Optimal Beamforming in Interference Networks with Perfect Local Channel Information

Rami Mochaourab, Student Member, IEEE, and Eduard Jorswieck, Senior Member, IEEE

Abstract

We consider multiuser settings in which systems simultaneously utilize the available communication resources. Since the performance of the systems is usually limited by mutual interference, efficient resource allocation in such scenarios is important. In general, the systems’ joint operation is desired to be Pareto optimal. However, designing Pareto optimal resource allocation schemes is known to be difficult. In this work, we assume all transmitters apply multiple antennas and have perfect local channel state information of their own channels to all single-antenna receivers. Each transmitter is associated with a power gain-region which is composed of all jointly achievable power gains at the receivers. We prove that the boundary of the power gain-region is convex and always achieved with single-stream beamforming. Thus, the efficient beamforming vectors that achieve its boundary part in a specific direction are characterized by real-valued parameters between zero and one. According to the network setting and monotonicity properties of each receiver’s utility function, the important boundary parts of the gain-regions are determined. Thereby, we characterize the set of Pareto efficient beamforming vectors which reduces the complexity of designing Pareto efficient resource allocation schemes. One example, including broadcast and multicast data, illustrates the results.

Part of this work has been performed in the framework of the European research project SAPHYRE, which is partly funded by the European Union under its FP7 ICT Objective 1.1 - The Network of the Future. This work is also supported in part by the Deutsche Forschungsgemeinschaft (DFG) under grant Jo 801/4-1.

The authors are with the Department of Electrical Engineering and Information Technology, Dresden University of Technology, 01062 Dresden, Germany. E-mail: {Mochaourab,Jorswieck}@ifn.et.tu-dresden.de. Phone: +49-351-46332239. Fax: +49-351-46337236.

Part of this work has been presented at the IEEE International Zurich Seminar on Communications, Zurich, Switzerland, March 3–5, 2010 [1], and submitted to International Workshop on Cognitive Information Processing, Elba Island, Italy, June 14–17, 2010 [2].
I. INTRODUCTION

Interference is known to be one of the major factors that limits the performance of a communication system in a wireless network. This situation is common in multiuser settings when the systems concurrently share the available communication resources. In general interference networks, the performance measure of individual users is described by a utility function. This function depends in a monotonic way on the received signal power, interference signal power and noise power. The joint operation of the systems is efficient if it is not possible to improve the performance of one system without degrading the performance of another. In this case, the operating point is said to be Pareto optimal. It is always desired to design resource allocation schemes that lead to Pareto optimal operation points. In this way, the available communication resources are utilized efficiently. However, developing efficient resource allocation schemes is not straightforward and proves to be difficult. For instance, the problem of finding the maximum sum-rate or the proportional-fair operating point in the multiple-input single-output (MISO) interference channel (IC) is proven to be strongly NP-hard.\[3\]

In this work, we characterize the transmission strategies that lead to Pareto optimal operating points. Moreover, we parameterize these efficient strategies by real values between zero and one. Thus, the relevant set of transmission strategies is confined and represented by low dimensional real parameters. This result tremendously reduces the complexity of designing efficient resource allocation schemes, and the parametrization can be utilized for low complexity coordination between transmitters.

A complex-valued parametrization of the Pareto boundary of the MISO IC rate-region is derived in [4]. The MISO IC is an example of an interference network where the systems consist of transmitter-receiver pairs. Real-valued parametrization is provided in [4] for the two-user setting. For this case, the efficient transmission strategies are proven to be a linear combination of the zero-forcing (ZF) strategy and the maximum ratio transmission (MRT) strategy. Based on this characterization, a monotonic optimization framework is developed in [5] to find maximum sum-rate, proportional-fair and minimax operating points. Moreover, the concept of combining the MRT and ZF strategies is important for developing so-called distributed bargaining algorithms. These algorithms improve the operation of the systems from the noncooperative outcome [6], [7]. In [6], a distributed bargaining algorithm is developed which requires one bit signaling between the transmitters. Extension to the precoding design in the multiple-input multiple-output (MIMO) case is given in [8]. In [7], similar distributed beamforming algorithms in the MISO IC are proposed for the case of statistical CSI at the transmitters. In [9], the high signal to interference plus

\[1\] Interestingly, these problems are efficiently solvable if rate requirements or interference constraints on each system are fixed.
noise ratio (SINR) approximation of the achievable sum-rate of a system pair is utilized to determine suboptimal joint transmission strategies. The achieved performance is shown to be better than the joint MRT and joint ZF strategies.

The parametrization in [4] relates to a parametrization using the virtual SINR framework in [10]. The use of this framework is motivated by the design of distributed algorithms that require local channel state information (CSI) at each transmitter. This framework is extended to the precoding design in MIMO settings in [11], and the optimality of the proposed scheme is justified in [12]. In [12], joint linear precoding is investigated taking into account the signaling overhead between the transmitters. The rate-region achieved with joint precoding is larger than the MISO IC rate-region, and the Pareto optimal beamforming vectors are proven to be a complex-valued combination of the MRT and ZF strategies. Linear precoding MIMO IC algorithms are moreover investigated in [13] for a two-user system.

In [14], the authors characterize the Pareto boundary of the MISO IC through controlling interference temperature constraints (ITC) at the receivers. Each Pareto optimal rate tuple is achieved iteratively when each transmitter optimizes its transmission constrained by the ITCs. ITC is a terminology used in cognitive radio scenarios under the underlay paradigm [15]. It quantifies the amount of interference from the secondary transmitters that is tolerated by the primary users. In order to fulfill the ITCs, a pricing mechanism can be included in the utility functions of the secondary systems. Consequently, these systems are penalized on imposing excessive interference on the primary users [16], [2]. Pricing mechanisms also lead to utility functions that have the monotonicity properties in the intended signal power, interference signal power and noise power. Hence, the framework developed in this paper can also be applied in such scenarios.

The contributions and outline of this paper are as follows:

- We investigate the properties of efficient transmission of a single transmitter. These properties are acquired on studying the transmitter’s power gain-region (Section III). The power gain-region is composed of all jointly achievable power gains at the receivers. Of interest are the transmission strategies which achieve its boundary part in a specific direction. We prove that the boundary of the power gain-region is convex and always achieved with single-stream beamforming (Lemma 3). Due to these properties, the corresponding strategies are then characterized by real-valued parameters (Theorem 1). Furthermore, we characterize under which conditions power control is needed for efficient transmission. (i) When the number of transmit antennas is greater than or equal to the number of receivers $K$ (Section III-A), we prove that full power transmission achieves all boundary points (Lemma 2). In this case, $K - 1$ real-valued parameters, each between zero and one, are needed
to parameterize the efficient beamforming vectors. (ii) When the number of transmit antennas is strictly less than the number of receivers (Section [III-B]), we characterize the efficient transmission strategies for which power control is needed. For this case, an additional real-valued parameter between zero and one is needed that varies the power level at the transmitter.

- We utilize the developed single-transmitter framework for the multiple-transmitter case (Section [IV]). Based on the network setting and the monotonicity properties of each receiver’s utility function, the important boundary part is determined for each transmitter’s gain-region. Consequently, each transmitter’s efficient strategies are parameterized (Theorem [2]). We provide an example setting which includes broadcast and multicast data, and we apply the developed framework to this setting (Section [IV-A]). Moreover, we apply the framework to the $K$-user MISO IC (Section [V-A]). As a special case, the result for the two-user MISO IC in [4] follows. In addition, we give an alternative characterization of the efficient transmission strategies (Corollary [1]). The characterization is motivated by the application of null-shaping constraints in underlay cognitive radio scenarios (Section [V-B]). We prove that all Pareto optimal operating points can be characterized through the design of null-shaping constraints on noncooperative secondary transmitters.

A. Notations

Column vectors and matrices are given in lowercase and uppercase boldface letters, respectively. The notation $x_{k,\ell}$ describes the $\ell$th component of vector $x_k$. The Euclidean norm of a vector $a, a \in \mathbb{C}^N$, is written as $\|a\|$, and the absolute value of $b, b \in \mathbb{C}$, is $|b|$. $(\cdot)^H$ denotes the Hermitian transpose. The $i$th eigenvalue of a matrix $Z$ is denoted by $\mu_i(Z)$. The eigenvector which belongs to the $i$th eigenvalue of the matrix $Z$ is denoted by $v_i(Z)$. We always assume that the eigenvalues are ordered in nondecreasing order such that $\mu_i(Z) \leq \mu_{i+1}(Z)$. Moreover, the eigenvectors corresponding to the largest and smallest eigenvalues of a matrix $Z$ are specified as $v_{\text{max}}(Z)$ and $v_{\text{min}}(Z)$, respectively. The notation $Z \succeq 0$ means that $Z$ is positive semidefinite. The rank and trace of a matrix $Z$ are given by rank($Z$) and tr($Z$), respectively. The orthogonal projector onto the column space of $Z$ is $\Pi_Z := Z(Z^HZ)^{-1}Z^H$. The orthogonal projector onto the orthogonal complement of the column space of $Z$ is $\Pi_{\perp Z} := I - \Pi_Z$, where $I$ is an identity matrix. $\mathbb{E}(\cdot)$ denotes statistical expectation. The set of non-negative real numbers is $\mathbb{R}_+$. The cardinality of a set $\mathcal{K}$ is written as $|\mathcal{K}|$. 

