Dynamic optimal portfolios benchmarking the stock market

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Abstract

The paper investigates dynamic optimal portfolio strategies of utility maximizing portfolio managers in the presence of risk constraints. Especially we consider the risk, that the terminal wealth of the portfolio falls short of a certain benchmark level which is proportional to the stock price. This risk is measured by the Expected Utility Loss. We generalize the findings of our previous papers to this case. Using the Black-Scholes model of a complete financial market and applying martingale methods, analytic expressions for the optimal terminal wealth and the optimal portfolio strategies are given. Numerical examples illustrate the analytic results.

Keywords: optimal portfolio, dynamic strategy, shortfall risk, benchmarking, martingale method

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1 Introduction

A typical problem in applied stochastic finance deals with optimal strategies for portfolios consisting of risky stocks and a risk-free bond. Given a finite planning horizon \([0, T]\) and starting with some initial endowment the aim is to maximize the expected utility of the terminal wealth of the portfolio by optimal selection of the proportions of the portfolio wealth invested in stocks and bond, respectively. We consider this problem in a dynamic setting and assume a continuous-time financial market allowing for permanent trading and rebalancing the portfolio. Then the portfolio proportions have to be found for every time \(t\) up to \(T\).

This problem has been solved in the context of the Black-Scholes model of a complete financial market, see e.g. Cox and Huang [5], Karatzas, Lehoczky and Shreve [13, 14]. Here, the portfolio can contain shares of a risk-free bond and of stocks whose prices follow a geometric Brownian motion.

Following the optimal portfolio strategy leads (by definition) to the maximum expected utility of the terminal wealth. Nevertheless, the terminal wealth is a random variable with a distribution which is often extremely skew and shows considerable probability in regions of small values of the terminal wealth. This means that strategy leading to the the optimal terminal wealth may exhibit large so-called shortfall risks. By the term shortfall we denote the event, that the terminal wealth falls short of a certain benchmark value.

Often portfolio managers are evaluated relative to a benchmark. A natural aim of an investor is to compete with the money market. Here, his portfolio manager should be evaluated relative to the result of an investment in a bond bearing a given risk-free interest rate. In this case the benchmark is a fixed and non-random quantity. On the other hand, investors have the aim to compete with the stock market, an market index or another portfolio. In this case the portfolio manager has to be evaluated relative to a benchmark which depends on the random dynamics of the risky assets of the financial market.

In order to incorporate shortfall risks of the above mentioned type into the optimization it is necessary to quantify them by using appropriate risk measures. Let us denote the wealth of the portfolio at time \(t\) by \(X_t\). Further, let \(Q\) be the (possibly random) benchmark value. Then the shortfall risk consists in the random event \(C = \{X_T < Q\}\).

Since the aim is to maximize the expected utility of the terminal wealth \(X_T\) one can also compare the utilities of \(X_T\) and of the benchmark \(Q\). Let \(u : [0, \infty) \to \mathbb{R} \cup \{-\infty\}\) denote a utility function which is assumed to be strictly increasing and strictly concave on \((0, \infty)\). Realizations of \(X_T\) with \(u(X_T)\) below the target utility \(u(Q)\) are those of an unacceptable shortfall. Then the random event \(C\) can also be written as \(C = \{X_T < Q\} = \{u(X_T) < u(Q)\}\).

Defining the random variable \(G = G(X_T, Q) = u(X_T) - u(Q)\) we have \(C = \{G < 0\}\). The random variable \(G\) can be interpreted as the utility gain of the terminal wealth relative to the benchmark. In order to quantify the shortfall risk we assign to the random variable
a real-valued risk measure \( R(G) \). Throughout this paper we restrict to the special risk measure
\[
R(G) = \text{EUL}(G) := E \left( \left( u(X_T) - u(Q) \right)^- \right)
\]
and call it \textit{Expected Utility Loss} (EUL).

Other definitions of risk measures use the Value at Risk (VaR) which is closely related to the shortfall probability
\[
R(G) = P(G < 0) = P(u(X_T) < u(Q)) = P(X_T < Q).
\]

Further risk measures can be found in the class of coherent measures introduced by Artzner, Delbaen, Eber and Heath [1] and Delbaen [6]. These are measures possessing the properties of monotonicity, subadditivity and positive homogeneity and the translation property. EUL and VaR do not belong to this class, since the first violates the translation property while the latter is not subadditive.

We refer to Basak, Shapiro et.al. [2, 3] and our paper [12], where VaR-based risk measures are used as constraints of portfolio optimization problems. In our papers [10, 12] we consider the case of a deterministic benchmark \( Q = q \) and use as an additional risk measure the Expected Loss \( E \left( [X_T - q]^- \right) \). The papers by Emmer, Klüppelberg, Korn [8] and by Dmitrasinovic-Vidovic, Lari-Lavassani, Li and T. Ware [7] consider the case \( u(x) = x \), i.e. the maximization of \( E X_T \) under the presence of risk constraints containing deterministic benchmarks.

An extension of this paper to financial markets with incomplete information about the drift parameter of the underlying Black-Scholes model can be found in [11]. An summary of our work is also contained in [9].

The paper is organized as follows. Section 2 introduces basic notation for the considered Black-Scholes model of the financial market and formulates the portfolio optimization problem. Thereby we restrict to the case of a financial market with only one risky asset. In Section 3, the unconstrained portfolio optimization problem is solved using martingale methods. Section 4 applies this procedure to the portfolio optimization in presence of a risk constraint. First, the case of an portfolio manager benchmarking the money market, where the benchmark is a fixed non-random quantity, is considered in Subsection 4.1. Then, Subsection 4.2 deals with the case of a random benchmark which is proportional to the result of an investment in a pure stock portfolio. Finally, Section 5 presents some numerical results.

2 The portfolio optimization problem

We consider a continuous-time economy with finite horizon \([0, T]\) which is built on a filtered probability space \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbf{P})\) on which is defined a one-dimensional Brownian motion \( W \). We assume that all stochastic processes are adapted to \((\mathcal{F}_t)\), the augmented filtration generated by \( W \). It is assumed through this paper that all inequalities as well as equalities hold \( \mathbf{P} \)-almost surely. Moreover, it is assumed that all stated processes are well defined without giving any regularity conditions ensuring this, since our focus is a characterization problem.
Financial investment opportunities are given by an instantaneously risk-free money market account and a risky stock as in the Black-Scholes model [4]. We suppose the money market provides a constant interest rate $r$. The stock price $S$ is represented by a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

(2.1)

where the drift parameter $\mu$ and the volatility $\sigma$ are assumed to be constants. For the sake of simplicity we set $S_0 = 1$. Moreover, throughout this paper we assume $\mu > r$. From the economic point of view this is a very natural assumption. It allows a shorter representation of some of the subsequent results. For $\mu \leq r$ similar results can be obtained.

The dynamic market completeness implies the existence of a unique state price density process $H_t$, given by

$$dH_t = -H_t(r dt + \kappa dW_t), \quad H_0 = 1,$$

(2.2)

where $\kappa := (\mu - r)/\sigma$ is the market price of risk in the economy. This quantity can be regarded as the driving economic parameter in a portfolio managers dynamic investment problem.

We assume that a portfolio manager in this economy is endowed at time zero with an initial wealth of $x > 0$. He chooses an investment policy $\theta$, where $\theta_t$ denotes the fraction of wealth invested in the stock at time $t$. The portfolio process $\theta_t$ is assumed to be self-financing so that the portfolio manager’s wealth process $X = X^\theta$ follows

$$dX^\theta_t = [r + \theta_t(\mu - r)]X^\theta_t dt + \theta_t\sigma X^\theta_t dW_t, \quad X^\theta_0 = x.$$  

(2.3)

At time $t = T$ the portfolio manager reaches the terminal wealth $X_T$. The portfolio process is assumed to be admissible in the following sense.

**Definition 2.1**

Given $x > 0$, we say that a portfolio process $\theta = (\theta_t)_{t \in [0,T]}$ is admissible at $x$, and write $\theta \in \mathcal{A}(x)$, if $\theta$ is $(\mathcal{F}_t)$-adapted, $\int_0^T \theta^2_s ds < \infty$ and if the wealth process $X^\theta_t$ starting at $X^\theta_0 = x$ satisfies $X^\theta_t \geq 0$, $0 \leq t \leq T$.

In this economy, the portfolio manager is assumed to derive from the terminal wealth $X_T$ a utility $u(X_T)$ and he is looking to maximize the expected utility by choosing an optimal strategy from the set of admissible strategies.

**The dynamic problem**

Find a strategy $\theta^* \in \mathcal{A}(x)$ that solves

$$\max_{\theta \in \mathcal{A}(x)} E[u(X^\theta_T)].$$

(2.4)

Thereby, the utility function $u : [0, \infty) \to \mathbb{R} \cup \{-\infty\}$ is assumed to be strictly increasing as well as strictly concave and twice continuously differentiable on $(0, \infty)$. It satisfies $\lim_{x \to 0} u'(x) = \infty$ and $\lim_{x \to \infty} u'(x) = 0$. 
3 The unconstrained optimization problem

Without additional restrictions such as risk management, the maximization problem (2.4) was solved in the case of a complete market, by Cox and Huang [5] and by Karatzas, Lehocky and Shreve [13] using martingale and duality approaches. The method consists of converting the dynamic optimization problem described in (2.4) into a static optimization problem of finding an optimal terminal wealth. Via a representation problem, one gets the optimal strategy associated with this optimal terminal wealth.

