contains no point from \( P \).

**Definition 8** (Delaunay complex). A Delaunay ball is a maximal empty ball. Specifically, \( B = B_{\infty}(x; r) \) is a Delaunay ball if any empty ball centred at \( x \) is contained in \( B \). A simplex \( \sigma \) is a Delaunay simplex, if there exists some Delaunay ball \( B \) such that the vertices of \( \sigma \) belong to \( \partial B \cap P \). The Delaunay complex is the set of Delaunay simplices, and is denoted \( \text{Del}(P) \).

The Delaunay complex has the combinatorial structure of an abstract simplicial complex, but \( \text{Del}(P) \) is embedded only when \( P \) satisfies appropriate genericity requirements, as discussed in Section 3.2. Otherwise, \( \text{Del}(P) \) contains degenerate simplices. We make here some observations that are not dependent on assumptions of genericity.

The union of the Delaunay simplices is \( \text{ch}(P) \). A simplex \( \sigma \in \text{Del}(P) \) is a *boundary simplex* if all its vertices lie on \( \partial \text{ch}(P) \). If none of the vertices of \( \sigma \) lie on \( \partial \text{ch}(P) \), then it is an *interior simplex*. We observe [BDG12b]:

**Lemma 9** (Maximal simplices). If \( \text{aff}(P) = \mathbb{R}^m \), then every Delaunay \( j \)-simplex, \( \sigma \), is a face of a Delaunay simplex \( \sigma' \) with \( \text{dim} \text{aff}(\sigma') = m \). In particular, if \( j \leq m \), then \( \sigma \) is a face of a Delaunay \( m \)-simplex. If \( \sigma \) is not a boundary simplex, and \( \text{dim} \text{aff}(\sigma) < m \), then there are at least two Delaunay \( (j + 1) \)-simplices that have \( \sigma \) as a face.

Lemma 9 gives rise to the following observation, which plays an important role in Section 3.3, where we argue that
Lemma 10 (Separation). If \( \tau \in \text{Del}(P) \) is a \( j \)-simplex that is not a boundary simplex, and \( q \in P \setminus \tau \), then there is a Delaunay \( m \)-simplex \( \sigma^m \) which has \( \tau \) as a face, but does not include \( q \).

Proof. Assume \( j < m \), for otherwise there is nothing to prove. If \( \sigma = q \times \tau \) is not Delaunay, the assertion follows from the first part of Lemma 9. Assume \( \sigma \) is Delaunay and let \( \sigma^m \) be a Delaunay \( m \)-simplex that has \( \sigma \) as a face. Thus \( \sigma^m = q \times \sigma^{m-1} \) for some Delaunay \( (m-1) \)-simplex, \( \sigma^{m-1} \). Since \( \tau \leq \sigma^{m-1} \) does not belong to the boundary of \( \text{ch}(P) \), neither does \( \sigma^{m-1} \), so by the second part of Lemma 9, there is another Delaunay \( m \)-simplex \( \sigma^m \) that has \( \sigma^{m-1} \) (and therefore \( \tau \)) as a face. Since \( \sigma^m \) is distinct from \( \hat{\sigma}^m \), it does not have \( q \) as a vertex.

3.1.1 The Delaunay complex in other metrics

We will also consider the Delaunay complex defined with respect to a metric \( d \) on \( R^m \) which differs from the Euclidean one. Specifically, if \( P \subset U \subset \mathbb{R}^m \) and \( d : U \times U \to \mathbb{R} \) is a metric, then we define the Delaunay complex \( \text{Del}_d(P) \) with respect to the metric \( d \).

The definitions are exactly analogous to the Euclidean case: A Delaunay ball is a maximal empty ball \( B(x; r) \) in the metric \( d \). The resulting Delaunay complex \( \text{Del}_d(P) \) consists of all the simplices which are circumscribed by a Delaunay ball with respect to the metric \( d \). The simplices of \( \text{Del}_d(P) \) are, possibly degenerate, geometric simplices in \( \mathbb{R}^m \). As for \( \text{Del}(P) \), \( \text{Del}_d(P) \) has the combinatorial structure of an abstract simplicial complex, but unlike \( \text{Del}(P) \), \( \text{Del}_d(P) \) may fail to be embedded even when there are no degenerate simplices.

3.1.2 Deep interior simplices

In order to focus on the local properties of Delaunay triangulations, we wish to identify a subcomplex of the interior simplices of \( \text{Del}(P) \) consisting of those simplices whose neighbourhood simplices are all interior simplices with small circumradius. An interior simplex near the boundary of \( \text{ch}(P) \) does not necessarily have its circumradius constrained by the sampling radius. However, we have the following:

Lemma 11. If \( P \) is an \( \epsilon \)-sample set, and \( \sigma \in \text{Del}(P) \) has a vertex \( p \) such that \( d_{\text{ch}}(p, \partial \text{ch}(P)) \geq 2\epsilon \), then \( R(\sigma) < \epsilon \) and \( \sigma \) is an interior simplex.

Proof. Let \( B_{\text{ch}}(p, \epsilon) \) be a Delaunay ball for \( \sigma \). We will show \( r < \epsilon \). Suppose to the contrary. Let \( x \) be the point on \([c, p]\) such that \( d_{\text{ch}}(p, x) = \epsilon \). Then \( p \) is the closest point in \( P \) to \( x \), and so the sampling criteria imply that \( d_{\text{ch}}(x, \partial \text{ch}(P)) \leq \epsilon \). But then \( d_{\text{ch}}(p, \partial \text{ch}(P)) \leq d_{\text{ch}}(x, \partial \text{ch}(P)) + d_{\text{ch}}(p, x) < 2\epsilon \), contradicting the hypothesis on \( p \).

Thus \( r < \epsilon \), and it follows that \( \sigma \) is an interior simplex because if \( q \in \sigma \), then \( d_{\text{ch}}(q, p) \leq 2r < d_{\text{ch}}(p, \partial \text{ch}(P)) \).

Definition 12. Suppose \( P \subset \mathbb{R}^m \) is an \( \epsilon \)-sample set. The subset \( P_1 \subset P \) consisting of all \( p \in P \) with \( d_{\text{ch}}(p, \partial \text{ch}(P)) \geq 4\epsilon \) is the set of safe interior points. The deep interior simplices are the simplices in \( \text{star}(P_1 \setminus \text{Del}(P)) \).

By Lemma 11, all the neighbour simplices of deep interior simplices will be interior simplices with small circumradius.

3.2 Protection

A Delaunay simplex \( \sigma \) is \( \delta \)-protected if it has a Delaunay ball \( B \) such that \( d_{\text{ch}}(q, \partial B) > \delta \) for all \( q \in P \setminus \sigma \). We say that \( B \) is a \( \delta \)-protected Delaunay ball for \( \sigma \). If \( \tau < \sigma \), then \( B \) is also a Delaunay ball for \( \tau \), but it cannot be a \( \delta \)-protected Delaunay ball for \( \tau \). We say that \( \sigma \) is protected to mean that it is \( \delta \)-protected for some unspecified \( \delta > 0 \).