April 27, 2010
II. SYSTEM AND CHANNEL MODEL

We consider $T$ transmitters and $K$ receivers sharing the same spectral band. Define the set of transmitters as $\mathcal{T} := \{1, \ldots, T\}$ and receivers as $\mathcal{K} := \{1, \ldots, K\}$. Each transmitter is equipped with $N$ transmit antennas and each receiver with a single antenna. The quasi-static block flat-fading instantaneous channel vector from transmitter $k$, $k \in \mathcal{T}$, to receiver $\ell$, $\ell \in \mathcal{K}$, is denoted by $h_{k\ell} \in \mathbb{C}^{N \times 1}$. The transmit covariance matrix of transmitter $k$ is given as $Q_k \in \mathbb{C}^{N \times N}$ and $Q_k \succeq 0$. We do not make any assumptions on the number of data streams applied at the transmitters. This leads to the following basic model for the matched-filtered, symbol-sampled complex baseband data received at receiver $\ell$:

$$y_\ell = \sum_{k=1}^{T} h_{k\ell}^H Q_k^{\frac{1}{2}} s_k + n_\ell,$$

where $s_k$ is the symbols vector transmitted by transmitter $k$ and $n_\ell$ are the noise terms which we model as independent and identically distributed (i.i.d.) complex Gaussian with zero mean and variance $\sigma^2$.

Each transmitter has a total power constraint of $P := 1$ which leads to the constraint $\text{tr}(Q_k) \leq 1$, $k \in \mathcal{T}$. Throughout, we define the signal to noise ratio (SNR) as $1/\sigma^2$. The feasible set of transmit covariance matrices is defined as

$$S := \{ Q \in \mathbb{C}^{N \times N} : Q \succeq 0, \text{tr}(Q) \leq 1 \}.$$  

Note that $S$ is compact and convex. We assume each transmitter has local CSI, i.e., it has perfect knowledge of the channel vectors only between itself and all receivers [11]. This ideal scenario serves as an upper bound to the more realistic case in which imperfect or partial CSI at the transmitters is available. Extensions in this direction are reported in [17], [18]. In these works, Pareto efficient transmission strategies are characterized for the two-user MISO IC with partial CSI at the transmitters.

We give an example of the described system model. Consider two transmitters using three transmit antennas, and three single antenna receivers as depicted in Fig. [1]. Transmitter one transmits different data to receivers one and two simultaneously. The setting between this transmitter and the two receivers corresponds to the broadcast channel (BC). Transmitter two sends data in a multicast to receivers two and three. Receiver two decodes the data transmitted from transmitter one and two successively such that this setting is described by the multiple access channel (MAC). Transmitter two induces interference on receiver one, while transmitter one induces interference on receiver three. This leads to an IC between these transmitter-receiver pairs.

2The general case in which one transmitter $k$ has $N_k$ antennas is a straightforward extension.

3We refer to a transmitter’s choice of transmit covariance matrix as a strategy.
Fig. 1. An example setting for the described system model. There exist two transmitters each equipped with three antennas, and three receivers each with a single antenna. The solid arrows refer to the intended receivers of a transmitter, while the dashed arrows refer to interference directions. The associated receiver sets of each transmitter are defined and determined later in Section IV and Section IV-A, respectively.

Accordingly, the solid and dashed arrows in Fig. 1 refer to useful and interference signal directions, respectively. Here, we assume transmitter one chooses the transmit covariance matrices $Q_{11}$ for receiver one and $Q_{12}$ for receiver two. With this choice of strategy, transmitter one can be considered as two virtual transmitters coupled by the total power constraint of transmitter one. The two virtual transmitters are represented as transmitters 11 and 12. The power allocation $p_1$ and $p_2$ for the transmit covariance matrices $Q_{11}$ and $Q_{12}$, respectively, satisfies $p_1 + p_2 \leq 1$. Therefore, in order to acquire the entire achievable rate-region the power allocation needs to be varied for all unit norm beamforming vectors of transmitters 11 and 12. The achievable rate at receiver one is

$$u_1(Q_{11}, Q_{12}, Q_2, p_1, p_2) = \log_2 \left( 1 + \frac{p_1 h_{11}^H Q_{11} h_{11}}{\sigma^2 + p_2 h_{11}^H Q_{12} h_{11} + h_{21}^H Q_2 h_{21}} \right).$$

(3)

The utility at receiver two is its sum capacity [20].

$$u_2(Q_{11}, Q_{12}, Q_2, p_1, p_2) = \log_2 \left( 1 + \frac{p_2 h_{12}^H Q_{12} h_{12} + h_{22}^H Q_2 h_{22}}{\sigma^2 + p_1 h_{12}^H Q_{11} h_{12}} \right),$$

(4)

and at receiver three as the achievable rate,

$$u_3(Q_{11}, Q_{12}, Q_2, p_1, p_2) = \log_2 \left( 1 + \frac{h_{23}^H Q_2 h_{23}}{\sigma^2 + p_1 h_{13}^H Q_{11} h_{13} + p_2 h_{13}^H Q_{12} h_{13}} \right).$$

(5)

This transmission strategy is suboptimal, however less complex and more robust than dirty paper coding [19].
Note that the transmission rate at transmitter two has to be chosen such that both receiver two and receiver three can decode the data successfully. We do not consider this requirement in (4) and (5) since this is beyond the scope of this paper. However, these rates can be achieved using rateless coding [21], [22] at transmitter two.

We return to this setting in Section IV-A after we formalize the framework for efficient beamforming and resource allocation.

III. POWER GAIN REGION

In this section, a single transmitter \( k, k \in \mathcal{T} \), is considered along with all \( K \) receivers. The subscript \( k \) in all terms referring to the single transmitter is omitted for convenience. For example, \( h_{\ell} \) is written instead of \( h_{k\ell} \). We return to include the subscript in the next sections when multiple transmitters are considered again.

The purpose of this section is to study the transmission effects of a single transmitter on all existing receivers. Thereby, we characterize its efficient transmission strategies. Define the power gain achieved by the transmitter at a receiver \( \ell, \ell \in \mathcal{K} \), as

\[
x_{\ell}(Q) = h_{\ell}^H Q h_{\ell},
\]

where \( x_{\ell}(Q) \in \mathbb{R}_+ \) since \( Q \) is positive semidefinite. For all feasible transmit covariance matrices, a power gain-region of a single transmitter is a set that contains all power gain tuples achievable at the receivers. The power gain-region is defined as

\[
\Omega := \{(x_1(Q), \ldots, x_K(Q)) : Q \in \mathcal{S}\} \subset \mathbb{R}_+^{K},
\]

where \( \mathcal{S} \) is defined in (2). We are only interested in points that lie on the boundary of the gain-region \( \Omega \). The reason for this is, these points characterize extreme power gains achievable at the receivers. At these points, the transmitter cannot increase the power gain in one direction of the gain-region without decreasing the gain in any other direction. An important property of the gain-region is its convexity. Having this property is convenient for characterizing the points on its boundary using simple programming problems based on the Hyperplane Separation theorem [23, Theorem 1.3].

Lemma 1: The set \( \Omega \) is a compact and convex set.

Proof: The proof is provided in Appendix A.

Next, we formalize the boundary of the set \( \Omega \) following the definitions in [5]. There, these definitions were used to derive the solution of monotonic optimization problems [24], [23].
Fig. 2. An illustration of a two-dimensional gain-region and its upper boundaries in directions $e_1 = [1, 1]$, $e_2 = [1, -1]$, and $e_3 = [-1, 1]$.