From Itô’s Formula it follows that the process $H_t X^\theta_t$ is a supermartingale which implies that the so called budget constraint

$$E[H_T X^\theta_T] \leq x$$

is satisfied for every $\theta \in \mathcal{A}(x)$. This means that the expected discounted terminal wealth cannot exceed the initial wealth. Here, the state price density $H_t$ serves as a discounting process.

In contrast to the dynamic problem, where the portfolio manager is required to maximize expected utility from terminal wealth over a set of processes, in a first step the static problem is considered. Here, the portfolio manager has the advantage to maximize only over a set of random variables.

Let us denote

$$B(x) := \{ \xi \geq 0 : \xi \text{ is } \mathcal{F}_T - \text{measurable and } E[H_T \xi] \leq x \}.$$

**The static problem**

Find a random variable $\xi^* \in B(x)$ that solves

$$\max_{\xi \in B(x)} E[u(\xi)]. \quad (3.1)$$

In a second step, the optimal strategy is found as the solution of the representation problem.

**The representation problem**

Given $\xi^* \in B(x)$ that solves (3.1), find an admissible strategy $\theta^* \in \mathcal{A}(x)$ such that $X^\theta^* = \xi^*$.

For $y > 0$, we denote by $I(y) := (u')^{-1}(y)$ the inverse function of the derivative of the utility function. Theorem 3.1 stated in [15] solves the static optimization problem (3.1).

**Theorem 3.1**

Consider the portfolio problem (2.4). Let $x > 0$ and $y$ be a solution of $E[H_T I(y H_T)] = x$. Then there exists for $\xi^* = \xi^N := I(y H_T)$ a self-financing portfolio process $\theta^* = \theta^N$, $t \in [0,T]$, such that

$$\theta^N_t \in \mathcal{A}(x), \quad X^\theta^N_T = \xi^N,$$

and the portfolio process solves the dynamic problem (2.4).
The strategy \( \theta^N \) is referred to as the normal strategy.

The representation problem can be solved using the fact that for \( \theta_t = \theta^N_t \) the supermartingale \( H_t X_t^\theta \) is a martingale, since the optimal terminal wealth fulfills the budget constraint with equality and we have \( x = E \left[ H_0 X_0^\theta \right] = E \left[ H_T X_T^\theta \right] \). The Markov property of solutions of stochastic differential equations allows the optimal wealth process before the horizon \( X^\theta_t \) to be written as a function of \( H_t \), for which we apply Itô’s Formula. By equating coefficients with the wealth process (2.3), one gets the optimal portfolio.

**Example 3.2** In this example we assume that the portfolio manager has a utility function with a constant relative risk aversion (CRRA) \( \gamma \). This parameter \( \gamma \) is contained in the definition of the so-called CRRA-utility function which is given by

\[
u(z) = \begin{cases} \frac{z^{1-\gamma}}{1-\gamma}, & \gamma \in (0, \infty) \setminus \{1\}, \\ \ln z, & \gamma = 1. \end{cases}
\]  

(3.2)

on \( (0, \infty) \) and \( u(0) = -\infty \). According to Theorem 3.1, the static problem (3.1) has the optimal solution \( \xi^N = I(y^N H_T) \) where \( I(z) = z^{-\frac{1}{\gamma}} \) is the inverse function of the derivative of the utility function \( u(\cdot) \) and

\[
y^N := \frac{1}{x^\gamma} e^{(1-\gamma)(r+\frac{\kappa^2}{2})T}.
\]

Let \( X_t^N \) be the optimal solution before the horizon. Equations (2.2) and (2.3) and Itô’s Lemma imply that the process \( H_t X_t^N \) is a \( \mathcal{F}_t \)-martingale, i.e., \( X_t^N = E \left[ \frac{H_T}{H_t} X_T^N | \mathcal{F}_t \right] \).

Here, the optimal terminal wealth \( X_T^N \) is given by Theorem 3.1 as \( X_T^N := \xi^N = I(y^N H_T) \). We apply Markov Property of the solution \( H_t \) of Equation (2.2) to compute this conditional expectation using the fact that given \( H_t \) the random variable \( \ln H_T \) is normally distributed with mean \( \ln H_t - (r + \frac{\kappa^2}{2})(T-t) \) and variance \( \kappa^2(T-t) \). We get for the optimal terminal wealth before the horizon the following form

\[
X_t^N = \frac{e^{-\Gamma(t)}}{(y^N H_t)^{\frac{1}{\gamma}}} \quad \text{with} \quad \Gamma(t) := \frac{1-\gamma}{\gamma}(r+\frac{\kappa^2}{2})(T-t).
\]

The optimal strategy is obtained by a representation approach. In this case, we have \( X_t^N = F(H_t, t) \) with \( F(z, t) = \frac{e^{-\Gamma(t)}}{(y^N z)^{\frac{1}{\gamma}}} \) for which we apply Itô’s Lemma to get

\[
dX_t^N = \left[ F_t(H_t, t) - r F_z(H_t, t) H_t + \frac{\kappa^2}{2} F_{zz}(H_t, t) H_t^2 \right] dt - \kappa F_z(H_t, t) H_t dW_t,
\]

where \( F_z, F_{zz} \) and \( F_t \) denote the partial derivatives of \( F(z, t) \) w.r.t. \( z \) and \( t \), respectively. If we equate the volatility coefficient of this equation with the volatility coefficient of Equation (2.3), we derive the following constant optimal strategy

\[
\theta_t^N = \frac{\kappa}{\gamma\sigma} = \frac{\mu - r}{\gamma\sigma^2} = \text{const}
\]

for the optimization in the absence of a risk-constraint.
4 Optimization with bounded Expected Utility Loss

In this section we present the portfolio optimization problem with an additional constraint of the type

$$\text{EUL}(G) := \mathbb{E}G = \mathbb{E}[(u(X_T) - u(Q))] \leq \varepsilon,$$

where $\varepsilon$ is a given bound for the Expected Utility Loss. The solutions to the resulting constrained dynamic optimization problem

$$\max_{\theta \in \mathcal{A}(\varepsilon)} \mathbb{E}[u(X^\theta_T)]$$

subject to $\mathbb{E}[(u(X_T) - u(Q))] \leq \varepsilon$

are called EUL-optimal. The corresponding EUL-optimal wealth at time $t \in [0, T]$ is denoted by $X^\text{EUL}_t$ and the EUL-optimal strategy by $\theta^\text{EUL} = (\theta^\text{EUL}_t)$.

The corresponding static problem reads as

$$\max_{\xi \in B(x)} \mathbb{E}[u(\xi)]$$

subject to $\mathbb{E}[(u(\xi) - u(Q))] \leq \varepsilon$.

In Subsection 4.1 we consider a portfolio manager who is benchmarking the money market. This case is related to a fixed non-random benchmark $Q = q$, where $q > 0$ is a given real number representing the shortfall level. A typical choice is

$$q = e^{\delta T} X^\theta_{T=0} = xe^{(r+\delta)T}, \quad \delta \in \mathbb{R}.$$ 

Here, the shortfall level is related to the result of an investment into the money market. $X^\theta_{T=0} = xe^{rT}$ is the terminal wealth of a pure bond portfolio where the portfolio manager follows the buy-and-hold strategy $\theta \equiv 0$. In this case shortfall means to reach not an target interest rate of $r + \delta$. For $\delta = -r$ we have $q = x$, i.e., the shortfall level is equal to the initial capital.

Subsection 4.2 considers a portfolio manager who is benchmarking the stock market. We use a benchmark $Q$, which is proportional to the result of an investment into a pure stock portfolio managed by the buy-and-hold strategy $\theta \equiv 1$. Especially we set

$$Q = e^{\delta T} X^\theta_{T=1} = e^{\delta T} xS_T, \quad \delta \in \mathbb{R}.$$ 

Contrary to Subsection 4.1, now the benchmark $Q$ is a random variable.

Following the normal strategy $\theta^N$ the portfolio manager reaches the terminal wealth $X^N_T = \xi^N$ given in Theorem 3.1 and in Example 3.2 for the case of a CRRA-utility function. Let

$$\varepsilon^N := \mathbb{E}[(u(\xi^N) - u(Q))]$$

be the corresponding Expected Utility Loss of the optimal terminal wealth $\xi^N$ measuring the risk of the normal strategy. Obviously, for $\varepsilon \geq \varepsilon^N$ the risk constraint is not binding.
and the normal strategy $\theta^N$ is optimal for the constrained problem, too. Therefore, we restrict to the case $\varepsilon < \varepsilon^N$.