Definition 13 (\( \delta \)-generic). A finite set of points \( P \subset \mathbb{R}^m \) is \( \delta \)-generic if all the Delaunay \( m \)-simplices are \( \delta \)-protected. The set \( P \) is simply generic if it is \( \delta \)-generic for some unspecified \( \delta > 0 \).

In his seminal work, Delaunay [Del34] demonstrated that if there is no empty ball with \( m + 2 \) points from \( P \) on its boundary, then \( \text{Del}(P) \) is realized as a simplicial complex in \( \mathbb{R}^m \). In other words it is a triangulation, the Delaunay triangulation. If \( P \) is generic according to Definition 13, then Delaunay’s criterion will be met. This is obvious if there are no degenerate \( m \)-simplices, and Definition 13 ensures that a degenerate \( m \)-simplex cannot exist in \( \text{Del}(P) \): If \( \sigma^m \) is degenerate, then by Lemma 9, there is a simplex \( \sigma \) with \( \text{aff}(\sigma) = \mathbb{R}^m \), and \( \sigma^m < \sigma \). An affinely independent set of \( m + 1 \) vertices from \( \sigma \) defines a non degenerate \( m \)-simplex \( \tilde{\sigma}^m \), and since its unique circumball is also a Delaunay ball for \( \sigma \), it cannot be protected, a contradiction.

In particular, if \( P \) is generic if and only if there are no Delaunay simplices with dimension higher than \( m \). We can say more. There are no degenerate Delaunay simplices. This can be inferred directly from Delaunay’s result [Del34], but is also easily established directly. In Section 3.3 we will quantify this observation.

Delaunay avoided boundary complications by assuming a periodic point set, but we will be working in a local setting where we cannot assume that our conditions apply globally. This is not a problem because Delaunay’s argument applies locally [BDG12b]:

Lemma 14 (Local Delaunay). If \( p \in P \) is an interior point, and the Delaunay \( m \)-simplices incident to \( p \) are protected, then \( \text{Del}(P) \) is a triangulation at \( p \).

For technical reasons it is inconvenient to demand that all the Delaunay \( m \)-simplices be \( \delta \)-protected. For our purposes it will be sufficient to have protection on the deep interior \( m \)-simplices and their neighbouring \( m \)-simplices.

Definition 15 (\( \delta \)-generic for \( P_1 \)). The set \( P \subset \mathbb{R}^m \) is \( \delta \)-generic for \( P_1 \) if all the \( m \)-simplices in \( \text{star}(\text{star}(P_1)) \) are \( \delta \)-protected.

If the following we will make assertions about deep interior simplices based on the assumption that \( P \) is \( \delta \)-generic for \( P_1 \). These assertions will be equally valid if \( P_1 \) is taken to be some subset of the safe interior points, provided “deep interior simplices” is restricted accordingly.

3.3 Thickness from protection

Our goal here is to demonstrate that the deep interior simplices on a \( \delta \)-generic point set are \( T_0 \)-thick. If \( \delta = \nu \epsilon \), for some constant \( \nu_0 \), then we obtain a constant \( T_0 \) which depends only on \( \nu_0 \). The key observation is that together with Lemma 10, protection imposes constraints on all the Delaunay simplices; they cannot be too close to being degenerate. In the particular case that \( j = 0 \), Lemma 10 immediately implies that any Delaunay edge has length greater than \( \delta \):
Lemma 16 (Sparsity from protection). If \( P \) is \( \delta \)-generic for \( P_1 \), then \( P_1 \) is \( \delta \)-sparse.

Figure 1: Diagram for Lemma 17. (a) When \( H \) separates \( q \) and \( c' \) then \( d_{\delta}(q, q') > \delta \). (b) Otherwise, a lower bound on the distance between \( q \) and its projection \( q' \) on \( H \) is obtained by an upper bound on the angle \( \angle qab \).

Lemma 17. Let \( P \) be \( \lambda \)-sparse with sampling radius \( \epsilon \) and \( \lambda \leq \epsilon \). Suppose \( B \) is a \( \delta \)-protected Delaunay ball for \( \sigma \), and \( \tau \) is a deep interior simplex that is a (not necessarily proper) face of \( \sigma \). If \( q = q + \tau \), is a Delaunay simplex that is not a face of \( \sigma \), and \( H = \text{aff}(\partial B \cap \partial B') \), where \( B' \) is a Delaunay ball for \( \sigma \), then \( d_{\delta}(q, H) > \frac{\sqrt{3}\lambda}{4\epsilon}(\lambda + \delta) \). In particular, if \( P \) is \( \delta \)-generic for \( P_1 \), then \( D(q, \sigma) > \frac{\sqrt{3}\lambda}{4\epsilon} \).

Proof. Let \( B = B_{\delta}(c, \tau) \) be the \( \delta \)-protected Delaunay ball for \( \sigma \), and \( B' = B_{\delta}(c', \tau') \) the Delaunay ball for \( \sigma \), so \( \tau' < \epsilon \). Our geometry will be performed in the plane, \( Q \), defined by \( c, c' \), and \( q \). This plane is orthogonal to the \((m-1)\)-flat \( H \), and it follows that the distance \( d_{\delta}(q, H) \) is realized by a segment in the plane \( Q \); the projection, \( q' \), of \( q \) onto \( H \) lies in \( Q \), and \( d_{\delta}(q, H) = d_{\delta}(q, q') \).

If \( H \) separates \( q \) from \( c' \), then \( \partial B \) separates \( q \) from \( q' \), and \( d_{\delta}(q, q') > d_{\delta}(q, \partial B) > \delta \), since \( B \) is \( \delta \)-protected (Figure 1a). The lemma then follows since \( \lambda \) and \( \delta \) are both less than \( \epsilon \). Thus assume that \( q \) and \( c' \) lie on the same side of \( H \), as shown in Figure 1(b). Let \( S = Q \cap \partial B \), and \( S' = Q \cap \partial B' \), and let \( a \) and \( b \) be the points of intersection \( S \cap S' \). Thus \( H \cap Q \) is the line through \( a \) and \( b \).

We will bound \( d_{\delta}(q, q') \) by finding an upper bound on the angle \( \gamma = \angle qab \). This is the same as the standard calculation for upper-bounding the angles in a triangle with bounded circumradius to shortest edge ratio. Without loss of generality, we may assume that \( \gamma \geq \angle qba \), and we will assume that \( \gamma \geq \pi/2 \) since otherwise \( d_{\delta}(q, q') > \delta \) and the lemma is again trivially satisfied.