**Definition 1:** A point $y \in \mathbb{R}^n_+$ is called upper boundary point of a compact convex set $C$ if $y \in C$ while the set

$$\{ y' \in \mathbb{R}^n_+ : y' > y \} \subset \mathbb{R}^n_+ \setminus C.$$  \hspace{1cm} (8)

The set of upper boundary points of $C$ is called the upper boundary of $C$ and is denoted by $\partial^+ C$. \hfill \square

Definition 1 describes only one boundary part of a compact convex set. The straightforward extension to describe all boundary parts of this set is to define its upper boundary in direction $e$, $e \in \{-1, +1\}^n$. For this purpose, we need first the following definition.

**Definition 2:** A vector $x$ dominates a vector $y$ in direction $e$, written as $x \geq^e y$, if $x_\ell e_\ell \geq y_\ell e_\ell$ for all $\ell$, $1 \leq \ell \leq n$, and the inequality has at least one strict inequality. \hfill \square

**Definition 3:** A point $y \in \mathbb{R}^n_+$ is called upper boundary point of a compact convex set $C$ in direction $e$ if $y \in C$ while the set

$$\{ y' \in \mathbb{R}^n_+ : y' \geq^e y \} \subset \mathbb{R}^n_+ \setminus C.$$  \hspace{1cm} (9)

We denote the set of upper boundary points in direction $e$ as $\partial^e C$. \hfill \square

An illustration of a two-dimensional power gain-region is given in Fig. 2 The direction vectors $e_1, e_2$, and $e_3$ refer to three different parts of the boundary. For the choice $e = 1$, the upper boundary in direction $e$ is the usual upper boundary as in Definition 1, i.e., $\partial^+ C = \partial^1 C$. 

April 27, 2010  \hspace{1cm} DRAFT
The important boundary part, and hence the important direction vector $e$, which the transmitter chooses for efficient operation depends on the network setting. A transmitter in an interference network is usually interested in maximizing its power gain at associated receivers. Moreover, for its efficient operation in the network, it should also be interested in minimizing the interference to the remaining receivers. The boundary that satisfies these requirements is identified by the direction vector that has all its components negative except to those receivers associated with the transmitter. For example, assume in Fig. 2 the transmitter is associated with receiver one and interferes on receiver two. Then, the important direction vector is $e_2$. This corresponds to the boundary part which includes the maximum achievable power gain at receiver one and also the minimum achievable power gain at receiver two. These extreme points correspond to MRT and ZF transmission strategies, respectively. According to Fig. 2, MRT achieves $\|h_1\|^2$ power gain at the first receiver and $x_2^*$ at the second receiver. On the other hand, ZF transmission achieves zero power gain at the second receiver and $x_1^0$ at the first receiver.

Since a single transmitter has at least one intended receiver, otherwise it will not operate, the direction vector with all components equal to $-1$ is not of interest. Therefore, we define the set of feasible directions as

$$\mathcal{E} := \{-1, 1\}^K \setminus \{-1\}^K.$$  (10)

Next, we study the transmission strategies that achieve the boundary points in $\partial_e \Omega, e \in \mathcal{E}$.

The relation between the number of existing receivers and the number of available antennas at the transmitter is important to distinguish whether power control is needed for efficient operation of the transmitter. This condition affects the number of required parameters to characterize efficient transmission strategies. First, we consider the case where the number of transmit antennas is greater than or equal to the number of existing receivers. For this case, a comprehensive study of the gain-region is given due to which we gain a link and some insights to a mathematical field of research in matrix analysis. Afterwards, we consider the case in which the number of antennas is strictly less than the number of receivers. This case is addressed more briefly since the tools needed for the analysis are similar to those in the first case.

A. The number of transmit antennas satisfies $N \geq K$

In this section, we assume $N \geq K$ and the channel vectors to the receivers are independently distributed. These assumptions lead to linear independence of the channel vectors with probability one. In this case, it

---

5Determining the transmitter’s important direction vector is provided later in Section IV.
is possible to achieve power gain on one receiver and simultaneously null the power gain at the remaining receivers. Therefore, the gain-region has boundary points that lie on each axis as the points \( x_0^0 \) and \( x_2^0 \) in Fig. 2. In the case that the number of antennas at the transmitter is strictly larger than the number of receivers, the gain-region has part of its boundary the points between \( x_1^0 \) and the origin. As a result, all interesting upper boundary points are achieved with full power.

**Lemma 2:** Transmit covariance matrices from the set
\[
\hat{S} := \{ Q : Q \succeq 0, \operatorname{tr}(Q) = 1 \},
\]
achieve all points in \( \partial^e \Omega, e \in E \).

**Proof:** The proof is provided in Appendix B.

Lemma 2 supports the result in [25], where it is shown that all Pareto efficient operating points in the MISO IC correspond to full power transmission strategies when the number of antennas is larger than or equal to the number of receivers. The power gain-region achieved with full power transmit covariance matrices is defined as
\[
\hat{\Omega} := \left\{ (x_1(Q), ..., x_K(Q)) : Q \in \hat{S} \right\} \subset \mathbb{R}_+^K.
\]

Since the power gain at a receiver \( \ell \) can equivalently be formulated as \( x_\ell(Q) = h_\ell^H Q h_\ell = \operatorname{tr}(Q h_\ell h_\ell^H) \), the set \( \hat{\Omega} \) is rewritten as
\[
\tilde{\Omega} = \left\{ (\operatorname{tr}(Q h_1 h_1^H), ..., \operatorname{tr}(Q h_K h_K^H)) : Q \succeq 0, \operatorname{tr}(Q) = 1 \right\}.
\]

This set is referred to in [26] as the *joint field of values* of the set of matrices \( h_1 h_1^H, ..., h_K h_K^H \). The set \( \tilde{\Omega} \) is compact and convex, and it is the convex hull of the *joint numerical range* of the matrices \( h_1 h_1^H, ..., h_K h_K^H \). The joint numerical range of a set of \( m \) matrices \( A_1, ..., A_m \) is defined as [27]
\[
\mathcal{W}(A_1, ..., A_m) := \left\{ (z^H A_1 z, ..., z^H A_m z) : z^H z = 1 \right\}.
\]

Convexity of the set \( \mathcal{W}(A_1, ..., A_m) \) is not always satisfied. Conditions for its convexity are studied in [28]. The next result shows that the boundary in any direction \( e \in E \) can be achieved with rank-1 transmit covariance matrices. This implies that the joint numerical range, if not convex, must have holes.

**Lemma 3:** Transmit covariance matrices from the set
\[
\tilde{S} := \{ Q : Q \in \tilde{S}, \operatorname{rank}(Q) = 1 \},
\]

\footnote{This is the reason for not initially assuming rank-1 transmit covariance matrices.}
achieve all points in \( \partial^e \hat{\Omega}, e \in \mathcal{E} \).

**Proof:** The proof is provided in Appendix C.

Lemma 3 supports the result in [29], where it is shown that single-stream beamforming is optimal in the MISO IC to achieve Pareto efficient operating points. Accordingly, efficient transmission strategies can be described by beamforming vectors. The next theorem characterizes the beamforming vectors that achieve the upper boundary of the set \( \hat{\Omega} \) in a specific direction \( e \).

**Theorem 1:** All upper boundary points of the set \( \hat{\Omega} \) in direction \( e \in \mathcal{E} \) can be achieved by

\[
w(\lambda) = v_{\max} \left( \sum_{\ell=1}^{K} \lambda_\ell e_\ell h_\ell h_\ell^H \right),
\]

with

\[
\lambda \in \Lambda := \left\{ \lambda \in [0, 1]^K : \sum_{\ell=1}^{K} \lambda_\ell = 1 \right\}.
\]

**Proof:** The proof is provided in Appendix D.

In case the largest eigenvalue of the Hermitian matrix in (16) has geometric multiplicity larger than one, i.e., there exist multiple linearly independent eigenvectors associated with that eigenvalue, we choose an eigenvector which lies in the span of \( H = [h_1, \ldots, h_K] \). This case can occur if the number of antennas is strictly greater than the number of receivers. The interesting observation from Theorem 1 is that all upper boundary points in direction \( e \) of the \( K \)-dimensional gain-region can be achieved by a parametrization using \( K - 1 \) real parameters.

In Fig. 3, a three dimensional gain-region is plotted where the transmitter uses three transmit antennas. The boundary points are calculated by generating the beamforming vectors characterized in Theorem 1 for each boundary part. For each boundary part, two real-valued parameters are required, and these are varied between zero and one with a step length of 0.02. The gain-region is shown to have a convex boundary. Moreover, since it is possible to null out the power gain at two receivers simultaneously, the boundary of the gain-region touches each axis in one point.

