Capacity of Fading Gaussian Channel with an Energy Harvesting Sensor Node

R Rajesh and Vinod Sharma

Abstract—Network life time maximization is becoming an important design goal in wireless sensor networks. Energy harvesting has recently become a preferred choice for achieving this goal as it provides near perpetual operation. We study such a sensor node with an energy harvesting source and compare various architectures by which the harvested energy is used. We find its Shannon capacity when it is transmitting its observations over a fading AWGN channel with perfect/no channel state information provided at the transmitter. We also obtain the capacity with a finite energy buffer via Markov decision theory.

Keywords: Energy harvesting, sensor networks, energy buffer, network life time, Markov decision theory.

I. INTRODUCTION

Sensor nodes deployed for monitoring a random field are characterized by limited battery power, computational resources and storage space. Often, once deployed the battery of these nodes are not changed. Hence when the battery of a node is exhausted, the node dies. When sufficient number of nodes die, the network may not be able to perform its designated task. Thus the life time of a network is an important characteristic of a sensor network ([1]) and it depends on the life time of a node.

The network life time can be improved by reducing the energy intensive tasks, e.g., reducing the number of bits to transmit ([2], [3]), making a node to go into power saving modes (sleep/listen) periodically ([4]), using energy efficient routing ([5], [6]), adaptive sensing rates and multiple access techniques ([7]). Network life time can also be increased by suitable architectural choices like the tiered system ([8]) and redundant placement of nodes ([9]).

Recently, energy harvesting techniques ([10], [11]) are gaining popularity for increasing the network life time. Energy harvester harnesses energy from environment or other energy sources (e.g., body heat) and converts them to electrical energy. Common energy harvesting devices are solar cells, wind turbines and piezo-electric cells, which extract energy from the environment. Among these, harvesting solar energy through photo-voltaic effect seems to have emerged as a technology of choice for many sensor nodes ([11], [12]). Unlike for a battery operated sensor node, now there is potentially an infinite amount of energy available to the node. However, the source of energy and the energy harvesting device may be such that the energy cannot be generated at all times (e.g., a solar cell). Furthermore the rate of generation of energy can be limited. Thus the new design criteria may be to match the energy generation profile of the harvesting source with the energy consumption profile of the sensor node. If the energy can be stored in the sensor node then this matching can be considerably simplified. But the energy storage device may have limited capacity. The energy consumption policy should be designed in such a way that the node can perform satisfactorily for a long time, i.e., energy starvation at least, should not be the reason for the node to die. In [10] such an energy/power management scheme is called energy neutral operation.

We study the Shannon capacity of such an energy harvesting sensor node transmitting over a fading Additive White Gaussian Noise (AWGN) Channel. We provide the capacity under various energy buffer constraints and perfect/no channel state information at the transmitter (CSIT). We show that the capacity achieving policies are also throughput optimal ([13]). We also study generalizations of this system with inefficiencies in the energy storage.

In the following we survey the relevant literature. Energy harvesting in sensor networks are studied in ([14] and [15]). Conditions for energy neutral operation for various models of energy generation and consumption are provided in [10]. A practical solar energy harvesting sensor node prototype is described in [16]. In [17] the authors study optimal sleep-wake cycles for event detection probability in sensor networks.

Energy harvesting architectures can be often divided into two major classes. In Harvest-use (HU), the harvesting system directly powers the sensor node and when sufficient energy is not available the node is disabled. In Harvest-Store-Use (HSU) there is a storage component that stores the harvested energy and also powers the sensor node. The storage can be single or double staged ([10], [16]).

Various throughput and delay optimal energy management policies for energy harvesting sensor nodes are provided in [13]. These energy management policies in [13] are extended in various directions in [18] and [19]. For example, [18] provides some efficient MAC policies for energy harvesting nodes. In [19] optimal sleep-wake policies are obtained for such nodes. Further more, [20] considers jointly optimal routing, scheduling and power control policies for networks of energy harvesting nodes.

In a recent contribution, optimal energy allocation policies over a finite horizon and fading channels are studied in [21]. An information theoretic analysis of an energy harvesting communication system is provided in [22].