Since \( d_{\delta}(a, q) > \delta \), we have \( d_{\delta}(q, q') = d_{\delta}(a, q) \sin \gamma > \delta \sin \gamma \). Also note that \( d(a, b) \geq 2R(\tau) \geq L(\tau) \geq \lambda \). Let \( \alpha = \angle qac \). Then \( \cos \alpha = \frac{d_{\delta}(a, q)}{2R(\tau)} \geq \frac{\lambda}{2\epsilon} \). Similarly, with \( \beta = \angle ca'b \), we have \( \cos \beta \geq \frac{\lambda}{2\epsilon} \). Thus since \( \gamma = \alpha + \beta \geq \pi/2 \), we have

\[
d_{\delta}(q, q') > \delta \sin \left( \arccos \frac{\delta}{2\epsilon} + \arccos \frac{\lambda}{2\epsilon} \right) \\
\geq \delta \left( \frac{\lambda}{2\epsilon} \sin \left( \arccos \frac{\delta}{2\epsilon} \right) + \frac{\delta}{2\epsilon} \sin \left( \arccos \frac{\lambda}{2\epsilon} \right) \right) \\
\geq \frac{\sqrt{3}\lambda}{4\epsilon}(\lambda + \delta),
\]

where the last inequality follows from \( \lambda \leq \epsilon \) and \( \delta \leq \epsilon \).

Since \( \text{aff}(\tau) \subset H \), it follows that \( D(q, \sigma) \geq d_{\delta}(q, H) \), and if \( P \) is \( \delta \)-generic for \( P_1 \), then \( \lambda \geq \delta \), and Lemma 10 ensures that there is a \( \delta \)-protected \( \sigma \) that contains \( \tau \) but not \( q \).

We thus obtain a bound on the thickness of the deep interior simplices when \( P \) is \( \delta \)-generic. Since Lemma 17 yields a lower bound of \( \frac{\sqrt{3}\lambda}{4\epsilon} \) on the altitudes of the deep interior simplices, and since \( \Delta(\sigma) \leq 2\epsilon \), we have that \( \Upsilon(\sigma) \geq \frac{\sqrt{3}\lambda}{4\epsilon} \) for all deep interior \( \sigma \). If \( \delta = \nu_0 \epsilon \), we obtain a constant thickness bound.

Theorem 18 (Thickness from protection). If \( P \subset \mathbb{R}^m \) is \( \delta \)-generic for \( P_1 \) with \( \delta = \nu_0 \epsilon \), where \( \epsilon \) is a sampling radius for \( P \), then the deep interior simplices are \( \Upsilon_0 \)-thick, with

\[
\Upsilon_0 = \frac{\sqrt{3}\lambda^2}{4}.
\]

4. DELAUNAY STABILITY

We find upper bounds on the magnitude of a perturbation for which a protected Delaunay ball remains a Delaunay ball. We consider both perturbations of the sample points in Euclidean space, and perturbations of the metric itself. The primary technical challenge is bounding the effect of a perturbation on the circumcentre of an \( m \)-simplex. We then find the relationship between the perturbation parameter \( \rho \) and the protection parameter \( \delta \) which ensures that a \( \delta \)-protected Delaunay simplex will remain a Delaunay simplex.

4.1 Perturbations and circumcentres

As expected, a bound on the displacement of the circumcentre requires a bound on the thickness of the simplex.

4.1.1 Almost circumcentres

If \( \sigma \) is thick, a point whose distances to the vertices of \( \sigma \) are all almost the same, will lie close to \( N(\sigma) \).

Lemma 19. If \( \sigma = \{p_0, ..., p_k\} \subset \mathbb{R}^m \) is a non-degenerate \( k \)-simplex, and \( x \in \mathbb{R}^m \) is such that

\[
||p_i - x||^2 - ||p_j - x||^2 \leq \beta^2 \quad \text{for all } i, j \in \{0, ..., k\},
\]

then there is a \( c \in N(\sigma) \) such that \( |c - x| \leq \eta \), where

\[
\eta = \frac{\beta^2}{2\Upsilon(\sigma)\Delta(\sigma)}.
\]

In particular, if \( \sigma \) is an \( m \)-simplex then \( x \in \overline{B}_{\delta}(C(\sigma); \eta) \).

If the inequalities in Equations (1) then the conclusions may also be stated with strict inequalities.

Proof. First suppose \( k = m \). The circumcentre of \( \sigma \) is given by the linear equations \( ||C(\sigma) - p_i||^2 = ||C(\sigma) - p_j||^2 \), or

\[
(p_i - p_0)^T C(\sigma) = \frac{1}{2} ||p_i||^2 - ||p_0||^2.
\]
Letting $b$ be the vector whose $i$th component is defined by the right hand side of the equation, and letting $P$ be the $m \times m$ matrix, whose $i$th column is $(p_i - p_0)$, we write the equations in matrix form:

$$P^T C(\sigma) = b.$$  \hspace{1cm} (2)

Without loss of generality, assume $p_0$ is the vertex that minimizes the distance to $x$. Then, defining $x_a$ to be the vector whose $i$th component is $\frac{1}{2}(\|p_i - x\|^2 - \|p_0 - x\|^2)$, we have $\|x_i - x\|^2 = \|p_0 - x\|^2 + 2(x_a)_i$, and we find

$$P^T x = b - x_a.$$  \hspace{1cm} (3)

From Equations (2) and (3) we have

$$\|C(\sigma) - x\| = \|P^{-T} x_a\| \leq \|P^{-1}\| \|x_a\|.$$  

From Equation (1), it follows that $\|x_a\| \leq \sqrt{m} \rho^2$, and from Lemmas 2 and 4 we have $\|P^{-1}\| \leq (\sqrt{m} Y(\sigma) \Delta(\sigma))^{-1}$, and thus the result holds for full dimensional simplices.

If $\sigma$ is a $k$-simplex with $k \leq m$, then we consider $\hat{x}$, the orthogonal projection of $x$ into aff($\sigma$). We observe that $\hat{x}$ also must satisfy Equation (1), and we conclude from the above argument that $\|C(\sigma) - \hat{x}\| \leq \eta$. Then letting $c = C(\sigma) + (x - \hat{x})$ we have that $c \in N(\sigma)$ and $\|c - x\| \leq \eta$.  

With a bound on the distance from $x$ to the vertices of $\sigma$, we obtain a bound without the squared distances:

**Lemma 20.** If $\sigma = [p_0, \ldots, p_i] \subset \mathbb{R}^m$ is non-degenerate, and $x$ satisfies $\|p_i - x\| < \epsilon$ for all $i, j \leq k$, and

$$\|p_i - x\| - \|p_j - x\| \leq \rho$$

for all $i, j \leq k$, then there exists a $c \in N(\sigma)$ such that $\|c - x\| < \eta$, where

$$\eta = \frac{\epsilon \rho}{Y(\sigma) \Delta(\sigma)}.$$  

In particular, if $\sigma$ is an $m$-simplex, then $x \in B_{r_m}(C(\sigma); \eta)$.

**Proof.** Let $R = \max_i \|p_i - x\|$ and $r = \min_i \|p_i - x\|$. Then $R^2 - r^2 = (R + r)(R - r) < 2\epsilon (R - r) \leq 2\epsilon \rho$, and the result then follows from Lemma 19.

It will be convenient to have a name for points $x$ which satisfy Equation (4): we call such a point a $\rho$-centre for $\sigma$.