On the other hand, if the bound $\varepsilon > 0$ is chosen too small it may happen, that it is impossible to find any strategy $\theta_t$ which (starting with the given initial capital $x$) generates a terminal wealth $X_T$ that fulfills the risk constraint, i.e., there is no admissible solution. Depending on the chosen parameters of the financial market this case can be observed, for instance in the case of a deterministic benchmark $q$ which is is chosen to be larger than $xe^{rT}$, i.e., the result of an investment in the risk-free bond. Then, there is some positive minimum (or infimum) value $\varepsilon_0$ which bounds the Expected Utility Loss from below. The same situation arises in case of a stochastic benchmark $Q = e^{\delta T} x S_T$ for $\delta > 0$.

Choosing the bound $\varepsilon$ for the risk constraint such that $0 \leq \varepsilon_0 < \varepsilon < \varepsilon^N$ provides that the risk constraint is binding and that there exist admissible solutions from which an optimal solution for the constrained optimization problem can be determined.

Remark 4.1 If we choose the logarithmic utility function $u(z) = \ln z$, i.e., we set $\gamma = 1$, then the maximization of the expected utility $E[u(X_T)]$ of the terminal wealth of the portfolio is equivalent to the maximization of the expected annual logarithmic return of the portfolio. The annual logarithmic return is defined as

$$\varrho(X_T) := \frac{1}{T} \ln \frac{X_T}{x}$$

where $x$ is the initial capital, i.e., $X_0 = x$. Hence, we find

$$E[\varrho(X_T)] = \frac{1}{T} (E[u(X_T)] - u(x)).$$

For the Expected Utility Loss we derive

$$E[(u(X_T) - u(Q))^+] = T E[(\varrho(X_T) - \varrho(Q))^+].$$

It can be seen, that bounding the Expected Utility Loss by $\varepsilon$ is equivalent to bounding the Expected Loss of the annual logarithmic return by $\frac{\varepsilon}{T}$.

4.1 Benchmarking the money market

In this section the benchmark $Q$ is assumed to be non-random, i.e., $Q = q$, where $q > 0$ is a given real number denoting the shortfall level. As we mentioned above, this level can be related to the result of an investment into the money market.

The following proposition characterizes the EUL-optimal terminal wealth.
Proposition 4.2
Let $Q = q > 0$ be a fixed benchmark. Moreover, let for $y_1, y_2 > 0$ be defined

$$h = h(y_1) := \frac{1}{y_1} u'(q),$$

$$\overline{h} = \overline{h}(y_1, y_2) := \frac{1 + y_2}{y_1} u'(q) = (1 + y_2)h$$

and

$$f(z) = f(z; y_1, y_2) :=
\begin{cases}
I(y_1 z) & \text{if } z < h, \\
q & \text{if } h \leq z < \overline{h}, \\
I\left(\frac{y_1}{1+y_2} z\right) & \text{if } \overline{h} \leq z,
\end{cases}$$

for $z > 0$. Finally, let the initial capital $x > 0$ and the bound for the risk constraint $\varepsilon \in (\varepsilon_0, \varepsilon^N)$ be such that there are strictly positive solutions $y_1, y_2$ of the following system of equations

$$E[H_T f(H_T; y_1, y_2)] = x$$

and

$$E\left[\left(\varepsilon (f(H_T; y_1, y_2)) - u(q)\right)^-\right] = \varepsilon.$$

Then the EUL-optimal terminal wealth is given by

$$\xi^{EUL} = f(H_T) = f(H_T; y_1, y_2).$$

Remark 4.3 In case of the CRRA-utility function (3.2) the relations $u'(z) = z^{-\gamma}$ and $I(z) = z^{-\gamma}$ imply

$$\xi^{EUL} = \begin{cases}
(y_1 H_T)^{-\frac{1}{\gamma}} & \text{if } H_T < h, \\
q & \text{if } h \leq H_T < \overline{h}, \\
\left(\frac{y_1}{1+y_2} H_T\right)^{-\frac{1}{\gamma}} & \text{if } \overline{h} \leq H_T,
\end{cases}$$

where $h = \frac{1}{y_1 q^{\gamma}}$, $\overline{h} = \frac{1 + y_2}{y_1 q^{\gamma}}$.

Remark 4.4 We note that $f(z; y_1, y_2) \rightarrow I(y_1 z)$ for $y_2 \uparrow 0$. This limit corresponds to $\varepsilon \uparrow \varepsilon^N$ and we derive the results for the unconstrained problem if we set $y_2 = 0$ and $f(z; y_1, 0) := I(y_1 z)$.

Proof. The assumption on the existence of solutions $y_1, y_2 > 0$ of the system of equations given in the proposition implies, that $\xi^{EUL}$ fulfills the risk constraint with equality. In order to solve the optimization problem under the risk constraint, we adopt the common convex-duality approach by introducing the convex conjugate of the utility function $u(\cdot)$ with an additional term capturing the risk constraint as it is shown in the following lemma. The proof can be found in Appendix A.
Lemma 4.5 Let $z, y_1, y_2, q > 0$. Then the solution of the optimization problem
\[
\max_{x \geq 0} \{ u(x) - y_1 x - y_2 (u(x) - u(q)) \}
\]
is $x^* = f(z; y_1, y_2)$.

Applying the above lemma pointwise for all $z = H_T(\omega)$ it follows that $\xi^* = f(H_T; y_1, y_2)$ is the solution of the maximization problem
\[
\max_{\xi \geq 0} \{ u(\xi) - y_1 H_T \xi - y_2 (u(\xi) - u(q)) \}.
\]

Obviously, $\xi^*$ is $\mathcal{F}_T$-measurable and if $y_1, y_2$ are chosen as solutions of the system of equations given in the proposition then it follows $\xi^* = \xi^{\text{EUL}}$.

To complete the proof, let $\eta \in \mathcal{B}(x)$ be any admissible solution satisfying the static budget constraint and the EUL-constraint (4.1). We have
\[
\mathbb{E}[u(\xi^{\text{EUL}})] - \mathbb{E}[u(\eta)] = \mathbb{E}[u(\xi^{\text{EUL}})] - \mathbb{E}[u(\eta)] - y_1 x + y_1 x - y_2 \varepsilon + y_2 \varepsilon \\
\geq \mathbb{E}[u(\xi^{\text{EUL}})] - y_1 \mathbb{E}[H_T \xi^{\text{EUL}}] - y_2 \mathbb{E}[(u(\xi^{\text{EUL}}) - u(q))] \\
- \mathbb{E}[u(\eta)] + y_1 \mathbb{E}[H_T \eta] + y_2 \mathbb{E}[(u(\eta) - u(q))] \\
\geq 0,
\]
where the first inequality follows from the static budget constraint and the constraint for the risk holding with equality for $\xi^{\text{EUL}}$, while holding with inequality for $\eta$. The last inequality is a consequence of the above lemma. Hence we obtain that $\xi^{\text{EUL}}$ is optimal.

\[ \square \]

In the following proposition we present the explicit expressions for the EUL-optimal wealth and portfolio strategies before the horizon.

Proposition 4.6
Let the assumptions of Proposition 4.2 be fulfilled. Moreover, let $u(.)$ be the utility function given in (3.2).

(i) The EUL-optimal wealth at time $t < T$ before the horizon is given by
\[
X_{t}^{\text{EUL}} = F(H_t, t) \tag{4.4}
\]
with $F(z, t) := \frac{e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{5}}} - \left[ \frac{e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{5}}} \Phi(-d_1(h, z, t)) - q e^{-r(T-t)} \Phi(-d_2(h, z, t)) \right] \\
+ \left[ \frac{(1 + y_2)^{\frac{2}{7}} e^{\Gamma(t)}}{(y_1 z)^{\frac{1}{5}}} \Phi(-d_1(h, z, t)) - q e^{-r(T-t)} \Phi(-d_2(h, z, t)) \right]$. 

for \( z > 0 \). Thereby, \( \Phi(\cdot) \) is the standard-normal distribution function, \( y_1, y_2 \) and \( h, \overline{h} \) are as in Proposition 4.2, and

\[
\Gamma(t) := \frac{1 - \gamma}{\gamma} \left(r + \frac{\kappa^2}{2\gamma}\right)(T - t),
\]
\[
d_2(u, z, t) := \frac{\ln \frac{u}{z} + (r - \frac{\kappa^2}{2})(T - t)}{\kappa\sqrt{T - t}},
\]
\[
d_1(u, z, t) := d_2(u, z, t) + \frac{1}{\gamma} \kappa \sqrt{T - t}.
\]

(ii) The EUL-optimal fraction of wealth invested in stock at time \( t < T \) before the horizon is

\[
\theta^EUL_t = \theta^N \Theta(H_t, t)
\]

where

\[
\Theta(z, t) := \left( 1 - \frac{q e^{-r(T-t)}}{F(z, t)} \left[ \Phi\left(-d_2(h, z, t)\right) - \Phi\left(-d_2(\overline{h}, z, t)\right) \right] \right)
\]

for \( z > 0 \). Thereby, \( \theta^N = \frac{\kappa}{\sigma^2} = \frac{\nu + T}{\sigma^2} \) denotes the normal strategy and \( \Theta(H_t, t) \) is the exposure to risky asset relative to the normal strategy.

Proof.