**B. The number of transmit antennas satisfies** \( N < K \)

In case the number of antennas at the transmitter is strictly less than the number of receivers, it is not possible for the transmitter to choose a full power transmission strategy which nulls out the power gain at all receivers except for one. Hence, the transmitter cannot perform a ZF strategy, and efficient operation of the transmitter involves reducing its transmission power. This is the reason why we study this case separately.
We start by assuming that the transmit covariance matrices are chosen from the set $\mathcal{S}$ in (2). The corresponding power gain-region is compact and convex according to Lemma 1. Hence the following programming problem, similar to the one formulated in (49) in Appendix C, achieves the boundary points of $\Omega$ in direction $e \in \mathcal{E}$:

$$\begin{align*}
\text{maximize} & \quad \sum_{\ell=1}^{K} \lambda_{\ell} e_{\ell} h_{\ell}^{H} Q h_{\ell} \\
\text{subject to} & \quad Q \succeq 0 \\
& \quad \text{tr}(Q) \leq 1,
\end{align*}$$

where $\lambda \in \Lambda$ is defined in (17). As in Appendix C, the problem in (18) can be equivalently written as

$$\begin{align*}
\text{maximize} & \quad \sum_{\ell=1}^{N} \mu_{\ell}(Q) \mu_{\ell} \left( \sum_{\ell=1}^{K} \lambda_{\ell} e_{\ell} h_{\ell}^{H} h_{\ell} \right) \\
\text{subject to} & \quad \mu_{\ell}(Q) \geq 0, \quad \text{for all } \ell = 1, \ldots, N \\
& \quad \sum_{\ell=1}^{N} \mu_{\ell}(Q) \leq 1.
\end{align*}$$

The solution of this problem is $\mu_{N}(Q) = p, p \in [0, 1]$, and $\mu_{\ell}(Q) = 0$ for $\ell \neq N$. Hence, the optimal transmit covariance matrices $Q$ are rank-1. In addition, the optimal power allocation is to allocate...
Fig. 4. An illustration of a three dimensional gain-region where the transmitter uses two antennas.

power only in the direction of the eigenvector corresponding to the largest eigenvalue of $Z$. Hence the formulation in Theorem 1 is still valid to determine the beamforming vectors that achieve the boundary points of $\Omega$. However, power control is to be applied on these beamforming vectors which is determined as follows. The impact of the assumption $N < K$ on the eigenvalues of $Z$ is that the largest eigenvalue can be negative, i.e., $Z$ can be negative definite. This is easily checked when $\lambda_\ell = 0$ whenever $e_\ell = +1$. In this case, the solution to problem (19) is to choose $p = 0$. In case the largest eigenvalue of $Z$ is zero, any feasible power allocation can be adopted, therefore, the power is to be varied between zero and one.

The power allocations that achieve the boundary points of the power gain-region $\Omega$ in direction $e \in E$ are summarized as follows:

$$p = \begin{cases} 
1 & \mu_N(Z) > 0 \\
[0, 1] & \mu_N(Z) = 0 \\
0 & \mu_N(Z) < 0 
\end{cases} \quad (20)$$

In Fig. 4 a three dimensional gain-region is plotted where two antennas are utilized at the transmitter. In comparison to the plot in Fig. 3 the region looks like a cone whose vertex is at the origin. The outermost boundary which is not flat is attained with full power transmission. The flat surfaces correspond to the case for which the transmission power is varied between zero and one.
IV. PARETO BOUNDARY CHARACTERIZATION

In this section, all transmitters in the set $\mathcal{T}$ are considered again. The analysis of the single transmitter case in the previous section builds the framework for the transmission strategies for each transmitter. Thus, each transmitter has an associated gain-region which is referred to by the corresponding subscript.

The performance measure of a system in an interference network is usually described by a utility function. This function depends on the power gains achieved by each transmitter on the corresponding receiver which builds the connection between the gains and utilities. The utility function of a receiver $\ell, \ell \in \mathcal{K}$, is defined by $u_\ell : \mathbb{R}_+^T \to \mathbb{R}_+$, where $T$ is the number of transmitters in the network. Next, we give properties of this utility function.

**Assumption 1:** The utility function $u_\ell$ has the following properties:

A. For each transmitter $k, k \in \mathcal{T}$, there is a set of receivers $\mathcal{K}(k) \subseteq \mathcal{K}$ such that for $\ell \in \mathcal{K}(k)$, $u_\ell$ is monotonically increasing in the power gain $x_{k,\ell}(Q_k)$, i.e.,

$$u_\ell(x_{1,\ell}(Q_1), ..., x_{T,\ell}(Q_T)) \leq u_\ell\left(x_{1,\ell}(\hat{Q}_1), ..., x_{k,\ell}(\hat{Q}_k), ..., x_{T,\ell}(Q_T)\right),$$

for $x_{k,\ell}(Q_k) \leq \hat{x}_{k,\ell}(\hat{Q}_k)$.

B. For each transmitter $k, k \in \mathcal{T}$, there is a set of receivers $\mathcal{K}(k) \subseteq \mathcal{K}$ such that for $\ell \in \mathcal{K}(k)$, $u_\ell$ is monotonically decreasing in the power gain $x_{k,\ell}(Q_k)$, i.e.,

$$u_\ell(x_{1,\ell}(Q_1), ..., x_{T,\ell}(Q_T)) \geq u_\ell\left(x_{1,\ell}(Q_1), ..., \hat{x}_{k,\ell}(\hat{Q}_k), ..., x_{T,\ell}(Q_T)\right),$$

for $x_{k,\ell}(Q_k) \leq \hat{x}_{k,\ell}(\hat{Q}_k)$.

The set $\mathcal{K}(k)$ is the set of receivers associated with transmitter $k$, whose utility is increased with more power gain from this transmitter. Increased power gain from transmitter $k$ reduces the utility of the receivers in the set $\mathcal{K}(k) = \mathcal{K} \setminus \mathcal{K}(k)$. Thus, the signals from this transmitter are regarded as interference on these receivers.

The **utility region** is the set of all utility tuples that can be achieved when the transmitters use transmission strategies from the set $\mathcal{S}$ defined in (2):

$$\mathcal{U} := \{(u_1(Q_1, ..., Q_T), ..., u_K(Q_1, ..., Q_T)) : Q_k \in \mathcal{S}, k \in \mathcal{T}\} \subset \mathbb{R}_+^K.$$  

(23)

The outer boundary of this region is called the **Pareto boundary**, because it consists of operating points for which it is impossible to improve one of the utilities, without simultaneously decreasing at least one of the other utilities. More precisely we define the **Pareto optimality** of an operating point as follows.
**Definition 4:** A utility tuple \((u_1, \ldots, u_K)\) is Pareto optimal if there is no other tuple \((u'_1, \ldots, u'_K)\) with \((u'_1, \ldots, u'_K) \succeq (u_1, \ldots, u_K)\) and \((u'_1, \ldots, u'_K) \neq (u_1, \ldots, u_K)\) (the inequality is component-wise). The set of Pareto optimal utility tuples is denoted by \(\mathcal{PB}\).

The Pareto boundary consists of optimal operating points in an interference network. The transmission strategies that achieve Pareto optimal points are characterized in the following theorem.

**Theorem 2:** All points of the Pareto boundary of the utility region \(\mathcal{U}\) can be achieved by beamforming vectors

\[
\mathbf{w}_k(\lambda) = p \mathbf{v}_{\text{max}} \left( \sum_{\ell=1}^{K} \lambda_k,\ell e_{k,\ell} \mathbf{h}_{k,\ell} \mathbf{h}_{k,\ell}^H \right),
\]

with \(\lambda_k \in \Lambda\) defined in (17) and

\[
e_{k,\ell} = \begin{cases} +1 & \ell \in \mathcal{K}(k) \\ -1 & \ell \in \overline{\mathcal{K}}(k) \end{cases},
\]

and

\[
p = \begin{cases} 1 & \mu_N(Z) > 0 \text{ or } N \geq K \\ [0,1] & \mu_N(Z) = 0 \text{ and } N < K \\ 0 & \mu_N(Z) < 0 \end{cases}.
\]

**Proof:** The proof is provided in Appendix E.

Theorem 2 characterizes the transmission strategies that achieve Pareto optimal points. The number of real-valued parameters, \(\Lambda\) and \(p\), that are required to characterize these beamforming vectors is: (i) \(T(K-1)\) in case no power control is needed, i.e., \(N \geq K\). (ii) \(TK\) in case power control is needed, i.e., \(N < K\). All these parameters take values between zero and one.