### 4.1.2 Circumcentres and metric perturbations

We will show here that for an $\Upsilon_0$-thick $m$-simplex $\sigma$ in $\mathbb{R}^m$, and a metric $d$ that is close to $d_{k_m}$, there will be a metric circumcentre $c$ near $C(\sigma)$. We require the metric $d$ to be topologically equivalent to $d_{k_m}$: both metrics define the same topology. Henceforth, whenever we refer to a perturbation of the Euclidean metric, a topologically equivalent metric will always be assumed. We will demonstrate the following:

**Lemma 21 (Circumcentres: Metric Perturbation).** Let $\sigma \subset \mathbb{R}^m$, and let $d : U \times U \to \mathbb{R}$ be a metric topologically equivalent to $d_{k_m}$ and such that for any $x, y \in U$ with $d_{k_m}(x, y) < 2\epsilon$, we have $d(x, y) - d_{k_m}(x, y) \leq \rho$, with $\rho < \frac{\epsilon}{2\sqrt{m}}$. If $\sigma = [p_0, \ldots, p_i] \subset U$ is an $\Upsilon_0$-thick $m$-simplex with $R(\sigma) < \epsilon$, and $L(\sigma) \geq \mu_0$e, and such that $d_{k_m}(p_i, \partial U) \geq 2\epsilon$, then there is a point $c \in B = B_{r_m}(C(\sigma); \frac{4\sqrt{m} \rho}{\Upsilon_0 \mu_0})$ such that $d(c, p_i) = d(c, p_j)$ for all $p_i, p_j \in \sigma$.

In order to prove Lemma 21, we will use a particular case of Lemma 20:

**Lemma 22.** Suppose $\sigma$ is an $\Upsilon_0$-thick $m$-simplex such that $L(\sigma) \geq \mu_0$e. If $x$ is a $\rho$-centre for $\sigma$ with $d_{k_m}(x, p) < 2\epsilon$ for all $p \in \partial \sigma$, then $x \in B_{r_m}(C(\sigma); \eta)$, where $\eta = \frac{\epsilon \rho}{\Upsilon_0 \mu_0}$.

Let $B = B_{r_m}(C(\sigma); \eta)$ be the open ball which contains the $\rho$-centres for $\sigma$. We wish to show that if $\tilde{\rho} = 2\sqrt{m} \rho$, then a circumcentre for $\sigma$ with respect to $d$ will also lie in $B$. Note that $\overline{B} \subset U$.

Consider the function $f_c : \overline{B} \to \mathbb{R}^m$ given by

$$f_c(x) = (d_{k_m}(x, p_1) - d_{k_m}(x, p_0), \ldots, d_{k_m}(x, p_m) - d_{k_m}(x, p_0))^{T}.$$  \hspace{1cm} (5)

Lemma 22 implies that $\|f_c(x)\| > \tilde{\rho}$, for all $x \in \partial B$. Also, $f_c$ maps the circumcentre of $\sigma$, and only the circumcentre, to the origin: $f_c^{-1}(0) = \{C(\sigma)\}$.

We construct a similar function for the metric $d'$,

$$f'(x) = (d(x, p_1) - d(x, p_0), \ldots, d(x, p_m) - d(x, p_0))^{T},$$  \hspace{1cm} (6)

and use a topological argument to show that there must be a $c \in f^{-1}(0) \subset B$. We first show that there is a homotopy between $f$ and $f_c$ such that the image of $\partial \overline{B}$ never touches the origin. Then we observe that $0$ is a regular value for $f_c$, and thus, since $f_c^{-1}(0) = \{C(\sigma)\}$, consideration of the degree of maps demonstrates that $f^{-1}(0) \neq \emptyset$.

**Lemma 23.** If $\rho = \frac{\tilde{\rho}}{2\sqrt{m}}$, there is a homotopy $F : \overline{B} \times [0, 1] \to \mathbb{R}^m$ with $F(x, t) \neq 0$ for all $x \in \partial B$ and $t \in [0, 1]$.

**Proof.** We define the homotopy $F : \overline{B} \times [0, 1] \to \mathbb{R}^m$ by $F(x, t) = (1 - t)f_c(x) + tf(x)$. By the bounds on $\rho$ and $R(\sigma)$, for every $x \in \overline{B}$, and $p \in \sigma$, we have $d_{k_m}(x, p) \leq \frac{\epsilon}{\sqrt{m}} + R(\sigma) < 2\epsilon$. Thus Lemma 22 yields $\|f_c(x)\| > \tilde{\rho}$ for all $x \in \partial \overline{B}$. Also, from the hypothesis on $d$, we have $\|f(x) - f_c(x)\| \leq \tilde{\rho}$, for all $x \in \partial \overline{B}$. It follows that $F(x, t) \neq 0$ when $x \in \partial B$.

By a direct calculation [BDG12b] we find:

**Lemma 24.** The origin is a regular value for the function $f_c$ defined in Equation (5).

Lemma 21 now follows from a consideration of the degree of the mappings $f$ and $f_c$ relative to zero. The degree of a smooth map $g : \overline{B} \to \mathbb{R}^m$ at a regular point $p \in g(B)$ is defined by

$$\deg(g, p, B) = \sum_{x \in g^{-1}(p)} \text{sign det } J(g)_{x},$$

where $J(g)_x$ is the Jacobian matrix of $g$ at $x$. The exposition by Dancer [Dan00] is a good reference for the degree of maps from manifolds with boundary. It is shown that the definition of $\deg(g, p, B)$ extends to continuous maps $g$ that are not necessarily differentiable. If $h : \overline{B} \to \mathbb{R}^m$ is homotopic to $g$ by a homotopy $H : \overline{B} \times [0, 1] \to \mathbb{R}^m$ such that $H(x, t) \neq p$ for all $t \in [0, 1]$, and $x \in \partial B$, then

$$\deg(g, p, B) = \deg(h, p, B).$$

Since $f_c^{-1}(0) = \{C(\sigma)\}$, it follows from Lemma 24 that $\deg(f_c, 0, B) = \pm 1$. Then Lemma 23 implies $\deg(f, 0, B) = \deg(f_c, 0, B)$, and since this is nonzero, it must be that $f^{-1}(0) \neq \emptyset$. The demonstration of Lemma 21 is complete.
4.1.3 Circumcentres and point perturbations

The exact same argument as was used to demonstrate Lemma 21 can be used to show that an $m$-simplex $\bar{\sigma} = [\bar{p}_0, \ldots, \bar{p}_m]$ whose vertices lie close to a thick $m$-simplex $\sigma$, will also have a circumcentre that lies close to $C(\sigma)$. We replace the function $f$ defined in Equation (6) by the function

\[
\tilde{f}(x) = (d_{\rho m}(x, \bar{p}_0) - d_{\rho m}(x, \bar{p}_0), \ldots, d_{\rho m}(x, \bar{p}_0) - d_{\rho m}(x, \bar{p}_0))^{\top},
\]

and the same argument goes through. We obtain:

**Lemma 25 (Circumcentres: point perturbation).** Suppose that $\sigma = [p_0, \ldots, p_m]$ is an $\Omega_0$-thick $m$-simplex with $R(\sigma) < \epsilon$ and $L(\sigma) \geq \mu_0 \epsilon$. Suppose also that $\bar{\sigma} = [\bar{p}_0, \ldots, \bar{p}_m]$ is such that $\|\bar{p}_i - p_i\| \leq \rho$ for all $i \in [0, \ldots, m]$. If $\rho \leq \frac{\sqrt{3\mu_0z}}{4\epsilon m}$, then $d_{\rho m}(C(\bar{\sigma}), C(\sigma)) < \frac{\sqrt{3\mu_0z}}{4\epsilon m}$.