(i) Equations (2.2) and (2.3), Itô’s Lemma and the fact that \( X^EUL_t \) satisfies the budget constraint with equality imply that the process \( H_tX^EUL_t \) is an \( \mathcal{F}_t \)-martingale. As a consequence we get

\[
X^EUL_t = \mathbb{E}\left[ H_t X^EUL_t \big| \mathcal{F}_t \right]
\]

\[
= \mathbb{E}\left[ H_T I\left(y_1 H_T\right) 1_{\{H_T < h\}} \big| \mathcal{F}_t \right] + \mathbb{E}\left[ H_T I\left(\overline{h} \leq H_T < \overline{h}\right) \big| \mathcal{F}_t \right] + \mathbb{E}\left[ H_T I\left(y_1 + y_2 H_T\right) 1_{\{H_T \leq \overline{h}\}} \big| \mathcal{F}_t \right]
\]

The three conditional expectations on the right hand side of Eq. (4.5) can be computed by applying Markov’s property of solutions of stochastic differential equations and using the fact that given \( H_t \) the random variable \( \ln H_T \) is normally distributed with mean \( \ln H_t - (r + \frac{\kappa^2}{2})(T - t) \) and variance \( \kappa^2(T - t) \). The details can be found in Appendix B. Hence, it results the given form of the optimal wealth before the horizon.

(ii) From Equality (4.4) it follows \( X^EUL_t = F(H_t, t) \). The process \( H_t \) satisfies the SDE (2.2). Applying Itô’s Lemma to the function \( F(H_t, t) \) we find that \( X^EUL_t \) satisfies the SDE

\[
dX^EUL_t = \left[ F_t(H_t, t) - rF_z(H_t, t)H_t + \frac{\kappa^2}{2} F_{zz}(H_t, t)H_t^2 \right] dt - \kappa F_z(H_t, t)H_t dW_t,
\]
where \( F_z, F_{zz} \) and \( F_t \) denote the partial derivatives of \( F(z, t) \) w.r.t. \( z \) and \( t \), respectively. Equating coefficients in front of \( dW_t \) in the above equation and Equation (2.3) leads to the following equality:

\[
\theta_{t}^{\text{EUL}} = -\frac{\kappa}{\sigma} \frac{F_z(H_t, t) H_t}{F(H_t, t)} = -\theta^N \frac{F_z(H_t, t) H_t}{F(H_t, t)}. \tag{4.6}
\]

Formal evaluation of the derivative \( F_z \) yields

\[
F_z(z, t) = \frac{1}{\gamma z} \left[ -F(z, t) + q e^{-r(T-t)} \left( \Phi(-d_2(h, z, t)) - \Phi(-d_2(\bar{h}, z, t)) \right) \right] - \frac{e^{\Gamma(t)}}{(y_1 z)^\frac{1}{2} \kappa \sqrt{T-t} z} \left[ \varphi(d_1(h, z, t)) - (1 + y_2) \frac{1}{\gamma} \varphi(d_1(\bar{h}, z, t)) \right] + q e^{-r(T-t)} \frac{\varphi(d_2(h, z, t)) - \varphi(d_2(\bar{h}, z, t))}{\kappa \sqrt{T-t} z}. \tag{4.7}
\]

Thereby, \( \varphi(\cdot) \) denotes the standard-normal probability density function. In Appendix E we show that the terms in the second and third line add to zero, hence \( F_z \) reduces to the expression given in the first line. Substituting into (4.6), we get the final form of the optimal strategy before the horizon.

The next proposition states two properties of the function \( \Theta(z, t) \) appearing in the definition of the above representation of the EUL-optimal strategy.

**Proposition 4.7**

Let the assumptions of Proposition 4.2 be fulfilled. Moreover, let \( u(\cdot) \) be the utility function given in (3.2). Then, for the function \( \Theta(z, t) \) defined in Proposition 4.6 (ii) there hold the following relations.

\[
\begin{align*}
(i) \quad 0 &< \Theta(z, t) < 1 \quad \text{for all } z > 0 \text{ and } t \in [0, T] \\
(ii) \quad \lim_{t \to T} \Theta(z, t) &= \begin{cases} 
1 & \text{if } z < h \text{ or } z > \bar{h} \\
0 & \text{if } h < z < \bar{h}, \\
\frac{1}{2} & \text{if } z = \bar{h}.
\end{cases}
\end{align*}
\]

**Proof.**

Using Eq. (4.4) the function \( F(z, t) \) can be written as

\[
F(z, t) = F_1(z, t) + F_2(z, t)
\]

where

\[
F_1(z, t) = \frac{e^{\Gamma(t)}}{(y_1 z)^\frac{1}{2}} \left[ 1 - \Phi(-d_1(h, z, t)) + (1 + y_2) \frac{1}{\gamma} \Phi(-d_1(\bar{h}, z, t)) \right]
\]

and

\[
F_2(z, t) = q e^{-r(T-t)} \left[ \Phi(-d_2(h, z, t)) - \Phi(-d_2(\bar{h}, z, t)) \right] \quad \text{for } z > 0.
\]
From the other hand we have from Proposition 4.6
\[ \Theta(z, t) = 1 - \frac{F_2(z, t)}{F(z, t)} = 1 - \frac{F_2(z, t)}{F_1(z, t) + F_2(z, t)}. \] (4.8)

The terms \( F_1(z, t) \) and \( F_2(z, t) \) are strictly positive since \( y_2 > 0 \) implies \( h < \tilde{h} \) and the functions \( d_{1/2}(u, \ldots) \) are strictly increasing w.r.t. \( u \) and \( \Phi \) is strictly increasing, too. Hence we have \( 0 < \Theta(z, t) < 1 \) and it follows assertion (i).

For the proof of the second assertion we consider the limits of the following functions for \( t \to T \).

<table>
<thead>
<tr>
<th>( d_{1/2}(h, z, t) )</th>
<th>( z &lt; h )</th>
<th>( z = h )</th>
<th>( h &lt; z &lt; \bar{h} )</th>
<th>( z = \bar{h} )</th>
<th>( z &gt; \bar{h} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( d_{1/2}(\bar{h}, z, t) )</td>
<td>( +\infty )</td>
<td>0</td>
<td>( -\infty )</td>
<td>( -\infty )</td>
<td>( -\infty )</td>
</tr>
<tr>
<td>( \Phi(-d_{1/2}(h, z, t)) )</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( \Phi(-d_{1/2}(\bar{h}, z, t)) )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \frac{1}{2} )</td>
<td>1</td>
</tr>
<tr>
<td>( F_1(z, t) )</td>
<td>( \frac{1}{(y_1 z)^{\frac{1}{y_1}}} )</td>
<td>( \frac{q}{2} )</td>
<td>0</td>
<td>( \frac{q}{2} )</td>
<td>( \frac{1+y_2}{y_1} \frac{1}{y_1 \bar{z}} )</td>
</tr>
<tr>
<td>( F_2(z, t) )</td>
<td>( 0 )</td>
<td>( \frac{q}{2} )</td>
<td>( q )</td>
<td>( \frac{q}{2} )</td>
<td>0</td>
</tr>
</tbody>
</table>

Here, the relations \( q = \left( \frac{1}{y_1 z} \right)^{\frac{1}{y_1}} = \left( \frac{1+y_2}{y_1 \bar{z}} \right)^{\frac{1}{y_1}} \) have been used. Substituting these limits into (4.8) yields the assertion.

\( \Box \)

**Remark 4.8** The second assertion of Proposition 4.7 shows that the lower and upper bounds for \( \Theta(z, t) \) given in the first assertion cannot be improved. The given bounds are reached (depending on the value of \( z \)) asymptotically if time \( t \) approaches the horizon \( T \).

From the proposition it follows that the EUL-optimal fraction of wealth \( \theta_{\text{EUL}}^T \) invested in the stock at the horizon is equal to the normal strategy \( \theta^N \) in the "bad" and "good" states and equal to 0 in the "intermediate" states of the market, which are described by \( H_T \).

Before horizon \( \theta_{\text{EUL}}^t \) is always strictly positive and never exceeds the normal strategy \( \theta^N \).

### 4.2 Benchmarking the stock market

Contrary to the above section now the benchmark \( Q \) is not a constant but is a random variable and chosen to be proportional to the result of an investment in a pure stock portfolio, i.e.
\[ Q = e^{\delta T} X_{\theta \equiv 1} = e^{\delta T} x S_T, \quad \delta \in \mathbb{R}. \]
The real number \( \delta \) measures the over- (\( \delta > 0 \)) or underperformance (\( \delta < 0 \)) of the stock market in terms of the annual logarithmic return.

As before, shortfall is related to the event, that the terminal wealth \( X_T \) of the portfolio is smaller than the benchmark \( Q = e^{\delta T} x S_T \) and we measure the shortfall risk using the Expected Utility Loss.

For the representation of the subsequent results we have to consider three cases of the underlying Black-Scholes model of the economy. These three cases result if we compare the quantity 
\[
\phi := \frac{\kappa}{\gamma \sigma^2} \text{ with } \frac{1}{\gamma}.
\]
Thereby, \( \phi \) is referred to as the sensitivity of the benchmark to economic conditions (see Basak, Shapiro, Tepla [3]) and \( \gamma \) is the parameter of the CRRA utility function characterizing the portfolio managers risk aversion. Using the relation 
\[
\theta^N = \frac{\mu - r}{\gamma \sigma^2} = \frac{\kappa}{\gamma \sigma} = \frac{1}{\gamma \nu},
\]
where \( \theta^N \) is the normal strategy which is the optimal strategy in the unconstrained optimization problem (see Example 3.2), the three cases are

- a) \( \nu < \frac{1}{\gamma} \iff \theta^N > 1 \)
- b) \( \nu > \frac{1}{\gamma} \iff \theta^N < 1 \)
- c) \( \nu = \frac{1}{\gamma} \iff \theta^N = 1 \).