The direction vector in (25) can be determined since each transmitter \(k, k \in T\), knows its receiver sets, \(\mathcal{K}(k)\) and \(\overline{\mathcal{K}}(k)\). This direction vector specifies the choice of the important boundary of the transmitter’s power gain-region.

**A. Example Revisited**

We will continue with the example started in Section II. The utility of receiver one in (3) is monotonically increasing in the power gain of transmitter 11, and monotonically decreasing in the power gains of the remaining transmitters. Hence, according to properties A and B in Assumption 1 receiver one belongs
to the following sets: \{1\} ∈ \mathcal{K}(11), \{1\} ∈ \mathcal{K}(12) and \{1\} ∈ \mathcal{K}(2). Similarly, from (4) and (5), we have \{2\} ∈ \mathcal{K}(12), \{2\} ∈ \mathcal{K}(11), \{2\} ∈ \mathcal{K}(12), \{3\} ∈ \mathcal{K}(2), \{3\} ∈ \mathcal{K}(11) and \{3\} ∈ \mathcal{K}(12). Accordingly, the receiver sets of each transmitter are stated as, \mathcal{K}(11) = \{1\}, \mathcal{K}(11) = \{2, 3\}, \mathcal{K}(12) = \{2\}, \mathcal{K}(12) = \{1, 3\}, \mathcal{K}(2) = \{2, 3\}, and \mathcal{K}(2) = \{1\}. These sets are shown in Fig. 1. The choice of the beamforming vectors with unit norm for transmitters 11, 12 and 2 which achieve Pareto optimal points are characterized in Theorem 2.

Since the number of antennas at the transmitters is equal to the number of receivers, the power allocation \(p_1\) and \(p_2\) on the beamforming vectors \(w_{11}\) and \(w_{12}\), respectively, is varied such \(p_1 + p_2 = 1\). The power allocation parameters \(p_1\) and \(p_2\) can be expressed as \(q\) and \(1 - q\), respectively, with \(q ∈ [0, 1]\). The characterization in Theorem 2 leads to the following nonnegative real-valued parameters:

- For transmitter 11: \(λ_{11,1}, λ_{11,2}, λ_{11,3}\), with \(λ_{11,1} + λ_{11,2} + λ_{11,3} = 1\).
- For transmitter 12: \(λ_{12,1}, λ_{12,2}, λ_{12,3}\), with \(λ_{12,1} + λ_{12,2} + λ_{12,3} = 1\).
- For transmitters 11 and 12: \(q ∈ [0, 1]\).
- For transmitter 2: \(λ_{2,1}, λ_{2,2}, λ_{2,3}\), with \(λ_{2,1} + λ_{2,2} + λ_{2,3} = 1\).

All seven parameters are in the interval \([0, 1]\), and the plot in Fig. 5 is obtained by varying these in a grid with 0.05 step length. The points obtained include the points that lie on the Pareto boundary of the utility region which satisfy Definition 4. Points corresponding to weak Pareto optimality are not included in the plot. Weak Pareto optimality is defined as in Definition 4 except that the corresponding inequality is
not strict. Weak Pareto optimal points complete the shape of the utility region by orthogonally projecting each Pareto optimal point in Fig. 5 onto the coordinate surfaces.

In the next section, we discuss two special applications of the developed framework.

V. APPLICATIONS

A. Multiple-input Single-output Interference Channel

The $K$-users MISO IC consists of $K$ transmitter-receiver pairs, i.e., the set of transmitters $\mathcal{T}$ can be equivalently described by the set of receivers $\mathcal{K}$. Each receiver has an intended transmitter, while all other transmitters induce interference on this receiver. We consider single-user decoding, i.e., interference is treated as additive noise at each receiver. For a given set of beamforming vectors $\{w_1, ..., w_K\}$, the following rate is achievable at receiver $k$, by using codebooks approaching Gaussian ones:

$$r_k(w_1, ..., w_K) = \log_2 \left( 1 + \frac{|w_k^H h_{kk}|^2}{\sum_{\ell \neq k} |w^H_{\ell} h_{\ell k}|^2 + \sigma^2} \right).$$

(27)

This utility function satisfies properties A and B in Assumption 1 which leads to the following receiver sets for each transmitter $k, k \in \mathcal{T}$: $\mathcal{K}(k) = \{k\}$, and $\mathcal{K}(\{k\}) = \mathcal{K}\setminus \{k\}$. All points on the Pareto boundary of the achievable rate-region of the MISO IC can be reached by beamforming vectors as given in Theorem. In [4], a characterization of the beamforming vectors that achieve the Pareto boundary of the achievable rate-region is provided by a complex linear combination of the MRT and ZF strategies. The parametrization in Theorem is real-valued with the same number of parameters, thus of lower dimension. Next, we identify special operation points of each transmitter.

1) Maximum ratio transmission for transmitter $k \in \mathcal{T}$ corresponds obviously to the parameters

$$\lambda_{k,\ell} = \begin{cases} 
1 & k = \ell \\
0 & \text{otherwise}
\end{cases}. $$

(28)

This transmission strategy is the unique Nash equilibrium (NE) strategy of each transmitter $k$, which is the outcome of a noncooperative game between the users.

2) Zero-forcing transmission is characterized with the following lemma:

**Lemma 4:** Zero Forcing transmission for transmitter $k \in \mathcal{T}$ corresponds to the parameters

$$\lambda_{k,\ell} : \begin{cases} 
= 0 & k = \ell \\
> 0 & \text{otherwise}
\end{cases}. $$

(29)

**Proof:** The proof is provided in Appendix [F].

April 27, 2010 DRAFT
3) Two-user MISO IC special case has an appealing form in terms of the parametrization of the Pareto boundary of the achievable rate-region. In [4, Corollary 2], real-valued parametrization of the transmission strategies that achieve Pareto optimal operating points are given as a linear combination of the MRT and ZF strategies. These two extreme strategies have the interpretation of a transmitter being either selfish or altruistic [30], [25]. The parameterized strategies for transmitter \( k \in \{1, 2\} \) are given as [4, Corollary 2]

\[
\begin{align*}
    w_k(\hat{\lambda}_k) &= \frac{\hat{\lambda}_k w_k^\text{MRT} + (1 - \hat{\lambda}_k) w_k^\text{ZF}}{\|\hat{\lambda}_k w_k^\text{MRT} + (1 - \hat{\lambda}_k) w_k^\text{ZF}\|},
\end{align*}
\]

where \( \hat{\lambda}_k \in [0, 1] \), and the extreme beamforming vectors are given as

\[
    w_k^\text{MRT} = \frac{h_{kk}}{\|h_{kk}\|},
\]

\[
    w_k^\text{ZF} = \frac{\Pi_{h_{kk}^\perp} h_{kk}}{\|\Pi_{h_{kk}^\perp} h_{kk}\|}, \quad k \neq \ell.
\]

Next, we prove that the parametrization in (30) has the same set of strategies as in Theorem 2 for \( K = 2 \). For this case, the eigenvalue equation for the hermitian matrix in Theorem 1 is written as

\[
\begin{align*}
    (\lambda_1 h_{11} h_{11}^H - (1 - \lambda_1) h_{12} h_{12}^H) w_1 &= \mu w_1. \quad (33)
\end{align*}
\]

Equivalently, the above equation can be formulated to

\[
\begin{align*}
    \lambda_1 \|h_{11}\|^2 h_{11} h_{11}^H w_1 - (1 - \lambda_1) \|h_{12}\|^2 h_{12} h_{12}^H w_1 &= \mu w_1. \quad (34)
\end{align*}
\]

Adding \((1 - \lambda_1) \|h_{12}\|^2 w_1\) on both sides of the equation leads to

\[
\begin{align*}
    \left(\lambda_1 \|h_{11}\|^2 \Pi_{h_{11}} + (1 - \lambda_1) \|h_{12}\|^2 \Pi_{h_{12}}\right) w_1 = (\mu + (1 - \lambda_1) \|h_{12}\|^2) w_1. \quad (35)
\end{align*}
\]

The LHS of the equation states that the eigenvector corresponding to the largest eigenvalue is a linear combination of its orthogonal projection on \( h_{11} \) and the orthogonal projection onto the orthogonal complement of \( h_{12} \). Since the largest eigenvalue \( \mu \) is larger or equal to zero, then the weight on the RHS of the equation is always positive. Hence, the optimal set of beamforming vectors can be equivalently characterized by (30).