We will also make use of a similar result [BDGO11, Lemma 3.7], which depends on the square of the thickness bound, but applies also to simplices that are not full-dimensional:

**Lemma 26.** Suppose $B = B_{\rho m}(c, r)$, with $r < \epsilon$, is a circumscribing ball for a j-simplex $\sigma = [p_0, \ldots, p_j]$. Suppose also that $\bar{\sigma} = [\bar{p}_0, \ldots, \bar{p}_j]$ is such that $\bar{p}_i = p_0$ and $\|\bar{p}_i - p_i\| \leq \rho$ for all $i \in [1, \ldots, j]$. If $\rho \leq \frac{\sqrt{3\mu_0z}}{4\epsilon j}$, then there is a circumscribing ball $B = B_{\delta}(\bar{c}; \bar{r})$ for $\bar{\sigma}$ with $|\bar{r} - r| \leq |\bar{c} - c| < \frac{\sqrt{3\mu_0z}}{4\epsilon j} \rho$.

4.2 Perturbations and protection

Suppose $\zeta : P \to P$ is a $\rho$-perturbation. If $\sigma$ is a $\delta$-protected $m$-simplex in Del($P$), then we want an upper bound on $\rho$ that will ensure that $\bar{\sigma} = \zeta(\sigma)$ is protected in Del($P$).

**Lemma 27 (Protection and point perturbation).** Suppose that $P \subset \mathbb{R}^m$ and $\sigma \in$ Del($P$) is a $\delta$-protected, $\Omega_0$-thick $m$-simplex with $R(\sigma) < \epsilon$ and $L(\sigma) \geq \mu_0 \epsilon$. If $\zeta : P \to P$ is a $\rho$-perturbation with $\rho \leq \frac{\sqrt{3\mu_0z}}{4\epsilon m} \delta$, then $\zeta(\sigma) = \bar{\sigma} \in$ Del($\tilde{P}$) and has a $\delta - \frac{\sqrt{3\mu_0z}}{4\epsilon m} \rho$-protected Delaunay ball.

**Proof.** Let $B = B_{\rho m}(c, r)$ be the $\delta$-protected Delaunay ball for $\sigma \in$ Del($P$), and let $\tilde{B} = B_{\rho m}(\bar{c}; \tilde{r})$ be the circumball for the corresponding perturbed simplicial $\tilde{\sigma}$. We wish to establish a bound on $\rho$ that will ensure that $\tilde{B}$ is protected with respect to $P$.

Let $q \in P$ be a point not in $\sigma$. We need to ensure that the corresponding $\tilde{q}$ lies outside the closure of $\tilde{B}$, i.e., that $d_{\rho m}(\tilde{q}, \tilde{c}) > \tilde{r}$.

Since $\delta \leq \epsilon$, the hypothesis of Lemma 25 is satisfied by $\rho$, and we have $d_{\rho m}(\tilde{c}, c) < \eta \rho$, where $\eta = \frac{\sqrt{3\mu_0z}}{4\epsilon m}$. Thus for $p \in \sigma$ and corresponding $\tilde{p} \in \sigma$ we have

\[
\tilde{r} \leq d_{\rho m}(c, p) + d_{\rho m}(p, \tilde{p}) + d_{\rho m}(\tilde{c}, c) < r + (\eta + 1) \rho.
\]

Also

\[
d_{\rho m}(\tilde{q}, \tilde{c}) \geq d_{\rho m}(q, c) - d_{\rho m}(\tilde{c}, c) - d_{\rho m}(\tilde{q}, q) > r + \delta - \rho(\eta + 1).
\]

Therefore $\tilde{q}$ will be outside of the closure of $\tilde{B}$ provided $r + \delta - \rho(\eta + 1) \geq r + (1 + \eta) \rho$, i.e., when $\delta \geq 2(\eta + 1) \rho$. The result follows from the definition of $\eta$ and the observation that $\mu_0$ and $\Omega_0$ are each no larger than one.

A similar argument [BDG12b] yields a bound on the metric perturbation that will ensure the Delaunay balls for the $m$-simplices remain protected.

**Lemma 28 (Protection and metric perturbation).** Suppose $U \subset \mathbb{R}^m$ contains ch($P$) and $d : U \times U \to \mathbb{R}$ is a metric such that $d_{\rho m}(x, y) - d(x, y) \leq \rho$ for all $x, y \in U$. Suppose also that $\sigma \in$ Del($P$) is a $\delta$-protected, $\Omega_0$-thick $m$-simplex with $R(\sigma) < \epsilon$ and $\Delta(\sigma) \geq \mu_0 \epsilon$. If $\rho \leq \frac{\sqrt{3\mu_0z}}{4\epsilon m} \delta$, and $d_{\rho m}(p, \partial U) \geq 2\epsilon$ for every vertex $p \in \sigma$, then $\sigma$ also belongs to Del($P$), and has a $(\delta - \frac{\sqrt{3\mu_0z}}{4\epsilon m} \rho)$-protected Delaunay ball in the metric $d$.

4.3 Perturbations and Delaunay stability

The results of Section 4.2 translate into stability results for Delaunay triangulations. In the case of point perturbations in Euclidean space, the connectivity of the Delaunay triangulation cannot change as long as the simplices corresponding to the initial $m$-simplices remain protected. This is a direct consequence of Delaunay’s original result [Del34], but we explicitly lay out the argument.

In the case of metric perturbation, we can no longer take for granted that the Delaunay complex cannot change its connectivity if the $m$-simplices remain protected. This is because we are no longer guaranteed that the Delaunay complex will be a triangulation. Using the consequences of the point-perturbation result, we establish bounds that ensure that the Delaunay complex in the perturbed metric will be the same as the original Delaunay triangulation.