Since the benchmark \( Q \) is related to the stock price \( S_T \) it can be expressed in terms of the state price density \( H_T \) at time \( T \). The SDEs (2.1) and (2.2) imply
\[
S_T = \exp \left( (\mu - \frac{\sigma^2}{2})T + \sigma W_T \right) \quad \text{and} \quad H_T = \exp \left( - \left( r + \frac{\kappa^2}{2} \right) T - \kappa W_T \right)
\]
which gives
\[
Q = e^{\delta T} x S_T = AH_T^{-\nu} \quad \text{where} \quad A := x \exp \left( [\delta + (\mu - \frac{\sigma^2}{2}) - (r + \frac{\kappa^2}{2}) \nu] T \right).
\]
The next proposition characterizes the form of the EUL -optimal terminal wealth.

**Proposition 4.9**

Let \( Q = e^{\delta T} x S_T \) with \( \delta \in \mathbb{R} \) be a random benchmark and \( u(.) \) be the utility function given in (3.2). Moreover, let for \( y_1, y_2 > 0 \) be defined
\[
\bar{h} = \bar{h}(y_1) := \left( \frac{1}{y_1 A \gamma} \right)^{\frac{1}{1 - \nu}},
\]
\[
\bar{\bar{h}} = \bar{\bar{h}}(y_1, y_2) := \left( \frac{1 + y_2}{y_1 A \gamma} \right)^{\frac{1}{1 - \nu}} = (1 + y_2)^{\frac{1}{1 - \nu}} \bar{h} \quad \text{and}
\]
(a) for economies with \( \nu < 1/\gamma \):
\[
f(z) = f(z; y_1, y_2) := \begin{cases} 
I(y_1 z) & \text{if } z < \bar{h}, \\
A z^{-\nu} & \text{if } \bar{h} \leq z < \bar{\bar{h}}, \\
I(\frac{y_1}{1 + y_2} z) & \text{if } \bar{\bar{h}} \leq z,
\end{cases}
\]
(b) for economies with $\nu > 1/\gamma$:

\[
f(z) = f(z; y_1, y_2) := \begin{cases} 
I(\frac{y_1}{1+y_2}) & \text{if } z < \bar{h}, \\
A z^{-\nu} & \text{if } \bar{h} \leq z < h, \\
I(y_1 z) & \text{if } h \leq z,
\end{cases}
\]

for $z > 0$. Finally, let the initial capital $x > 0$ and the bound for the risk constraint $\varepsilon \in (\varepsilon_0, \varepsilon^N)$ be such that there are strictly positive solutions $y_1, y_2$ of the following system of equations

\[
E[H_T f(H_T; y_1, y_2)] = x \\
E[(u(f(H_T; y_1, y_2)) - u(A H_T^{-\nu}))^-] = \varepsilon.
\]

Then the EUL-optimal terminal wealth is

\[
\xi_{\text{EUL}} = f(H_T) = f(H_T; y_1, y_2).
\]

**Proof.**

The proof is analogous to the proof of Proposition 4.2 if Lemma 4.5 is replaced by the following lemma which is proven in Appendix C.

**Lemma 4.10** Let $z, y_1, y_2 > 0$. Then the solution of the optimization problem

\[
\max_{x \geq 0} \{u(x) - y_1 z x - y_2(u(x) - u(A z^{-\nu}))^- \}
\]

is $x^* = f(z; y_1, y_2)$.

\[\Box\]

In the following proposition we present the explicit expressions for the EUL-optimal wealth and portfolio strategies before the horizon.

**Proposition 4.11**

Let the assumptions of Proposition 4.9 be fulfilled.

(i) The EUL-optimal wealth at time $t < T$ before the horizon is given by

\[
X^\text{EUL}_t = F(H_t, t),
\]

where $F(z, t)$ for $z > 0$ is defined by

(a) for economies with $\nu < 1/\gamma$:

\[
F(z, t) = \frac{e^{\Gamma_1(t)}}{(y_1 z)^{\gamma}} - \left[ \frac{e^{\Gamma_1(t)}}{(y_1 z)^{\gamma}} \Phi \left( -d_1(\bar{h}, z, t) \right) - A \frac{e^{\Gamma_2(t)}}{z^{\nu}} \Phi \left( -d_2(\bar{h}, z, t) \right) \right] + \left[ (1 + y_2)^{\gamma} \frac{e^{\Gamma_1(t)}}{(y_1 z)^{\gamma}} \Phi \left( -d_1(\bar{h}, z, t) \right) - A \frac{e^{\Gamma_2(t)}}{z^{\nu}} \Phi \left( -d_2(\bar{h}, z, t) \right) \right]
\]

\[\text{(4.9)}\]
(b) for economies with $\nu > \frac{1}{\gamma}$:

$$F(z, t) = (1 + y_2)^{\frac{1}{\gamma}} \left( \frac{e^{\Gamma_1(t)}}{(y_1 z)^{\frac{1}{\gamma}}} \Phi(-d_1(h, z, t)) - A \frac{e^{\Gamma_2(t)}}{z^\nu} \Phi(-d_2(h, z, t)) \right)$$

$$- \left( 1 + y_2 \right)^{\frac{1}{\gamma}} \left( \frac{e^{\Gamma_1(t)}}{(y_1 z)^{\frac{1}{\gamma}}} \Phi(-d_1(\bar{h}, z, t)) - A \frac{e^{\Gamma_2(t)}}{z^\nu} \Phi(-d_2(\bar{h}, z, t)) \right).$$

Here, $\Phi(\cdot)$ is the standard-normal distribution function, $y_1, y_2$ and $h, \bar{h}, A$ are as in Proposition 4.9, and

$$\Gamma_1(t) := \frac{1 - \gamma}{\gamma} (r + \frac{\kappa^2}{2\gamma})(T - t)$$

$$\Gamma_2(t) := \frac{\sigma - \kappa}{\kappa} (r + \frac{\sigma \kappa}{2})(T - t)$$

$$d_1(u, z, t) := \frac{\ln \frac{u}{z} + (r - z^2)(T - t)}{\kappa \sqrt{T - t}} + \frac{\kappa}{\gamma} \sqrt{T - t},$$

$$d_2(u, z, t) := \frac{\ln \frac{u}{z} + (r - z^2)(T - t)}{\kappa \sqrt{T - t}} + \sigma \sqrt{T - t}.$$

(ii) The EUL-optimal fraction of wealth invested in stock at time $t < T$ before the horizon is

$$\theta^EUL_t = \theta^N \Theta(H_t, t).$$

Here, $\theta^N = \frac{\kappa}{\sigma} = \frac{\kappa - \gamma r}{\gamma \sigma^2}$ denotes the normal strategy and $\Theta(z, t)$ is the exposure to the risky asset relative to the normal strategy, which is defined for $z > 0$ as follows

(a) for economies with $\nu < \frac{1}{\gamma}$:

$$\Theta(z, t) := 1 - (1 - \gamma \nu) \frac{A e^{\Gamma_2(t)}}{z^\nu F(z, t)} \left[ \Phi(-d_2(h, z, t)) - \Phi(-d_2(\bar{h}, z, t)) \right],$$

(b) for economies with $\nu > \frac{1}{\gamma}$:

$$\Theta(z, t) := 1 + (1 - \gamma \nu) \frac{A e^{\Gamma_2(t)}}{z^\nu F(z, t)} \left[ \Phi(-d_2(h, z, t)) - \Phi(-d_2(\bar{h}, z, t)) \right].$$

Remark 4.12 The case of an economy with $\nu = \frac{1}{\gamma}$ is considered in Proposition 4.15.

Proof.

We give the proof for case (a), i.e., economies with $\nu < \frac{1}{\gamma}$, the proof for case (b) is similar.
(i) Equations (2.2) and (2.3), Itô’s Lemma and the fact that $X_t^{EUL}$ satisfies the budget constraint with equality imply that the process $H_tX_t^{EUL}$ is an $\mathcal{F}_t$-martingale. As a consequence we get

$$X_t^{EUL} = \mathbb{E} \left[ \frac{H_T}{H_t} X_t^{EUL} \mid \mathcal{F}_t \right]$$

$$= \mathbb{E} \left[ \frac{H_T}{H_t} I(y_1H_T)1_{\{H_T<\bar{H}\}} \mid \mathcal{F}_t \right] + \mathbb{E} \left[ \frac{H_T}{H_t} AH_T^{-\nu} 1_{\{\bar{H} \leq H_T \}} \mid \mathcal{F}_t \right] + \mathbb{E} \left[ \frac{H_T}{H_t} I \left( \frac{y_1}{1+y_2H_T} \right) 1_{\{\bar{H} \leq H_T \}} \mid \mathcal{F}_t \right].$$

(4.10)

The three conditional expectations on the right hand side of Eq. (4.10) can be computed by applying Markov’s property of solution of stochastic differential equation and using the fact that given $H_t$ the random variable $H_T$ is normally distributed with mean $\ln H_t - (r + \frac{\kappa^2}{2})(T - t)$ and variance $\kappa^2(T - t)$. The details can be found in Appendix D. Hence, it results the given form of the optimal wealth before the horizon.