The Pareto boundary of a three user MISO IC rate-region is plotted in Fig. 6 for SNR = -10 dB and in Fig. 7 for SNR = 30 dB. The generated points correspond to the beamforming vectors characterized in Theorem 2. The real-valued parameters are varied in a 0.05 step length. Since we are only interested in revealing the Pareto boundary of the achievable rate region, we do the following. We randomly choose
ten thousand generated points and remove all points that are dominated by these. In other words, for a randomly chosen rate tuple, all points corresponding to joint rates less than the chosen one are removed. In addition, the algorithm provided in [31] is applied. The algorithm reduces the number of plotted points in removing the points that are not visible from the viewed angle of the figure. This algorithm further reduces the complexity of rendering the generated points.
B. Noncooperative Underlay Cognitive Radio

Motivated by the concept of null-shaping constraints in cognitive radio scenarios [32], the next result gives an alternative characterization of efficient transmission strategies. In order to be able to fulfill the null-shaping constraints, the number of applied antennas at the transmitter has to be greater than or equal to the number of primary receivers. We assume there exists virtual single-antenna primary receivers, and consider the efficient design of the null-shaping constraints.

**Corollary 1**: Assume \( N \geq K \) and define the matrix
\[
Z_k(\lambda_k) = \left[ z_1(\lambda_k), \ldots, z_{|\mathcal{K}(k)|}(\lambda_k), z_{N-|\mathcal{K}(k)|+1}(\lambda_k), \ldots, z_{N-1}(\lambda_k) \right],
\]  
where
\[
z_i(\lambda_k) = v_i \left( \sum_{\ell=1}^{K} \lambda_{k,\ell} e_{k,\ell} h_{k,\ell} h_{k,\ell}^H \right),
\]
with \( \lambda_k \in \Lambda \) defined in (17) and \( e_{k,\ell} \) defined in (25). All points on the Pareto boundary of the utility region \( \mathcal{U} \) can be reached by beamforming vectors
\[
w_k(\lambda_k) = \frac{\Pi_{\|Z_k(\lambda_k)h_{k,\ell}\|^2}}{\Pi_{\|Z_k(\lambda_k)h_{k,\ell}\|^2}},
\]
where \( \ell \in \mathcal{K}(k) \).

**Proof**: The proof is provided in Appendix G.

In Corollary 1, the design of the null-shaping constraints is given in (36), and the efficient transmission strategies are given in (38). Here, \( K - 1 \) null-shaping constraints are to be applied on each transmitter. The number of required real-valued parameters in Corollary 1 is the same as in Theorem 2. Hence, the complexity of parameterizing the Pareto boundary is the same.

The form of the transmission strategy in (38) has a relevant interpretation. Consider a MISO IC setting as in the previous section. In this case, rewriting (38) for transmitter \( k \) gives
\[
w_k(\lambda_k) = \frac{\Pi_{\|Z_k(\lambda_k)h_{kk}\|^2}}{\Pi_{\|Z_k(\lambda_k)h_{kk}\|^2}},
\]
This transmission strategy is MRT which also satisfies null-shaping constraints in \( Z_k(\lambda_k) \). The MRT strategy is the unique noncooperative strategy of transmitter \( k \), and corresponds to the NE strategy in game theoretical terms [33]. Through the design of the null-shaping constraints in Corollary 1 all Pareto optimal points of the utility region are characterized by transmission strategies that are NEs. These NEs have to satisfy the characterized null-shaping constraints. The interesting observations are as follows. Null-shaping constraints are sufficient to characterize the Pareto boundary. Moreover, the transmitters are
required to be noncooperative in order to achieve efficient operating points with these constraints [34], [2]. This result is convenient since noncooperative operation of the systems is desired when designing distributed resource allocation schemes.

VI. CONCLUSIONS

This work presents a characterization of Pareto efficient transmission strategies. By studying the single transmitter’s power gain-region, the properties of efficient transmission are acquired. We prove that the boundary of the gain-region is convex and achieved with single-stream beamforming. Due to the convexity of the boundary of the gain-region, the efficient transmit beamforming vectors can be parameterized by real-valued parameters. We determine and distinguish the conditions under which power control is needed for efficient transmission. When the number of antennas at the transmitter is greater than or equal to the number of existing receivers, we show that full power transmission achieves all boundary points of the gain region. In this case, the parameterizations of efficient beamforming vectors requires \( K - 1 \) real-valued parameters between zero and one. When the number of antennas at the transmitter is strictly less than the number of receivers, we characterize the transmission strategies for which power control is required. For this case, an additional real-valued parameter is needed that varies the power level at the transmitter. We apply the single-transmitter framework to the multiple-transmitter case. On determining the important boundary part of each transmitter’s gain-region, all Pareto efficient beamforming vectors are characterized. This parameterizations simplifies the design of Pareto efficient resource allocation schemes.

ACKNOWLEDGMENT

The authors would like to thank Yiu Tung Poon, Christian Scheunert, Johannes Lindblom, and ELEFTHERIOS KARIPIDIS for interesting discussions.

APPENDIX A

PROOF OF LEMMA I

Note that \( \Omega \) does not correspond to the joint field of values [25] which is known to be convex and compact. Therefore, we provide a proof of the convexity of \( \Omega \) and show that it is bounded and closed, thus compact. It is simple to show that \( \Omega \) is bounded because the power gain at the \( \ell \)th receiver has a finite maximum which is achieved when the transmitter performs MRT to that receiver, i.e.,

\[
x_{\ell}(Q) \leq x_{\ell}\left(\frac{h_{\ell}h_{\ell}^H}{h_{\ell}^Hh_{\ell}}\right) = \|h_{\ell}\|^2.
\]
Therefore, the box described by the set
\[ \mathcal{Y} := \left\{ x : 0 \leq x_k \leq \|h_k\|^2 \right\}, \]  
contains \( \Omega \), i.e., \( \Omega \subset \mathcal{Y} \) as illustrated in Fig. 2. The set \( \Omega \) is closed because the feasible set of transmission strategies, \( Q \succeq 0 \) and \( \text{tr}(Q) \leq 1 \), is compact and convex. Since every pre-image of a closed set is closed for continuous functions [35], follows that \( \Omega \) is a closed set.

It remains to prove that \( \Omega \) is convex. For any two points \( x(Q_x) \in \Omega \) and \( x(Q_y) \in \Omega \), we prove that \( x(Q_z) \in \Omega \), where \( x(Q_z) = tx(Q_x) + (1 - t)x(Q_y) \) and \( t \in [0, 1] \). Any component of \( x(Q_z) \) is written as
\[ x_t(Q_z) = tx_t(Q_x) + (1 - t)x_t(Q_y) = th_t^H Q_z h_t + (1 - t)h_t^H Q_y h_t = h_t^H (tQ_x + (1 - t)Q_y) h_t. \]  
(42)

Hence, the transmit covariance matrices that achieve the line segment between \( x(Q_x) \) and \( x(Q_y) \) are given as
\[ Q_z(t) = tQ_x + (1 - t)Q_y. \]  
(43)

Accordingly, since \( Q_x \) and \( Q_y \) are positive semidefinite, then \( Q_z(t) \) is positive semidefinite for all \( t \in [0, 1] \). In addition, \( Q_z(t) \) fulfills the trace constraint, \( \text{tr}(Q_z(t)) \leq 1 \), for all \( t \in [0, 1] \) since
\[ \text{tr}(Q_z) = \text{tr}(tQ_x + (1 - t)Q_y) = t\text{tr}(Q_x) + (1 - t)\text{tr}(Q_y) \leq 1. \]  
(44)

Therefore, \( x(Q_z(t)) \) also lies in \( \Omega \) for all \( t \in [0, 1] \). Hence, the set \( \Omega \) is a compact and convex set.

**Appendix B**

**Proof of Lemma 2**

In order to prove that the transmit covariance matrices from \( \hat{S} \) in (11) achieve points on the boundary of the set \( \Omega \) in direction \( e, e \in \mathcal{E} \), we show that for any transmit covariance matrix \( P \) with \( \text{tr}(P) < 1 \), a transmit covariance matrix \( Q \) from \( \hat{S} \) can be constructed in which \( x(Q) \) dominates \( x(P) \) in direction \( e \) according to Definition 2.