The capacity of a fading Gaussian channel with channel
state information (CSI) at the transmitter and receiver and at the receiver alone are provided in [23]. It was shown that optimal power adaptation when CSI is available both at the transmitter and the receiver is ‘water filling’ in time. An excellent survey on fading channels is provided in [24].

We consider the problem of determining the information-theoretic capacity of an energy harvesting sensor node transmitting its observation over a fading AWGN channel. We also compute the capacity when the energy buffer is finite or there may be inefficiency in energy storage. We address the infinite buffer case without fading in [25]. Our results can be useful in the context of green communication ([26], [27]) when solar and/or wind energy can be used by a base station ([28]).

The paper is organized as follows. Section II describes the system model. Section III provides the capacity for a single node under idealistic assumptions. We show that the capacity achieving policy is also throughput-optimal. Section IV obtains the capacity with inefficiencies in the energy storage system. Section V takes into account the finite buffer and Section VI concludes the paper.

II. MODEL AND NOTATION

In this section we present our model for a single energy harvesting sensor node.

![Fig. 1. The model](image)

We consider a sensor node (Fig. 1) which is sensing and generating data to be transmitted to a central node via a discrete time AWGN channel. We assume that transmission consumes most of the energy in a sensor node and ignore other causes of energy consumption (this is true for many low quality, low rate sensor nodes ([12])). The sensor node is able to replenish energy by $Y_k$ at time $k$ via an energy harvesting source. The energy available at the node at time $k$ is $E_k$. This energy is stored in an energy buffer with an infinite capacity.

The node uses energy $T_k$ at time $k$ which depends on $E_k$ and $T_k \leq E_k$. The process $\{E_k\}$ satisfies

$$E_{k+1} = (E_k - T_k)^+ + Y_k. \quad (1)$$

We assume that $\{Y_k\}$ is stationary ergodic. This assumption is general enough to cover most of the stochastic models developed for energy harvesting. Often the energy harvesting process is time varying (e.g., solar cell energy harvesting will depend on the time of day). Such a process can be approximated by piecewise stationary processes. As in [13], we can indeed consider $\{Y_k\}$ to be periodic stationary ergodic.

The encoder receives a message $S$ from the node and generates an $n$-length codeword to be transmitted on the fading AWGN channel. We assume flat fast fading. At time $k$ the channel gain is $H_k$ and takes values in $\mathcal{H}$. For simplicity we assume $\mathcal{H}$ to be countable. It can be easily extended to the case with general $\mathcal{H}$. The sequence $\{H_k\}$ is assumed independent identically distributed (iid), independent of the energy generation sequence $\{W_k\}$. The channel output at time $k$ is $W_k = H_kX_k + N_k$ where $X_k$ is the channel input at time $k$ and $\{N_k\}$ is iid Gaussian noise with zero mean and variance $\sigma^2$. The decoder receives $Y^n = (Y_1, \ldots, Y_n)$ and reconstructs $S$ such that the probability of decoding error is minimized. Also, the decoder has perfect knowledge of the channel state $H_k$.

We obtain the information-theoretic capacity of this channel. This of course assumes that there is always data to be sent at the sensor node. This channel is essentially different from the usually studied systems in the sense that the transmit power and coding scheme depend on the energy available in the energy buffer at that time. In later part of the paper we will relax some of the assumptions made in this section.

III. CAPACITY FOR THE IDEAL SYSTEM

In this section we obtain the capacity of the channel with an energy harvesting node under ideal conditions: infinite energy buffer, energy consumed in transmission only, no inefficiencies in storage and perfect CSI at the transmitter. Several of these assumptions will be removed in later sections. We always assume that the receiver has perfect CSI.

The system starts at time $k = 0$ with an empty energy buffer and $E_k$ evolves with time depending on $Y_k$ and $T_k$. Thus $\{E_k, k \geq 0\}$ is not stationary and hence $\{T_k\}$ may also not be stationary. In this setup, a reasonable general assumption is to expect $\{T_k\}$ to be asymptotically stationary. One can further generalize it to be Asymptotically Mean Stationary (AMS), i.e.,

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} P[T_k \in A] = \mathcal{P}(A) \quad (2)$$

exists for all measurable $A$. In that case $\mathcal{P}$ is also a probability measure and is called the stationary mean of the AMS sequence ([29]).