4.3.1 Point perturbations

**Lemma 27** establishes bounds on a $\rho$-perturbation $\zeta : P \to P$ which will guarantee that $\zeta(\text{star}(P_1)) \subseteq \text{Del}(\tilde{P})$. Since Lemma 27 also guarantees that, if $\rho$ is small enough, the vertices of $\text{star}(\zeta(P_1); \text{Del}(\tilde{P}))$ will be generic, we know from Delaunay’s Lemma 14 that $\text{star}(\zeta(P_1); \text{Del}(\tilde{P}))$ will be a triangulation at each $\tilde{p} \in \zeta(P_1)$, and thus $\zeta(\text{star}(P_1)) = \text{star}(\text{star}(P_1))$ follows from Lemma 7. Explicitly, we have:

\[
\text{Lemma 29.} \quad \text{Suppose } P \subset \mathbb{R}^m \text{ is a generic sample set, and } \zeta : P \to P \text{ is a perturbation such that } \zeta(\text{star}(P_1); \text{Del}(P)) \subseteq \text{star}(\zeta(P_1); \text{Del}(P)), \text{ and every } \tilde{\sigma} \in \zeta(\text{star}(P_1)) \text{ is protected in } \text{Del}(\tilde{P}). \text{ Then } \zeta(\text{star}(P_1)) = \text{star}(\text{star}(P_1)).
\]

Thus, considering Theorem 18, Lemma 16, and Lemma 27, we have the following stability theorem for Delaunay triangulations of $\delta$-protected points:

**Theorem 30 (Stability under point perturbation).** Suppose $P \subset \mathbb{R}^m$ is $\delta$-generic for $P_1$, with sampling radius $\epsilon$ and $\delta = \nu_0 \epsilon$. If $\zeta : P \to P$ is a $\rho$-perturbation, with

\[
\rho \leq \frac{\nu_0}{24\sqrt{m}} \delta = \frac{\nu_0}{24\sqrt{m}} \epsilon,
\]

then

\[
\text{star}(P_1; \text{Del}(P)) \subseteq \text{star}(\zeta(P_1); \text{Del}(P)).
\]

The $\rho$-relaxed Delaunay complex for $P$ was defined by de Silva [dS08] by the criterion that $\sigma \in \text{Del}(P)$ if and only if there is a ball $B = B_{\rho m}(c, r)$ such that $\sigma \subseteq B$, and $d_{\rho m}(c, q) \geq r - \rho$ for all $q \in P$. Thus the simplices in Del($P$) all have “almost empty”, balls centred on a $\rho$-centre for $\sigma$. We have the following consequence of Theorem 30:
Corollary 31 (Stability under relaxation). Let \( P \subset \mathbb{R}^m \) be \( \delta \)-generic for \( P \), with sampling radius \( \epsilon \) and \( \delta = \nu_0 \epsilon \). If \( \rho \leq \frac{\nu_0^3}{48 \sqrt{m}} \epsilon = \frac{\nu_0^3}{48 \sqrt{m}} r \), then \( \text{star}(P; \text{Del}^\rho(P)) = \text{star}(P; \text{Del}(P)). \)

Proof. Suppose that \( \sigma \in \text{star}(P; \text{Del}^\rho(P)) \). Then there is a ball \( B \) for \( \sigma \) such that any point \( q \in B \) is within a distance \( \rho \) from \( B \). Project all such points radially out to \( B \). Then we have a \( \rho \)-perturbation \( \zeta : P \to \mathbb{R} \), and \( \sigma \) has become \( \bar{\sigma} \in \text{star}(\zeta(P); \text{Del}(P)) \). By Theorem 30, \( \text{star}(\zeta(P); \text{Del}(P)) \) \( \subseteq \text{star}(P; \text{Del}(P)) \), and therefore \( \sigma \in \text{star}(P; \text{Del}(P)) \) \( \square \).

4.3.2 Metric perturbation

For a perturbation of the metric, we can exploit the stability results obtained for perturbations of points in the Euclidean metric to ensure that no simplices can appear in \( \text{star}(P; \text{Del}^\rho(P)) \) that do not already exist in \( \text{star}(P; \text{Del}(P)) \).

Lemma 32. Suppose \( P \) is \( \delta \)-generic for \( P \), with sampling radius \( \epsilon \) and \( \delta = \nu_0 \epsilon \). Suppose also that \( \text{ch}(P) \subseteq U \) and \( d : U \times U \to \mathbb{R} \) is such that \( |d(x, y) - d_m(x, y)| \leq \rho \) for all \( x, y \in U \).

Proof. Let \( B(c, r) \) be a Delaunay ball for simplex \( \sigma \in \text{star}(P; \text{Del}(P)) \). Then \( d(c, p) \leq d(c, q) \) for all \( p \in \sigma \), and \( q \in P \). This results from the hypothesis on \( d \). By the hypothesis on \( d \), this implies that \( d_m(c, p) \leq d_m(c, q) + 2 \rho \) for all \( p \in \sigma \) and \( q \in P \), and therefore \( \sigma \in \text{Del}(P) \). The result now follows from Corollary 31 \( \square \).

The perturbation bounds required by Lemma 32, also satisfy the requirements of Lemma 28, when Theorem 18 and Lemma 16 are taken into account. This gives us the reverse inclusion, and thus we can quantify the stability of metric perturbation for Delaunay triangulations on \( \delta \)-generic point sets:

Theorem 33 (Stability under metric perturbation). Suppose \( P \) is \( \delta \)-generic for \( P \), with sampling radius \( \epsilon \) and \( \delta = \nu_0 \epsilon \). Suppose also that \( \text{ch}(P) \subseteq U \) and \( d : U \times U \to \mathbb{R} \) is such that \( |d(x, y) - d_m(x, y)| \leq \rho \) for all \( x, y \in U \). If

\[
\rho \leq \frac{\nu_0^3}{48 \sqrt{m}} \epsilon = \frac{\nu_0^3}{48 \sqrt{m}} r,
\]

then

\[
\text{star}(P; \text{Del}(P)) = \text{star}(P; \text{Del}(P)).
\]

5. EQUATING DELAUNAY STRUCTURES

We apply the results of the previous sections to the task of triangulating a compact \( m \)-manifold, \( M \) embedded in \( \mathbb{R}^N \). We wish to build a Delaunay complex on a finite set \( P \subset M \).

The restricted Delaunay complex is the Delaunay complex \( \text{Del}_{\mathbb{R}^N \setminus \mathcal{M}}(P) \) obtained when distances on the manifold are measured with the metric \( d_{\mathbb{R}^N \setminus \mathcal{M}} \). This is the Euclidean metric of the ambient space, restricted to the manifold. In other words, \( d_{\mathbb{R}^N \setminus \mathcal{M}}(x, y) = d_{\mathbb{R}^N}(x, y) \). We use this notation to avoid ambiguities in conjunction with the local Euclidean metrics discussed below. The Delaunay complex \( \text{Del}_{\mathbb{R}^N \setminus \mathcal{M}}(P) \) is a substructure of \( \text{Del}_{\mathbb{R}^N}(P) \).

Alternatively, distances on the manifold may be measured with \( d_{\mathcal{M}} \), the intrinsic metric of the manifold. This metric defines the distance between \( x \) and \( y \) as the infimum of the lengths of the paths on \( \mathcal{M} \) which connect \( x \) and \( y \). This metric is also induced from \( d_{\mathbb{R}^N} \). The intrinsic Delaunay complex is the Delaunay structure \( \text{Del}_{\mathcal{M}}(P) \) associated with this metric.

Although neither of these metrics are Euclidean, the idea is that locally, in a small neighbourhood of any point, these metrics may be well approximated by \( d_{\mathbb{R}^N} \). Then, if the sampling satisfies appropriate \( \delta \)-generic and \( \epsilon \)-dense criteria in these local Euclidean metrics, the global Delaunay complex in the metric of the manifold will coincide locally with a Euclidean Delaunay triangulation, and we can thus guarantee a manifold complex.