Assume \( e_\ell = +1 \), we can construct \( Q \) as
\[ Q = P + (1 - \text{tr}(P)) \frac{\Pi_{\hat{Z}}^\perp h_\ell h_\ell^H \Pi_{\hat{Z}}^\perp}{\|\Pi_{\hat{Z}}^\perp h_\ell h_\ell^H \Pi_{\hat{Z}}^\perp\|}, \]  
(45)

where \( Z = [h_1, ..., h_{\ell-1}, h_{\ell+1}, ..., h_K] \). Since \( K \leq N \), the dimension of the null space of \( Z \) is greater or equal to one, therefore the projection \( \Pi_{\hat{Z}}^\perp \) is not equal to the zero vector. Clearly, \( Q \) in (45) is in the set \( \hat{S} \). The power gain achieved with \( Q \) at the \( k \)th receiver, \( k \in \mathcal{K} \), is
\[ x_k(Q) = h_k^H \left( P + (1 - \text{tr}(P)) \frac{\Pi_{\hat{Z}}^\perp h_\ell h_\ell^H \Pi_{\hat{Z}}^\perp}{\|\Pi_{\hat{Z}}^\perp h_\ell h_\ell^H \Pi_{\hat{Z}}^\perp\|} \right) h_k. \]  
(46)
We distinguish two cases. For \( k \neq \ell \), the power gain at the \( k \)th receiver is

\[
x_k(Q) = h_k^H \left( P + (1 - \text{tr}(P)) \Pi_Z h_k h_k^H \Pi_Z \right) h_k
\]

\[
= h_k^H Ph_k + (1 - \text{tr}(P)) h_k^H \frac{\Pi_Z h_k h_k^H \Pi_Z}{\| \Pi_Z h_k h_k^H \Pi_Z \|} h_k = x_k(P). \tag{47}
\]

This implies that the gain at receiver \( k \neq \ell \) has not changed. For \( k = \ell \), the power gain at that receiver is

\[
x_\ell(Q) = h_\ell^H \left( P + (1 - \text{tr}(P)) \Pi_Z h_\ell h_\ell^H \Pi_Z \right) h_\ell
\]

\[
= h_\ell^H Ph_\ell + (1 - \text{tr}(P)) h_\ell^H \frac{\Pi_Z h_\ell h_\ell^H \Pi_Z}{\| \Pi_Z h_\ell h_\ell^H \Pi_Z \|} h_\ell \tag{48}
\]

\[
= h_\ell^H Ph_\ell + (1 - \text{tr}(P)) \left| h_\ell^H \frac{\Pi_Z h_\ell}{\| \Pi_Z h_\ell h_\ell^H \Pi_Z \|} \right|^2 > x_\ell(P).
\]

According to the above results we can construct \( Q, Q \in \hat{S} \), such that \( x(Q) \geq x(P) \) for any given \( P \), with \( \text{tr}(P) < 1 \). Therefore, the boundary set \( \partial e \Omega, e \in \mathcal{E} \), can be achieved with transmit covariance matrices that fulfill the total power constraint with equality.

APPENDIX C

PROOF OF LEMMA

Since the set \( \hat{\Omega} \) is convex and compact, the boundary in direction \( e, \partial e \Omega, e \in \mathcal{E} \), can be achieved using the Supporting Hyperplane theorem [23, Theorem 1.5] by the following programming problem

\[
\text{maximize } \sum_{\ell=1}^{K} \lambda_{\ell} e_\ell^H Q h_\ell
\]

subject to \( Q \succeq 0 \),

\[
\text{tr}(Q) = 1,
\]

where \( \lambda \in \Lambda \) defined in (17). The objective in (49) can be written as

\[
\sum_{\ell=1}^{K} \lambda_{\ell} e_\ell^H Q h_\ell = \sum_{\ell=1}^{K} \lambda_{\ell} e_\ell \text{tr}(Q h_\ell h_\ell^H) = \text{tr} \left( Q \sum_{\ell=1}^{K} \lambda_{\ell} e_\ell h_\ell h_\ell^H \right) \tag{50}
\]

\[
= \text{tr}(QZ) \leq \sum_{\ell=1}^{N} \mu_{\ell}(Q) \mu_{\ell}(Z)
\]
where the last inequality holds according to the von Neumann trace inequality of product of matrices \[36\]. Hence, the problem in (49) can be equivalently written as

\[
\begin{align*}
\text{maximize} & \quad \sum_{\ell=1}^{N} \mu_{\ell}(Q) \mu_{\ell}(Z) \\
\text{subject to} & \quad \mu_{\ell}(Q) \geq 0, \quad \text{for all } \ell = 1, \ldots, N \\
& \quad \sum_{\ell=1}^{N} \mu_{\ell}(Q) = 1.
\end{align*}
\]

(51)

The solution of this problem is $\mu_N(Q) = 1$ and $\mu_\ell(Q) = 0$ if $\ell \neq N$. Thus, $Q$ is rank-1 and the transmit covariance matrices from $\tilde{S}$ achieve the boundary of the region $\Omega$.

**APPENDIX D
PROOF OF THEOREM 1**

As in the proof of Lemma 3, the convex boundary of the set $\hat{\Omega}$ in direction $e$ can be characterized by the solution of the following programming problem

\[
\begin{align*}
\text{maximize} & \quad \sum_{\ell=1}^{K} \lambda_{\ell} e_{\ell} |w^H h_{k\ell}|^2 \\
\text{subject to} & \quad \|w\|^2 = 1,
\end{align*}
\]

(52)

The objective function in (52) can be rewritten as

\[
y(w) = \sum_{\ell=1}^{K} \lambda_{\ell} e_{k,\ell} |w^H h_{k\ell}|^2 = w_k^H \left( \sum_{\ell=1}^{K} \lambda_{\ell} e_{k,\ell} h_{k\ell} h_{k\ell}^H \right) w_k.
\]

(53)

Note that the matrix $Z$ in (53) is not necessarily positive semidefinite because the directional vector $e$ can contain negative components. However, it is Hermitian and therefore, the solution to (52) is the eigenvector which corresponds to the largest eigenvalue of $Z$.

**APPENDIX E
PROOF OF THEOREM 2**

The proof is by contradiction. Assume that there exists a beamforming vector $w_k$ for transmitter $k$ which is not at the upper boundary of its power gain-region in direction $e_k$, i.e.

\[
x_k(w_k) \notin \partial^{e_k} \Omega_k,
\]

(54)

April 27, 2010 DRAFT
and achieves a point on the Pareto boundary of \( U \), i.e.
\[
(u_1(w_1, ..., w_T), ..., u_K(w_1, ..., w_T)) \in \mathcal{PB},
\]
according to Definition 4. By assuming (54), we can find another beamforming vector \( v_k \) with \( \|v_k\|^2 = 1 \) and the property
\[
x_{k,\ell}(w_k)e_{k,\ell} \leq x_{k,\ell}(v_k)e_{k,\ell}, \quad \text{for all } \ell \in \mathcal{K}.
\]
(56)

Next we distinguish two cases for the inequality in (56) corresponding to \( \ell \in \bar{K}(k) \) or \( \ell \in K(k) \).

1) Assume the inequality in (56) is strict for \( \ell \in \bar{K}(k) \) with \( e_{k,\ell} = +1 \) as given in (25), then \( x_{k,\ell}(w_k) < x_{k,\ell}(v_k) \). The gains to all other receivers are to stay unchanged such that \( x_{k,j}(w_k) = x_{k,j}(v_k) \) for \( j \neq \ell \). Then,
\[
u_{\ell}(w_1, ..., w_T) < u_{\ell}(w_1, ..., v_\ell, ..., w_T)
\]
holds according to property A in Assumption 1 in section IV. This result contradicts (55).

2) Assume the inequality in (56) is strict for \( j \in K(k) \) with \( e_{k,j} = -1 \). Then, (56) changes to \( x_{k,j}(w_k) > x_{k,j}(v_k) \). Assuming \( x_{k,\ell}(w_k) = x_{k,\ell}(v_k) \) for \( \ell \neq j \), then
\[
u_j(w_1, ..., w_T) < u_j(w_1, ..., v_j, ..., w_T)
\]
holds according to property B in Assumption 1 in section IV. This result contradicts (55).

Therefore, all points on the Pareto boundary belong to the boundary of the gain-region in direction specified in (25). The transmission strategies that achieve the boundary of the gain-region are characterized in Theorem 1 in Section III-A and for the optimal power control in (20) in Section III-B, which lead to the formulation in Theorem 2.