If the channel input $\{X_k\}$ is AMS, then it can be easily shown that for the fading AWGN channel $\{(X_k, W_k), k \geq 0\}$ is also AMS. Thus the channel capacity of our system is ([29])

$$C = \sup_{P_x} I(X; W) = \sup_{P_x} \limsup_{n \to \infty} \frac{1}{n} I(X^n, W^n) \quad (3)$$

where $\{X_n\}$ is an AMS sequence, $X^n = (X_1, \ldots, X_n)$ and the supremum is over all possible AMS sequences $\{X_n\}$. For a fading AWGN channel capacity achieving $X_k$ is zero mean with variance $T_k$ where $T_k$ depends on the power control policy used and is assumed AMS. Then $E[T] \leq E[Y]$ where $E[T]$ is the mean of $T$ under its stationary mean. If $R < C$ then one can find a sequence of codewords with code length $n$ and rate $R$ such that the average probability of error goes to zero as $n \to \infty$ ([29]).
In the following we obtain $C$ in (3) for our system.

We need the following definition.

**Pinskar Information Rate** (29): Given an AMS random process $\{X_n, W_n\}$ with standard alphabets (Borel subsets of Polish spaces) $A_X$ and $A_{W}$, the Pinskar information rate is defined as $I^*(X; W) = \sup_{q}I(q(X); r(W))$ where the supremum is over all quantizers $q$ of $A_X$ and $r$ of $A_{W}$.

It is known that, $I^*(X; W) \leq T(X; W)$. The two are equal if the alphabets are finite. The capacity in (3) involve $T(X; W)$. But $I^*(X; W)$ is easier to handle.

We also need the following Lemma for proving the achievable delivery rate of the capacity of the channel. This Lemma holds for $I^*$ but not for $T$.

**Lemma 1** (29 Lemma 6.2.2): Let $\{X_n, W_n\}$ be AMS with distribution $P$ and stationary mean $\mathcal{P}$. Then $I^p_r(X; W) = I^p_r(X; W)$

**Theorem 1** For the energy harvesting system with perfect CSIT $C = 0.5 E_H[\log(1 + H^2 T^*(H)/\sigma^2)]$ where

$$
T^*(H) = \left(\frac{1}{H_0} - \frac{1}{H}\right)^+,
$$

and $H_0$ is chosen such that $E_H[T^*(H)] = E[Y]$.

**Proof:** Achievability: Let $T^*_k = T^*(H_k)$ with $T^*$ defined in (4) with $E[T^*(H)] = E[Y] - \epsilon$ where $\epsilon > 0$ is a small constant. Since $\{H_k\}$ is iid, $\{T^*_k\}$ is also iid. We take $T_k = \min(E_k, T^*_k)$. Then from (30), $E_k \rightarrow \infty$ as $T^*$ is upper bounded, $\lim_{n \rightarrow \infty} \sup_{k \geq n}[T_k - T^*(H_k)] \rightarrow 0$ a.s. Infact $T_k = T^*(H_k)$ for all $k$ after some time.

Let $\{X^*_k\}$ be iid with mean zero and variance one. The channel codeword $X_k = \sqrt{T_k}X^*_k$ is an AMS sequence with the stationary mean being the distribution of $T^*(H_k)X^*_k$. Then since AWGN channel under consideration is AMS ergodic (29), $(X, W) \overset{d}{=} \{(X_k, W_k), k \geq 1\}$ is AMS ergodic.

By using Lemma 1

$$
I^*(X; W) = \sup_{q, r} \lim_{k \rightarrow \infty} \frac{1}{K} I(q(X_k); r(W_k)) = I^*(X, W)
$$

Corresponding to the iid sequence $\{X_k, W_k\}$ with $X_k \sim \mathcal{N}(0, E[Y] - \epsilon)$ and $W_k$ is the corresponding channel output.