5.1 Local Euclidean metrics

A local parameterization at a point \( p \in \mathcal{M} \) is a pair \( (U, \psi_p) \), where \( U \subset \mathbb{R}^m \) is an open neighbourhood of the origin, and \( \psi_p : U \to \psi_p(U) = W \subset \mathcal{M} \) is a homeomorphism onto its image, and maps the origin to \( p \). We will use \( \psi_p \) to pull back the metric of the manifold to \( U \), and to simplify the notation we will write \( d_{\mathcal{M}}(x, y) \) for \( x, y \in U \), where it is to be understood that this means \( d_{\mathcal{M}}(\psi_p(x), \psi_p(y)) \), and likewise for \( d_{\mathbb{R}^N \setminus \mathcal{M}}(x, y) \). Indeed, once \( W \) and \( U \) have been coupled together by a homeomorphism, we can transfer the metrics between them and the distinction becomes only one of perspective. We refer to the standard metric \( d_{\mathbb{R}^N} \) on \( U \) as a local Euclidean metric for \( p \) on \( W \). Clearly this metric depends upon the choice of \( \psi_p \); there are different ways to impose a Euclidean metric on \( W \). In this work the only local parameterization we consider is defined by the projection map \( \pi_p \) described in Section 5.3.

We wish to generate a sample set \( P \subset \mathcal{M} \) that will allow us to exploit the results of Section 4. We will require that for each \( p \in P \) there be a local parameterization \( (U, \psi_p) \) such that \( B = B_{\mathbb{R}^N}(0; 3 \epsilon) \subset U \), and \( \mathcal{M} = \psi_p(P \cap W) \) is an \( \epsilon \)-sample set for \( B \) in the Euclidean metric. We identify \( p \) with the origin, and note that \( p \in P \), since \( d_{\mathbb{R}^N}(p, \text{ch}(P)) \geq 4 \epsilon \). We will demand that \( \sigma \) be \( \delta \)-generic for \( \{p\} \), with \( \delta = \nu_0 \epsilon \).

In order to exploit Theorem 33 we need to find a bound on the metric distortion induced by \( \psi_p \), and this will depend on the sampling radius \( \epsilon \), through the requirement that \( B \subset U \).

5.2 Background results for manifolds

We will make use of some established nomenclature and properties related to a smooth compact \( m \)-dimensional submanifold of \( \mathbb{R}^N \).

For \( p \in \mathcal{M} \), \( T_p \mathcal{M} \) denotes the tangent space at \( p \), which we may identify with an \( m \)-flat in the ambient space. The normal space, \( N_p \mathcal{M} \), is the orthogonal complement of \( T_p \mathcal{M} \) in \( T_p \mathbb{R}^N \), and we likewise treat it as the affine space of dimension \( m - k \) orthogonal to \( T_p \mathcal{M} \subset \mathbb{R}^N \).

A ball \( B = B_{\mathbb{R}^N}(c, r) \) is a medial ball at \( p \) if \( B \cap \mathcal{M} = \emptyset \), it is tangent to \( \mathcal{M} \) at \( p \), and it is maximal in the sense that any ball which is centred on the line through \( p \) and \( c \) and contains \( B \), either coincides with \( B \) or intersects \( \mathcal{M} \). The local reach at \( p \) is the infimum of the radii of the medial balls at \( p \), and the reach of \( \mathcal{M} \), denoted \( \text{rch}(\mathcal{M}) \), is the infimum of the local reach over all points of \( \mathcal{M} \). In order to approximate the geometry and topology with a simplicial complex, manifolds with small reach require a higher sampling density than those with a larger reach. As is typical, an upper bound on our sampling radius will be proportional to \( \text{rch}(\mathcal{M}) \). We require \( \mathcal{M} \) to have positive reach.
Giesen and Wagner [GW04, Lemma 6] estimate how the tangent space locally deviates from the manifold:

**Lemma 34 (Distance to Tangent Space).** If \( x, y \in M \subset \mathbb{R}^N \) and \( d_{R^N}(x, y) \leq r < \text{rch}(M) \), then \( d_{R^N}(y, T_x M) \leq \frac{r^2}{2 \text{rch}(M)} \), and thus \( \sin \alpha \leq \frac{r}{\text{rch}(M)} \), where \( \alpha \) is the angle between \([x, y]\) and \(T_x M\).

A convenient bound on the angle between nearby tangent spaces [BDGO11, Prop. A.3]:

**Lemma 35 (Tangent Space Variation).** Let \( x, y \in M \) be such that \( d_{R^N}(x, y) \leq \frac{\text{rch}(M)}{2} \). Then, \( \sin \angle(T_x M, T_y M) \leq \frac{d_{R^N}(x, y)}{\text{rch}(M)} \).

Niyogi et al [NSW08, Prop 6.3] demonstrate a bound on the geodesic distance between nearby points, with respect to the ambient distance. We will use a modified statement [BDG12b] of this result:

**Lemma 36 (Geodesic Distance Bound).** Let \( x, y \in M \) be such that \( d_{R^N}(x, y) \leq \frac{\text{rch}(M)}{2} \). Then

\[
d_M(x, y) \leq d_{R^N}(x, y) \left( 1 + \frac{2d_{R^N}(x, y)}{\text{rch}(M)} \right).
\]

### 5.3 Sampling criteria for manifolds

We will confine ourselves to smooth compact manifolds without boundaries. A local parameterization at \( p \in M \) will be constructed with the aid of the orthogonal projection \( \pi_p : \mathbb{R}^N \to T_p M \), restricted to \( M \). Niyogi et al. [NSW08, Lemma 5.4] demonstrated that, when \( r < \frac{\text{rch}(M)}{\text{rch}(M)} \), \( \pi_p \) is a diffeomorphism from \( W = B_{R^N \setminus M}(p, r) \) onto its image \( U \subset T_p M \). We will identify \( T_p M \) with \( \mathbb{R}^m \), and define

\[
\psi_p = \pi_p^{-1} : U \xrightarrow{\approx} W.
\]

Using \( \psi_p \) to pull back the metrics \( d_M \) and \( d_{R^N \setminus M} \) to \( \mathbb{R}^m \), we can view them as perturbations of \( d_{R^N} \). The magnitude of the perturbation is governed by the radius of the ball used to define \( W \).

**Lemma 37.** Using the local parameterization \((U, \psi_p)\), with \( \mathbb{R}^m \ni U \ni W \subset M \), if \( W \subset B_{R^N \setminus M}(p, r) \), with \( r < \frac{\text{rch}(M)}{16} \), then for all \( x, y \in U \), \( |d_M(x, y) - d_{R^N \setminus M}(x, y)| \leq \frac{10r^2}{\text{rch}(M)^2} \).