(ii) It holds $X_t^{EUL} = F(H_t, t)$. The process $H_t$ satisfies the SDE (2.2). Applying Itô’s Lemma to the function $F(H_t, t)$ we find that $X_t$ satisfies the SDE

$$dX_t^{EUL} = \left[ F_t(H_t, t) - rF_z(H_t, t)H_t + \frac{\kappa^2}{2} F_{zz}(H_t, t)H_t^2 \right] dt - \kappa F_z(H_t, t)H_t dW_t,$$

where $F_z, F_{zz}$ and $F_t$ denote the partial derivatives of $F(z, t)$ w.r.t. $z$ and $t$, respectively. Equating coefficients in front of $dW_t$ in the above equation and Equation (2.3) leads to the following equality:

$$\theta_t^{EUL} = -\frac{\kappa F_z(H_t, t)H_t}{\sigma F(H_t, t)} = -\theta N \gamma F_z(H_t, t)H_t / F(H_t, t).$$

(4.11)

Formal evaluation of the derivative $F_z$ yields

$$F_z(z, t) = \frac{1}{\gamma z} \left[ F(z, t) - (1 - \gamma \nu) \frac{A e^{\Gamma_1(t)}}{z^\nu} \left( \Phi(-d_2(h, z, t)) - \Phi(-d_2(\bar{h}, z, t)) \right) \right]$$

$$- \frac{e^{\Gamma_1(t)}}{(y_1)^{\frac{1}{2}} \kappa \sqrt{T - t}} \left[ \varphi(d_1(h, z, t)) - (1 + y_2)^{\frac{1}{2}} \varphi(d_1(\bar{h}, z, t)) \right]$$

$$+ \frac{A e^{\Gamma_2(t)}}{z^\nu \kappa \sqrt{T - t}} \left[ \varphi(d_2(h, z, t)) - \varphi(d_2(\bar{h}, z, t)) \right].$$

(4.12)

In Appendix F we show that the terms in the second and third line add to zero, hence $F_z(z, t)$ reduces to the expression given in the first line.

Substituting into (4.11), we get the final form of the optimal strategies before the horizon.
The next proposition states two properties of the term $\Theta(z, t)$ which describes the exposure to risky assets relative to the normal strategy $\theta^N$ and which appears in the above representation of the EUL-optimal strategy.

**Proposition 4.13**

Let the assumptions of Proposition 4.9 be fulfilled. Then, for the function $\Theta(z, t)$ defined in Proposition 4.11 (ii) there hold the following relations.

(i) (a) for economies with $\nu < \frac{1}{\gamma}$: $\frac{1}{\varrho^\nu} < \Theta(z, t) < 1$

(b) for economies with $\nu > \frac{1}{\gamma}$: $1 < \Theta(z, t) < \frac{1}{\varrho^\nu}$

for all $z > 0$ and $t \in [0, T)$.

(ii)

\[
\lim_{t \to T} \Theta(z, t) = \begin{cases} 
1 & \text{if } z < \underline{h} \text{ or } z > \overline{h} \\
\frac{1}{\varrho^\nu} & \text{if } \underline{h} < z < \overline{h}, \\
\frac{1}{2}(1 + \frac{1}{\varrho^\nu}) & \text{if } z = \underline{h}, \overline{h}
\end{cases}
\]

**Proof.**

We prove the claim for economies with $\nu < \frac{1}{\gamma}$, economies with $\nu > \frac{1}{\gamma}$ are treated similarly. Using Eq. (4.9) the function $F(z, t)$ can be written as

\[
F(z, t) = F_1(z, t) + F_2(z, t)
\]

where

\[
F_1(z, t) = e^{\Gamma(t)} \left[ 1 - \Phi(-d_1(\underline{h}, z, t)) + (1 + y_2) \Phi(-d_1(\overline{h}, z, t)) \right]
\]

and

\[
F_2(z, t) = \frac{A e^{\Gamma(t)}}{z^\nu} \left[ \Phi(-d_2(\underline{h}, z, t)) - \Phi(-d_2(\overline{h}, z, t)) \right]
\]

for $z > 0$.

From the other hand we have from Proposition 4.11

\[
\Theta(z, t) = 1 - \left(1 - \gamma \nu \right) \frac{F_2(z, t)}{F(z, t)} = 1 - \left(1 - \frac{1}{\theta^N} \right) \frac{F_2(z, t)}{F_1(z, t) + F_2(z, t)}.
\]

Thereby, it the relation $\gamma \nu = \frac{1}{\varrho^\nu}$ has been used. The terms $F_1(z, t)$ and $F_2(z, t)$ are strictly positive since $y_2 > 0$ implies $\underline{h} < \overline{h}$ and the functions $d_{1/2}(u, .., )$ are strictly increasing w.r.t. $u$ and $\Phi$ ist strictly increasing, too. Hence we have $\frac{1}{\varrho^\nu} < \Theta(z, t) < 1$ and it follows assertion (i).

The proof of the second assertion is analogous to the proof of Proposition 4.7. \qed
Remark 4.14 As in Proposition 4.7 dealing with the case of a deterministic benchmark the second assertion of Proposition 4.13 shows that the lower and upper bounds for $\Theta(z, t)$ given in the first assertion can not be improved. The given bounds are reached (depending on the value of $z$) asymptotically if time $t$ approaches the horizon $T$.

From the proposition it follows that the EUL-optimal fraction of wealth $\theta^{EUL}_t$ invested in the stock at the horizon is equal to the normal strategy $\theta^N$ in the „bad“ and „good“ states and equal to 1 in the „intermediate“ states of the market, which are described by $H_T$. Before horizon $\theta^{EUL}_t$ is always bounded by 1 and the normal strategy $\theta^N$.

Finally, the case of an economy with $\nu = \frac{1}{\gamma}$ is considered. In this case the optimal strategy of the unconstrained optimization problem is $\theta^N = 1$, i.e., the optimal portfolio is a pure stock portfolio and $\theta^N$ is a "buy-and-hold" strategy. The following proposition is proven in Appendix G.

Proposition 4.15
Let $u(\cdot)$ be the utility function given in (3.2) and $\nu = \frac{1}{\gamma}$. Moreover, let

$$y := y^N = \frac{1}{x^\gamma} e^{(1-\gamma)(t+\frac{\gamma}{2})^T} = \frac{1}{x^\gamma} e^{(1-\gamma)(\frac{\mu+\lambda}{\gamma})^T}.$$ 

Then the EUL-optimal terminal wealth is $\xi^{EUL} = \xi^N = I(y^N H_T)$ and the EUL-optimal strategy coincides with the normal strategy, i.e., $\theta^{EUL}_t \equiv \theta^N = 1$, provided that the risk constraint

$$E \left[ (u(\xi^N) - u(Q))^- \right] \leq \varepsilon$$

is fulfilled. Otherwise, there is no admissible solution.

5 Numerical results

This section illustrates the findings of the preceding sections with an example. Table 5.1 shows the parameters for the portfolio optimization problem and the underlying Black-Scholes model of the financial market. In this example the aim is to maximize the expected logarithmic utility ($\gamma = 1$) of the terminal wealth $X_T$ of the portfolio with the horizon $T = 20$ years. The benchmark $Q$ is set to be equal to 100% of the terminal wealth of a pure stock portfolio, i.e., we set $\delta = 0$ and $Q = xS_T$. We bound the Expected Utility Loss by $\varepsilon = 0.02$.