Also, since the mutual information between two random variables is the limit of the mutual information between their quantized version (11), $I^*(X, W) = I(X, W) = 0.5 E_H[\log(1 + H^2 T^*(H)/\sigma^2)]$. We can show that as $\epsilon \rightarrow 0$, $0.5 E_H[\log(1 + H^2 T^*(H)/\sigma^2)] \rightarrow C$ defined in the statement of the theorem.

**Converse Part:** Let there be a sequence of codebooks for our system with rate $R$ and average probability of error going to 0 as $n \rightarrow \infty$. If $\{X_k(S), k = 1, ..., n\}$ is a codeword for message $S \in \{1, ..., 2^nR\}$ then $1/n \sum_{k=1}^{n} X_k(S)^2 \leq 1/n \sum_{k=1}^{n} Y_k \leq E[Y] + \delta$ for any $\delta > 0$ with a large probability for all $n$ large enough. Hence by the converse in the fading AWGN channel case (23), $\lim_{K \rightarrow \infty} \frac{1}{K} I(X^k; W^k) \leq 0.5 E_H[\log(1 + H^2 T^*(H)/\sigma^2)]$ for $T^*(H)$ given in (4).

Combining the direct part and converse part completes the proof.

Thus we see that the capacity of this fading channel is same as that of a node with average power constraint $E[Y]$ and instantaneous power allocated according to 'water filling' power allocation, i.e., the hard energy constraint of $E_k$ at time $k$ does not affect its capacity. The capacity achieving signaling for our system is independent zero mean Gaussian $X_k$ with $X_k^2 = \min(E_k, T^*(H_k))$, $T^*(H)$ defined in (4).

When no CSI is available at the transmitter, take $T_k(H) = \min(E_k, E[Y] - \epsilon)$ and as in Theorem 1 this approaches the capacity of $0.5 E_H[\log(1 + H^2 E[Y]/\sigma^2)]$ in this case always power $E[Y] - \epsilon$ is used irrespective of the fade state.

If there is no energy buffer to store the harvested energy (Harvest-Use) and no CSIT provided, the best one can do is to use $T_k(H) = Y_k$, irrespective of the channel state for all $k$. Then the capacity of this AWGN channel is $C = 0.5 E_H[\log(1 + E[Y]/\sigma^2)]$. Providing perfect CSIT does not improve the capacity as there is no buffer to store the energy and use later on. Also, $0.5 E_H[\log(1 + E[Y]/\sigma^2)] \leq 0.5 E_H[\log(1 + E[Y]/\sigma^2)] \leq 0.5 E_H[\log(1 + H^2 T^*(H))/\sigma^2]$. The first inequality is by concavity of log function and the second by the power allocation. Thus, having energy buffer to store the harvested energy and perfect CSIT strictly increases the capacity of the system under ideal conditions.

In (13), a system with a data buffer at the node which stores data sensed by the node before transmitting it over the fading AWGN channel, is considered. The stability region (for the data buffer) for the 'no energy-buffer' and 'infinite energy-buffer' corresponding to the harvest-use and harvest-store-use architectures with perfect/no CSIT are provided. The throughput optimal policies in (13) are the same as the Shannon capacity achieving energy management policies provided here for both the buffer architectures and perfect/no CSIT. Also the capacity is the same as the maximum throughput obtained in the data-buffer case in (13) for both buffer architectures and perfect/no CSIT.

An advantage of the above capacity/throughput optimal policies is that they are easy to implement online and do not require exact information about $E_k$ or the queue lengths.

**IV. CAPACITY WITH ENERGY INEFFICIENCIES**

In this section we make our model more realistic by taking into account the inefficiency in storing energy in the energy buffer and the leakage from the energy buffer (15) for HSU architecture.

We assume that if energy $Y_k$ is harvested at time $k$, then only energy $\beta_1 Y_k$ is stored in the buffer and energy $\beta_2$ gets leaked in each slot where $0 \leq \beta_1 < 1$ and $0 \leq \beta_2 < 1$. Then (1) becomes

$$
E_{k+1} = ((E_k - T_k) - \beta_2) + \beta_1 Y_k.
$$

The energy can be stored in a super capacitor and/or in a battery. For a supercapacitor, $\beta_1 \geq 0.95$ and for the Ni-MH battery (the most commonly used battery) $\beta_1 \sim 0.7$. The leakage $\beta_2$ for the super-capacitor as well as the battery is
close to 0 but for the super capacitor it may be somewhat larger.