**Proof.** Let \( u, v \in W \subset B_{R^N \setminus M}(p, r) \), and let \( \theta \) be the angle between the line segments \([u, v]\) and \([\pi_p(u), \pi_p(v)]\), \( \theta_1 \) the angle between \([u, v]\) and \(T_u M\), and \( \theta_2 \) the angle between \(T_p M\) and \(Tu M\). Thus

\[
\theta \leq \theta_1 + \theta_2 \tag{8}
\]

and \( d_{R^N}(\pi_p(u), \pi_p(v)) = d_{R^N}(u, v) \cos \theta \). Defining \( \eta = \frac{r}{\text{rch}(M)} \), Lemma 36 yields \( d_M(u, v) \leq d_{R^N}(u, v) (1 + 4\eta) \), and so

\[
d_{R^N}(\pi_p(u), \pi_p(v)) \geq \frac{d_M(u, v) \cos \theta}{1 + 4\eta}.
\]

Using Lemma 34, we find \( \theta_1 \leq \sin \theta_1 \leq \eta \), and Lemma 35, yields \( \theta_2 \leq \sin \theta_2 \leq 5\eta \). Therefore from Equation (8), \( \theta \leq 6\eta < \frac{r}{2} \), and thus \( \cos \theta \geq 1 - \theta \geq 1 - 6\eta \). We get

\[
d_{R^N}(\pi_p(u), \pi_p(v)) \geq d_M(u, v) \left( \frac{1 - 6\eta}{1 + 4\eta} \right)
\]

\[
\geq d_M(u, v)(1 - 6\eta)(1 - 4\eta)
\]

\[
\geq d_M(u, v)(1 - 10\eta)
\]

\[
\geq d_M(u, v) - \frac{10r^2}{\text{rch}(M)}
\]

The result follows since \( d_M(u, v) \geq d_{R^N}(\pi_p(u), \pi_p(v)) \).

We call a neighbourhood \( W \) of \( p \) admissible if it satisfies the hypothesis of Lemma 37.

We need to define the local parameterization on a domain that is large enough to contain an \( \epsilon \)-sampled ball around \( p \) that will ensure that \( p \) is a safe interior point. Thus we translate Lemma 37 into a sampling density requirement:

**Lemma 38.** If \( 1 < a \leq 400 \) and \( \epsilon \leq \frac{\text{rch}(M)}{1000} \), and \( U = B_{R^N}(p; (a - 1)\epsilon) \), then \( \psi_p(U) = W \subset B_{R^N \setminus M}(p; a\epsilon) \), and

\[
|d_{R^N \setminus M}(x, y) - d_{R^N}(x, y)| \leq |d_M(x, y) - d_{R^N}(x, y)| \leq \frac{10a^2\epsilon^2}{\text{rch}(M)}.
\]

**Proof.** Since \( a\epsilon \leq \frac{\text{rch}(M)}{1000} \), the perturbation bound follows from Lemma 37, and the fact that

\[
d_{R^N}(x, y) \leq d_{R^N \setminus M}(x, y) \leq d_M(x, y),
\]

for any \( x, y \in U = \pi_p(W) \). It remains to show that \( B_{R^N}(p; (a - 1)\epsilon) \subset \pi_p(W) \). Using Lemma 34, we have that \( B_{R^N}(p; r) \subset \pi_p(W) \) if

\[
r^2 \leq a^2\epsilon^2 - \left( \frac{2\epsilon^2}{\text{rch}(M)} \right)^2 = a^2\epsilon^2 \left( 1 - \left( \frac{a\epsilon}{\text{rch}(M)} \right)^2 \right)
\]

\[
\leq a^2\epsilon^2 \left( 1 - \left( \frac{1}{20} \right)^2 \right).
\]

Thus we require \( r \leq \sqrt{\frac{399}{400}a\epsilon} \), which is satisfied by \( r = (a - 1)\epsilon \) if \( a \leq 400 \).

With Lemma 38 we can establish the sampling requirements that ensure that we locally meet the criteria needed to apply Theorem 33.

**Theorem 39.** (Equating structures). Suppose \( \mathcal{P} \subset M \) is such that for every \( p \in \mathcal{P} \), the set \( \mathcal{P} = B_{R^N \setminus M}(p; \tau \epsilon) \cap \mathcal{P} \) is an \( \epsilon \)-sample set for \( B_{R^N}(p; 5\epsilon) \) in the local Euclidean metric on \( W = B_{R^N \setminus M}(p; 7\epsilon) \), and \( \mathcal{P} \) is \( \delta \)-generic for \( \{p\} \), with \( \delta = \nu_0 \epsilon \). If

\[
\epsilon \leq \frac{\nu_0 \text{rch}(M)}{23520 \sqrt{m}}
\]

then \( \text{Del}_{R^N \setminus M}(\mathcal{P}) = \text{Del}_M(\mathcal{P}) \) and they are manifold complexes.

**Proof.** As discussed in Section 5.1, if \( P \) is an \( \epsilon \)-sample set for \( B_{R^N}(p; 5\epsilon) \), then \( p \in P \). Theorem 33 guarantees that \( \text{star}(p; \text{Del}(P)) = \text{star}(p; \text{Del}_{R^N \setminus M}(P)) = \text{star}(p; \text{Del}_M(P)) \) if \( \rho \leq \frac{499}{400 \sqrt{m}} \), and, by Lemma 38, this will be true with \( \alpha = \frac{7}{4} \) if \( \frac{499}{400 \sqrt{m}} \leq \frac{\epsilon^2}{\text{rch}(M)} \), and we obtain the required bound on \( \epsilon \). Thus the star of every vertex in \( \text{Del}_M(\mathcal{P}) \) is equal to the star of that point in the local Euclidean metric, and likewise for
The awkwardness of the stated sampling conditions in Theorem 39 can be alleviated somewhat by recognizing that if $d$ is a metric on $U$ such that $d(x, y) \geq d_{\delta}(x, y)$ for all $x, y \in U$, then if $P$ is an $\epsilon$-sample with respect to $d$, then it is also an $\epsilon$-sample with respect to $d_{\delta}$. Thus if $P \subset M$ is an $\epsilon$-sample with respect to the intrinsic metric $d_M$, with a sampling radius satisfying Equation (9), then it will satisfy the density requirements in the local Euclidean metrics.

The condition of $\delta$-genericity is not so easily transferred to the metric of the manifold. Our results in Section 4 depend on the $m$-simplices of Del$(P)$ being $\delta$-protected, and exploit the fact that Del$(P)$ is then necessarily a triangulation. It is not clear that similar assertions can be made if protection is assumed only on the $m$-simplices of Del$_d$(P).

6. CONCLUSIONS

We have quantified the close relationship between the genericity of a point set, the quality of the simplices in the Delaunay complex, and its stability under perturbation.

We have produced extrinsic sampling conditions which will guarantee that the intrinsic Delaunay complex is a manifold and coincides with the restricted Delaunay complex. In a companion paper [BDG12a], we have adapted an algorithm for the tangential Delaunay complex so as to meet these sampling conditions, guaranteeing that the tangential Delaunay complex also coincides with the intrinsic Delaunay complex. In this way we are able to exploit existing structural results [BG11] and demonstrate that the intrinsic Delaunay complex is a triangulation.

We have relied on an embedding of $M$ in $\mathbb{R}^N$. In future work we aim to develop intrinsic sampling conditions. Another possible direction is to develop sampling conditions for manifolds that are not smooth, since our stability results impose no such constraints on the metric.

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7. REFERENCES


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