APPENDIX F

PROOF OF LEMMA 4

The zero-forcing condition on the beamforming vector of transmitter \( k \), denoted as \( w_{ZF}^k \), \( k \in \mathcal{K} \), is
\[
x_{k,\ell}(w_{ZF}^k) : \begin{cases} 
\geq 0 & k = \ell \\
0 & \text{otherwise}
\end{cases}
\]
(57)

It is possible to fulfill the condition in (57) if \( N \geq K \) and all channel vectors from transmitter \( k \) to all receivers are linearly independent. Here, we give a direct proof. Assume the conditions in (29) hold, we have to show that these conditions lead to beamforming vectors that satisfy (57). In order to do this, we
have to study the matrix in (24) whose eigenvector, corresponding to the largest eigenvalue, determines
the used beamforming vector. Define the matrix $M_k$, as

$$M_k = \sum_{\ell=1}^{K} \lambda_{k,\ell} e_{k,\ell} h_{k,\ell} h_{k,\ell}^H$$

$$= \underbrace{\lambda_{k,k} h_{k,k} h_{k,k}^H}_{A_k} - \sum_{\ell \neq k} \lambda_{k,\ell} h_{k,\ell} h_{k,\ell}^H, \quad \text{(58)}$$

where the direction vector $e$ is specified as for the MISO IC application. According to the conditions in
(29), $A_k$ in (58) is equal to zero, and hence $M_k$ is negative semidefinite. The largest eigenvalue of $M_k$
is therefore zero since $N \geq K$, and we can write

$$\left( \sum_{\ell \neq k} \lambda_{k,\ell} h_{k,\ell} h_{k,\ell}^H \right) w_k = 0. \quad \text{(59)}$$

Since all channel vectors are linearly independent, the eigenvector associated with the largest eigenvalue
produces zero gain on any of the interference channel vectors. It is clearly seen that if $\lambda_{\ell} = 0$ for any
$\ell \neq k$, then the largest eigenvector does not necessarily produce zero gain on this receiver.

**Appendix G**

**Proof of Corollary 1**

We prove that the gains achieved by the beamforming vectors in (38) are equal to the gains achieved
by the beamforming vectors given in (24). Define the matrix $M_k$, with the direction vector $e_{k,\ell}$ given
in (25), as

$$M_k = \sum_{\ell=1}^{K} \lambda_{k,\ell} e_{k,\ell} h_{k,\ell} h_{k,\ell}^H$$

$$= \sum_{\ell \in \mathcal{K}(k)} \lambda_{k,\ell} h_{k,\ell} h_{k,\ell}^H + \sum_{\ell \in \mathcal{K}(k)} -\lambda_{k,\ell} h_{k,\ell} h_{k,\ell}^H \quad \text{(60)}$$

The matrices $M_k$, $A_k$ and $B_k$ are Hermitian matrices of size $N \times N$. The eigenvalues of $M_k$ are real
and we always consider them ordered in nondecreasing order, i.e.,

$$\mu_1(M_k) \leq \mu_2(M_k) \leq \ldots \leq \mu_N(M_k). \quad \text{(61)}$$

$A_k$ consists of sum of positive semidefinite matrices. Hence, $A_k \succeq 0$ and rank $(A_k) \leq |\mathcal{K}(k)|$, i.e.,

$$0 \leq \mu_{|\mathcal{K}(k)|+1}(A_k) \leq \mu_N(A_k). \quad \text{(62)}$$
and
\[ \mu_1(\mathbf{A}_k) = \ldots = \mu_{N-|\mathcal{K}(k)|}(\mathbf{A}_k) = 0. \]  
(63)

\( \mathbf{B}_k \) consists of the sum of the negative of positive semidefinite matrices. Hence, \( \mathbf{B}_k \preceq 0 \) and \( \text{rank}(\mathbf{B}_k) \leq |\mathcal{K}(k)| \), which leads to the following properties on the eigenvalues:
\[ \mu_1(\mathbf{B}_k) \leq \ldots \leq \mu_{|\mathcal{K}(k)|}(\mathbf{B}_k) \leq 0, \]
(64)

and
\[ \mu_{|\mathcal{K}(k)|+1}(\mathbf{B}_k) = \ldots = \mu_N(\mathbf{B}_k) = 0. \]
(65)

Next, we study the eigenvalues of \( \mathbf{M}_k = \mathbf{A}_k + \mathbf{B}_k \). According to Weyl's inequality of the eigenvalues of the sum of Hermitian matrices \([37, \text{Theorem } 4.3.7]\) the following properties are gained:
\[ \mu_{N-|\mathcal{K}(k)|}(\mathbf{M}_k) \leq \mu_{N-|\mathcal{K}(k)|}(\mathbf{A}_k) + \mu_N(\mathbf{B}_k) = 0 \]
(66)
\[ \mu_{|\mathcal{K}(k)|+1}(\mathbf{M}_k) \geq \mu_1(\mathbf{A}_k) + \mu_{|\mathcal{K}(k)|+1}(\mathbf{B}_k) = 0 \]
(67)

The eigenvalues of \( \mathbf{M}_k \) are ordered in nondecreasing order. Therefore, the following eigenvalues of \( \mathbf{M}_k \) are always equal to zero,
\[ \mu_{|\mathcal{K}(k)|+1}(\mathbf{M}_k) = \ldots = \mu_{N-|\mathcal{K}(k)|}(\mathbf{M}_k) = 0. \]
(68)

In addition, the smallest \( |\mathcal{K}(k)| \) eigenvalues of \( \mathbf{M}_k \) are nonpositive.

If the dimension of space is larger than the number of receivers, i.e., \( N \geq |\mathcal{K}| \), then there would be at least \( N - |\mathcal{K}(k)| - |\mathcal{K}(k)| \) eigenvalues of \( \mathbf{M}_k \) that are zero. For the eigenvectors corresponding to those eigenvalues, the eigenvalue equation is written as
\[ \left( \sum_{\ell \in \mathcal{K}(k)} \lambda_{k,\ell} \mathbf{h}_{k\ell} \mathbf{h}_{k\ell}^H + \sum_{\ell \in \mathcal{K}(k)} -\lambda_{k,\ell} \mathbf{h}_{k\ell} \mathbf{h}_{k\ell}^H \right) \mathbf{v}_i = 0, \]
(69)

for all \( i = |\mathcal{K}(k)| + 1, \ldots, N - |\mathcal{K}(k)| \). Then, for all \( \ell \in \mathcal{K} \),
\[ (\lambda_{k,\ell} \mathbf{h}_{k\ell} \mathbf{h}_{k\ell}^H) \mathbf{v}_i = 0, \quad \text{for all } i = |\mathcal{K}(k)| + 1, \ldots, N - |\mathcal{K}(k)|. \]
(70)

The set of eigenvectors of \( \mathbf{M}_k \), \( \{\mathbf{v}_1, \ldots, \mathbf{v}_N\} \), form an orthonormal set, i.e. \( \|\mathbf{v}_i\| = 1 \) for all \( i = 1, \ldots, N \) and \( \mathbf{v}_i^H \mathbf{v}_j = 0 \) for \( i \neq j \). Therefore, we can write
\[ \sum_{\ell=1}^N \mathbf{v}_\ell \mathbf{v}_\ell^H = \mathbf{I} \]
(71)
\[ \mathbf{v}_N(\lambda) \mathbf{v}_N^H(\lambda) = \mathbf{I} - \sum_{\ell=1}^{N-1} \mathbf{v}_\ell(\lambda) \mathbf{v}_\ell^H(\lambda) = \mathbf{I} - \mathbf{G}_k(\lambda) \mathbf{G}_k^H(\lambda) = \mathbf{\Pi}_{\mathcal{K}(k)}, \]
where \( G_k(\lambda) = [v_1(\lambda), \ldots, v_{N-1}(\lambda)] \). Let the matrix \( Z_k(\lambda) \) consist of the eigenvectors of \( G_k(\lambda) \) excluding the eigenvectors that satisfy (70), i.e.,

\[
Z_k(\lambda) = \left[ v_1(\lambda), \ldots, v_{|\mathcal{K}(k)|+1}(\lambda), v_{N-|\mathcal{K}(k)|+1}(\lambda), \ldots, v_{N-1}(\lambda) \right],
\]

then for any \( g \in \mathbb{C}^N \) we can write

\[
g^H \frac{\Pi_{Z_k(\lambda)} h_{k\ell}}{\|\Pi_{Z_k(\lambda)} h_{k\ell}\|} = g^H \frac{\Pi_{G_k(\lambda)} h_{k\ell}}{\|\Pi_{G_k(\lambda)} h_{k\ell}\|} = g^H \frac{v_N(\lambda) v_N^H(\lambda) h_{k\ell}}{\|v_N(\lambda) v_N^H(\lambda) h_{k\ell}\|} \]

\[
= \left| g^H v_N(\lambda) \right|^2,
\]

where \( \ell \in \mathcal{K} \). Hence, the same power gains are achieved with the beamforming vectors \( \frac{\Pi_{Z_k(\lambda)} h_{k\ell}}{\|\Pi_{Z_k(\lambda)} h_{k\ell}\|} \) as with \( v_N(\lambda) \).

**REFERENCES**