In this case, similar to the achievability of Theorem 1 we can show that the following rates are achievable in the no CSIT and Perfect CSIT case respectively

\[ R_{S-NC} = 0.5 E_H[\log(1 + \frac{H^2(\beta_1 E[Y] - \beta_2)}{\sigma^2})] \]

(6)

\[ R_{S-CS} = 0.5 E_H[\log(1 + \frac{H^2(\beta_1 T(H) - \beta_2)}{\sigma^2})] \]

(7)

where \( T(H) \) is a power allocation policy such that the maximized subject to \( E_H[T(H)] \leq E[Y] \) This policy is neither capacity achieving nor throughput optimal [13].

An achievable rate when there is no buffer is

\[ R = 0.5 E_{HY}[\log(1 + \frac{H^2 Y}{\sigma^2})]. \]

(8)

In Harvest-Use architecture since the energy harvested is used immediately, there is no loss due to storage inefficiency and leakage. Thus the achievable rate does not depend on \( \beta_1, \beta_2 \). Also, it does not depend on the availability of CSIT. Hence, unlike Section III, [3] may be larger than [6] and [7] for certain range of parameter values. We will illustrate this by an example.

Another achievable policy for the system with an energy buffer with storage inefficiencies is to use the harvested energy \( Y_k \) immediately instead of storing in the buffer. The remaining energy after transmission is stored in the buffer. We call this Harvest-Use-Store (HUS) architecture. For this case, [5] becomes

\[ E_{k+1} = ((E_k + \beta_1(Y_k - T_k))^{+} - (T_k - Y_k))^{+} - \beta_2)^{+}. \]

(9)

Find the largest constant \( c \) such that \( \beta_1 E[(Y_k - c)^{+}] \geq E[(c - Y_k)^{+}] + \beta_2 \). Of course \( c < E[Y] \). When there is no CSIT, this is the largest \( c \) such that taking \( T_k = min(c - \delta, E_k) \), where \( \delta > 0 \) is any small constant, will make \( E_k \rightarrow \infty \) a.s. and hence \( T_k \rightarrow c \) a.s. Then, as in Theorem 1, we can show that

\[ R_{US-NC} = 0.5 E_H[\log(1 + \frac{H^2 c}{\sigma^2})] \]

(10)

is an achievable rate.

When there is perfect CSIT, 'water filling' power allocation can be done subject to average power constraint of \( c \) and the achievable rate is

\[ R_{US-CS} = 0.5 E_H[\log(1 + \frac{H^2 T^*(H)}{\sigma^2})] \]

(11)

where \( T^*(H) \) is the 'water filling' power allocation with \( E[T^*(H)] = c \).

Equation [5] approximates the system where we have only rechargeable battery while [9] approximates the system where the harvested energy is first stored in a supercapacitor and after initial use transferred to the battery.

Let \( \{X_k\} \) be an AMS signalling scheme with \( \mathcal{P} \) as its stationary mean. Then we can show that \( E_{\mathcal{P}}[X^2] \leq c \) where \( c \) is defined below [9]. Then again from the converse argument used in proof of Theorem 1 we can show that

\[ \limsup_{n \to \infty} \frac{1}{n} I(X^n; W^n) \leq 0.5 E_H[\log(1 + \frac{T^*(H)H^2}{\sigma^2})] \]

where \( T^*(H) \) is defined as in [4] with \( E[T^*(H)] = c \).

When there is no CSIT \( T(H) \) will not depend on \( H \) and hence for \( E_{\mathcal{P}}[X^2] = c \) we take \( T(H) = c \) and \( \{X_k\} \) iid distributed as \( \mathcal{N}(0, c) \).

Hence capacity achieving distribution is Gaussian with transmit power given by \( c \) when there is no CSIT and \( T^*(H) \) when there is CSIT.

We illustrate the achievable rates mentioned above via an example.