For the present case of a logarithmic utility function $u(z) = \ln(z)$ the risk constraint can be reformulated in terms of the annual logarithmic return $\varrho(z) = \frac{1}{T} \ln \frac{x}{x}$ where $x$ is the initial capital (see remark 4.1). It holds

$$E \left[ (u(X_T) - u(Q))^- \right] \leq \varepsilon = 0.02 = 2\% \iff E \left[ (\varrho(X_T) - \varrho(Q))^- \right] \leq \frac{\varepsilon}{T} = 0.001 = 0.1\%,$$

i.e., we bound the expected Loss of the annual logarithmic return by 0.1%.
Table 5.1: Parameters of the optimization problem

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
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</thead>
<tbody>
<tr>
<td>stock</td>
<td>$\mu = 9%, \sigma = 20%$</td>
</tr>
<tr>
<td>bond</td>
<td>$r = 4%$</td>
</tr>
<tr>
<td>horizon</td>
<td>$T = 20$</td>
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<tr>
<td>initial capital</td>
<td>$x = 1$</td>
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<tr>
<td>utility function</td>
<td>$u(z) = \ln z$ i.e., $\gamma = 1$</td>
</tr>
<tr>
<td>benchmark</td>
<td>$Q = xS_T, \delta = 0$</td>
</tr>
<tr>
<td>constraint</td>
<td>$\varepsilon = 0.02$</td>
</tr>
</tbody>
</table>

The chosen parameters correspond to an economy with

$$\nu = \frac{\sigma}{\kappa} = \frac{\sigma^2}{\mu - r} = 0.8 < \frac{1}{\gamma} = 1$$

or equivalently

$$\theta^N = \frac{\kappa}{\gamma \sigma} = \frac{\mu - r}{\gamma \sigma^2} = 1.25 > 1.$$  

This is case (a) in the propositions of Subsection 4.2. The form of the EUL-optimal terminal wealth $X_{T}^{\text{EUL}}$ as a function $f(H_T)$ of the state price density $H_T$ at the horizon $T$ is given in Proposition 4.9 (a). For the interpretation of the result it seems to be more convenient to express $X_{T}^{\text{EUL}}$ as a function $\tilde{f}(S_T)$ of the terminal stock price $S_T$. This dependence can easily derived from the following relation which follows from the SDEs (2.1) and (2.2)

$$H_t = G(S_t, t) := \exp \left( \left[ \frac{1}{\nu} \left( \mu - \frac{\sigma^2}{2} \right) - \left( r + \frac{\kappa^2}{2} \right) \right] t \right) S_t^{-\frac{1}{\nu}} \text{ for } t \in [0, T] \tag{5.1}$$

which implies

$$S_T = g(H_T) := \exp \left( \left[ \left( \mu - \frac{\sigma^2}{2} \right) - \left( r + \frac{\kappa^2}{2} \right) \nu \right] T \right) H_T^{-\nu}.$$  

**Remark 5.1** For an optimal view of the subsequent figures we recommend a colored hardcopy or the electronic version of this paper which is available at http://archiv.tu-chemnitz.de/pub/2005/ or http://www.fh-wickau.de/~raw.

Figure 5.1 shows the EUL-terminal wealth $X_{T}^{\text{EUL}}$ as a function of the terminal stock price $S_T$. Moreover, the terminal wealth of the

- pure bond portfolio $X_{T}^{\theta_{t}=0}$,
- pure stock portfolio $X_{T}^{\theta_{t}=1} = S_T = Q$, which is equal to the chosen benchmark $Q$,
- the optimal portfolio of the unconstrained problem $X_{T}^{N} = X_{T}^{\theta^{N}}$, (see Example 3.2).

are drawn as functions of $S_T$.  

Dynamic optimal portfolios benchmarking the stock market

It can be seen, that for states with large $S_T$, i.e., $S_T > g(\bar{h})$, the EUL-optimal portfolio overperforms the stock market, it holds $X_T^{EUL} > Q = S_T$. For states with intermediate stock prices, i.e., $g(\bar{h}) \leq S_T \leq g(\bar{h})$, the EUL-optimal wealth coincides with the benchmark $Q$. An underperformance or shortfall of the EUL-optimal portfolio occurs in the states with small stock prices, i.e., $S_T < g(\bar{h})$. On the other hand, it can be seen that the terminal wealth $X_T^N$, resulting from the optimization without risk constraint, underperforms the stock market in considerable more states of the market.

Table 5.2 gives the expected utilities of the terminal wealth $E_u(X_T)$ (which have to be maximized in the optimization), the corresponding expected annual return $E\rho(X_T)$, the expected terminal wealth $EX_T$ and the risk measure Expected Utility Loss for the above considered portfolios. The comparison of these values demonstrates, that imposing an additional risk constraint leads only to small losses of the expected utility but to considerable gains of the risk measure.

Figure 5.2 shows the probability density functions of the random terminal wealth $X_T^{\theta_t=0}$, $X_T^{\theta_t=1} = S_T = Q$, $X_T^N$ and $X_T^{EUL}$ considered above. While the terminal wealth of the pure bond portfolio $X_T^{\theta_t=0}$ is concentrated in the single point $e^{rT}$, the random variables
\( X_t^{\theta_t = 0} \), \( X_t^{\theta_t = 1} = Q \), \( X_t^N \), \( X_t^{\text{EUL}} \)

<table>
<thead>
<tr>
<th></th>
<th>( X_t^{\theta_t = 0} )</th>
<th>( X_t^{\theta_t = 1} = Q )</th>
<th>( X_t^N )</th>
<th>( X_t^{\text{EUL}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \mathbb{E}u(X_T) )</td>
<td>0.800</td>
<td>1.400</td>
<td>1.425</td>
<td>1.415</td>
</tr>
<tr>
<td>( \mathbb{E}g(X_T) )</td>
<td>0.040</td>
<td>0.070</td>
<td>0.071</td>
<td>0.071</td>
</tr>
<tr>
<td>( \mathbb{E}X_T )</td>
<td>2.226</td>
<td>6.050</td>
<td>7.758</td>
<td>6.893</td>
</tr>
<tr>
<td>( \mathbb{E}[(u(X_T) - u(Q))^-] )</td>
<td>0.734</td>
<td>0.000</td>
<td>0.077</td>
<td>0.020</td>
</tr>
</tbody>
</table>

Table 5.2: Expected utility \( \mathbb{E}u(X_T) \), expected annual logarithmic return \( \mathbb{E}g(X_T) \), expected terminal wealth \( \mathbb{E}X_T \) and Expected Utility Loss \( \mathbb{E}[(u(X_T) - u(Q))^-] \)

Figure 5.2: Probability density functions of terminal wealth

\( Q \) and \( X_T^N \) possess logarithmic normal distributions. The distribution of \( X_T^{\text{EUL}} \) is also absolutely continuous but its probability density function shows jumps at \( g(h) \) and \( g(-h) \). This is a consequence of the form of \( X_T^{\text{EUL}} \) as a function of \( H_T \) given in Proposition 4.9 (a).

Next, the EUL-optimal strategy \( \theta_t^{\text{EUL}} \), which leads to the terminal wealth \( X_T^{\text{EUL}} \) discussed above, is considered. This strategy is a feedback strategy, i.e., it depends on time \( t \) as well as of the state of the financial market, which can be expressed in terms of the state price density \( H_t \), the stock price \( S_t \) or of the wealth \( X_t \) of the portfolio at time \( t \), respectively. In Proposition 4.11, (ii) the EUL-optimal strategy has been given as a function of time \( t \) and state price density \( H_t \). This is convenient from the mathematical point of view. However, for practical purposes it seems to be more convenient to express the strategy in
terms of $t$ and the stock price $S_t$ at time $t$, i.e., $\theta^EUL_t = \theta^EUL(t, S_t)$. Such a representation can be easily derived from the representation given in Proposition 4.11, (ii) using the relation $H_t = G(t, S_t)$ given in (5.1). The value of $\theta^EUL_t$ gives the EUL-optimal fraction of wealth, which at time $t$ has to be invested in the stock, if the stock price $S_t$ is observed.

The upper plot in Figure 5.3 shows $\theta^EUL(t, S_t)$ as a function of the stock price $S_t$ for the fixed times $t = \frac{1}{4}T = 5$ years, $t = \frac{3}{4}T = 15$ years and its limit for $t \to T - 0 = 20$ years (i.e., the time just before the horizon). Moreover, the upper plot displays the strategies of the pure bond and stock portfolio, which are no feedback strategies but constants, namely $\theta_t \equiv 0$ and $\theta_t \equiv 1$, respectively, and the normal strategy $\theta_t = \theta^N = \frac{\mu - r}{\sigma^2} = 1.25$, which is also constant.

The lower plot shows a 3D-plot of $\theta^EUL(t, S_t)$ as a function of both time $t$ and stock price $S_t$. For $t = \frac{1}{4}T, \frac{3}{4}T$ and $T - 0$ one gets the plots of the upper picture as intersection of the 3D-plot with planes parallel to the $(S, \theta)$-plane. Moreover, the lower picture contains the path of a simulated stock price $S_t = S_t(\omega)$, $0 \leq t < T$, in the $(S, t)$-plane and the
corresponding path of the EUL-optimal strategy \( \theta_{t}^{EUL}(\omega) = \theta^{EUL}(t, S_t(\omega)) \).

The figure shows that the EUL-optimal fraction of wealth \( \theta_{t}^{EUL} \) which has to be invested in the stock is bounded from below by 1, i.e., the strategy of the pure stock portfolio. It is bounded from above by \( \theta^N = 1.25 \), i.e., the normal strategy. Let us note, that \( \theta_t > 1 \) corresponds to a short position in the bond, since the fraction of wealth invested in the bond is \( 1 - \theta_{t}^{EUL} \).

For the present parameters of the financial market, especially the large positive difference of the mean return \( \mu = 9\% \) of the stock and the risk-free interest rate \( r = 4\% \) relative to the volatility of \( \sigma = 20\% \), it is optimal to borrow money and invest this money into the stock earning the "high" mean return \( \mu \) while paying the "low" interest rate \( r \).

If time \( t \) approaches the horizon then corresponding to Proposition 4.13, (ii) the strategy tends to the normal strategy \( \theta^N \) in the states with small and large stock prices. In the states with intermediate stock prices the EUL-optimal strategy tends to 1, in order to reach the corresponding EUL-optimal terminal wealth, which is in this case the stock price, i.e., \( X_{T}^{EUL} = Q = S_T \).