A. Example 1

Let \( \{Y_k\} \) process be iid taking values in \{2, 6\} with probability \{0.6, 0.4\}. We take the loss due to leakage \( \beta_2 = 0 \). The fade states are iid taking values in \{0.2, 1, 1.5\} with probability \{0.4, 0.5, 0.1\}. In Figure 2 we compare the various architectures discussed in this section for varying storage efficiency \( \beta_1 \).

Fig. 2. Rates for various architectures

From the figure it can be seen that if the storage efficiency is very poor it is better to use the HU policy than HUS when there is no CSIT. This requires no storage buffer and has a simpler architecture. However HUS gives good performance compared to both the cases. For \( \beta = 1 \), the HUS policy and HUS policy are the same for both perfect CSIT and no CSIT. Unlike the ideal system, the HUS (which uses infinite energy buffer) performs worse than the HU (which uses no energy buffer) when storage efficiency is poor when there is no CSIT. When there is perfect CSIT, the HUS performs the best, like in the no CSIT case.
Thus if we judiciously use a combination of a super capacitor and a battery with perfect CSIT one may obtain a better performance.

V. CAPACITY WITH FINITE BUFFER

In the previous sections, we considered the cases when there is infinite energy buffer or when there is no buffer at all. However, in practice often there is a finite energy buffer to store the harvested energy. Then in general the capacity achieving $\{T_k\}$ will not be available in closed form. The optimal policy finds a sequence $\{T_k\}$ that maximizes $\limsup_{n \to \infty} \frac{1}{n} I(X^n, W^n)$. It can be found via the Markov decision theory ([32]). For finite buffer (FB) (i.e, $E_k$ takes values in a finite set) the optimal policy always exists and can be computed via policy/value iteration algorithms ([32]).

Let the maximum storage capacity of the battery be $N$ units. In the Markov decision theory terminology, when there is no CSIT, the state of the system is the energy $E_k$ in the buffer, where $E_k \leq N$. The randomness in the system is due to the energy arrival process $Y_k$. The set of actions at each state corresponds to the power $T_k$ used in transmission. The transmitted power $T_k \leq E_k$ for the HSU system and $T_k \leq E_k + Y_k$ for the HUS system. The reward obtained at time $k$ is the rate at which data is transmitted using power $T_k$ and is $E[g(T_k)]$ where the expectation is over the fade states and $g(x) = 0.5 \log(1 + x/\sigma^2)$.

However when perfect CSIT is provided the state of the system is the product space of energy $E_k$ in the buffer and the fade state $H_k$. Also, the reward obtained at time $k$ is $g(T_k(H_k))$.

We solve both these problems by value iteration and the optimal policy is obtained that maximizes the expected reward.

In the following we consider finite buffer cases with buffer size $N$ and action $T_k$ takes values in integers. $Y_k$ takes values in $\{2, 6\}$ with probability $\{p, 1-p\}$. The fade states are the same as in Example 1.

In Fig.3 we compare both the HUS and HSU architectures for the perfect CSIT and no CSIT case with $N = 15$. The zero buffer and infinite buffer cases are also given for comparison.

From Fig.3 we find that the HU architecture performs the worst. When no CSIT is provided, the HSU and the HUS architectures for $N = 15$ are close and also closer to the infinite buffer case. However, when CSIT is provided, HUS outperforms the HU architecture. Also, the capacity for the HUS is not very close to the infinite buffer case.

In Fig.4 we evaluate the capacity with varying buffer sizes for HSU and HUS architectures when perfect CSIT is provided.

It can be seen from Fig.4 that the HSU performs bad as compared to HUS when buffer size is small. This is due to the fact that in HSU the harvested energy can be stored only up to the maximum size of the buffer and rest of the harvested energy is wasted. However as the buffer size increases the gap between both reduces and both architectures achieve the same performance of the infinite buffer case.

VI. CONCLUSIONS

In this paper the Shannon capacity of an energy harvesting sensor node transmitting over a fading AWGN Channel is provided. It is shown that the capacity achieving policies are also throughput optimal. The capacity achieving policy is also provided when there are inefficiencies in energy storage. Capacity with various storage architectures and finite buffer are also obtained.

REFERENCES