Next, we change the value of the mean return of the stock \( \mu \) from 9\% to the lower value 7\%. The other parameters remain unchanged. Now we have an economy with

\[
\nu = \frac{\sigma}{\kappa} = \frac{\sigma^2}{\mu - r} = 1.33 > \frac{1}{\gamma} = 1 \quad \text{or equivalently} \quad \theta^N = \frac{\kappa}{\gamma \sigma} = \frac{\mu - r}{\gamma \sigma^2} = 0.75 < 1.
\]

This is case (b) in the propositions of Subsection 4.2.

Figure 5.4 shows the terminal wealth of the EUL-optimal portfolio and of the other considered portfolios as a function of the terminal stock price \( S_T \). Note that in case (b) contrary to case (a) it holds \( \bar{h} > \bar{h} \) and consequently \( g(\bar{h}) < g(\bar{h}) \).

It can be seen, that contrary to the first example where \( \mu = 9\% \), for states with large \( S_T \), i.e., \( S_T > g(\bar{h}) \), the EUL-optimal portfolio underperforms the stock market, it holds \( X_{T}^{EUL} < Q = S_T \). These are the states where the shortfall happens. For states with intermediate stock prices, i.e., \( g(\bar{h}) \leq S_T \leq g(\bar{h}) \), the EUL-optimal wealth coincides with the benchmark \( Q \). An overperformance of the EUL-optimal portfolio occurs in the states with small stock prices, i.e., \( S_T < g(\bar{h}) \). As in the first example it can be seen, that the terminal wealth \( X_{T}^{N} \), resulting from the optimization without risk constraint, underperforms the stock market in considerable more states of the market.

Table 5.3 gives the expected utilities of the terminal wealth \( E_u(X_T) \), the corresponding expected annual logarithmic return \( E\% (X_T) \), the expected terminal wealth \( EX_T \) and the risk measure Expected Utility Loss for the above considered portfolios. Once again the comparison of of these values demonstrates, that imposing an additional risk constraint leads only to small losses of the expected utility but to considerable gains of the risk measure.

Figure 5.5 shows the probability density functions of the random terminal wealth \( X_T^{\theta_i \equiv 0}, X_T^{\theta_i \equiv 1} = S_T = Q, X_T^{N} \) and \( X_T^{EUL} \) considered above.
Figure 5.4: EUL-optimal terminal wealth

<table>
<thead>
<tr>
<th></th>
<th>$X_T^{\theta_t=0}$</th>
<th>$X_T^{\theta_t=1} = Q$</th>
<th>$X_T^N$</th>
<th>$X_T^{EUL}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pure bond</td>
<td>0.800</td>
<td>1.000</td>
<td>1.025</td>
<td>1.015</td>
</tr>
<tr>
<td>Pure stock</td>
<td>0.040</td>
<td>0.050</td>
<td>0.051</td>
<td>0.051</td>
</tr>
<tr>
<td>Normal</td>
<td>2.226</td>
<td>4.055</td>
<td>3.489</td>
<td>3.791</td>
</tr>
<tr>
<td>$E[u(X_T) - u(Q)^{-}]$</td>
<td>0.466</td>
<td>0.000</td>
<td>0.077</td>
<td>0.020</td>
</tr>
</tbody>
</table>

Table 5.3: Expected utility $E[u(X_T)$, expected annual logarithmic return $E[\theta_t(X_T)$, expected terminal wealth $EX_T$ and Expected Utility Loss $E[(u(X_T) - u(Q)^{-}]]$
Figure 5.5: Probability density functions of terminal wealth

Figure 5.6 is analogous to Figure 5.3 and shows $\theta^{EUL}(t, S_t)$ as a function of time $t$ and the stock price $S_t$. In the upper picture three times $t = \frac{1}{3}T = 5$ years, $t = \frac{2}{3}T = 15$ years and the time just before the horizon $t = T - 0 = 20$ years are fixed. The lower picture shows a 3D-plot of $\theta^{EUL}(t, S_t)$ as a function of both time $t$ and stock price $S_t$.

The figure shows that the EUL-optimal fraction of wealth $\theta^{EUL}$ is bounded from below by $\mu_N = 0.75$, i.e., the normal strategy. It is bounded from above by 1, i.e., the strategy of the pure stock portfolio. Contrary to the first example with the larger mean return $\mu = 9\%$ of the stock, now it is not necessary to go short into the bond, since we have $0 \leq 1 - \theta^{EUL} \leq 1 - \theta^N = 0.25$. We note, that in the present example according to Proposition 4.13, (i) the fraction of wealth invested in the risky stock by the EUL-optimal portfolio manager is in all states of the financial market at least as much as the corresponding fraction of the manager following the normal strategy $\theta^N$. Due to this riskier strategy the EUL-optimal portfolio manager reaches a slightly smaller expected utility of the terminal wealth but he satisfies the risk constraint, which is violated by the manager following the normal strategy.

As in the first example with $\mu = 9\%$ we observe, that if time $t$ approaches the horizon then the strategy tends to the normal strategy $\theta^N$ in the states with small and large stock prices. In the states with intermediate stock prices the EUL-optimal strategy tends to 1, in order to reach the corresponding EUL-optimal terminal wealth, which is in this case the stock price, i.e., $X_T^{EUL} = Q = S_T$. 

Figure 5.6
Figure 5.6: EUL-optimal strategy $\theta_t^{\text{EUL}} = \theta^{\text{EUL}}(t, S_t)$ as a function of time $t$ and stock price $S_t$

References


Appendix

A Proof of Lemma 4.5

Consider the function

\[ h(x) := u(x) - y_1zx - y_2(u(x) - u(q))^- \]

Defining the functions

\[ h_1(x) := u(x) - y_1zx \]
\[ h_2(x) := u(x) - y_1zx + y_2(u(x) - u(q)) = (1 + y_2)u(x) - y_1zx - y_2u(q), \]
the function $h$ can be written as

$$
h(x) = \begin{cases} 
h_1(x) & \text{for } x \geq q, \\
h_2(x) & \text{for } x < q.
\end{cases}
$$

Since $h_1$ and $h_2$ are strictly concave and continuously differentiable, the function $h$ is a continuous and strictly concave function which is differentiable in $[0, q)$ and $(q, \infty)$ and possesses different one-sided derivatives in the point $x = q$ which are $h'(q - 0) = h'_2(q)$ and $h'(q + 0) = h'_1(q)$.

The functions $h_1$ and $h_2$ attain their maximum values at $x_1 := I(y_1 z)$ and $x_2 := I\left(\frac{y}{1+y_2} z\right)$, respectively. Since the function $I(\cdot)$ is strictly decreasing and $y_2 > 0$ it follows $x_1 < x_2$. If $h$ possesses a maximum then it is unique because $h$ is strictly concave. To find the maximum of $h$ one has to consider the following three cases.

(i) $q < x_1$:
Since $u'$ is strictly decreasing we have $u'(q) > u'(x_1) = u'(I(y_1 z)) = y_1 z$. Considering the one-sided derivatives at $x = q$ one obtains

$$
h'(q - 0) = h'_2(q) = (1 + y_2)u'(q) - y_1 z > (1 + y_2)y_1 z - y_1 z = y_1 y_2 z > 0
$$

and

$$
h'(q + 0) = h'_1(q) = u'(q) - y_1 z > y_1 z - y_1 z = 0,
$$

i.e., the function $h$ is increasing at $x = q$. It follows that the function $h$ attains its maximum on $(q, \infty)$ where $h(x) = h_1(x)$, i.e., the maximum is at $x^* = x_1 = I(y_1 z)$.

Solving the inequality $u'(q) > y_1 z$ for $z$ it yields

$$
z < \frac{u'(q)}{y_1} = h.
$$

(A.1)

(ii) $x_1 \leq q < x_2$:
Now the relation $q \geq x_1$ implies $u'(q) \leq y_1 z$ while $q < x_2$ leads to

$$
u'(q) > u'(x_2) = u'\left( I\left(\frac{y_1}{1+y_2} z\right) \right) = \frac{y_1}{1+y_2} z.
$$

For the one-sided derivatives at $x = q$ we find

$$
h'(q - 0) = h'_2(q) = (1 + y_2)u'(q) - y_1 z > (1 + y_2)\frac{y_1}{1+y_2} z - y_1 z = y_1 y_2 z > 0
$$

and

$$
h'(q + 0) = h'_1(q) = u'(q) - y_1 z \leq y_1 z - y_1 z = 0.
$$

From the strict concavity of $h$ we deduce that

$$
h'(x) = h'_2(x) > h'_2(q) > 0 \text{ for } x < q
$$

and

$$
h'(x) = h'_1(x) < h'_1(q) \leq 0 \text{ for } x > q.
$$

Thus the function $h$ is strictly increasing for $x < q$ and strictly decreasing for $x > q$, hence $h$ attains its maximum at $x^* = q$. 

The relations
\[ \frac{y_1}{1+y_2} z < u'(q) \leq y_1 z \]
imply
\[ h \leq z < \overline{h} = \frac{1+y_2}{y_1} u'(q). \] (A.2)

(iii) \( q \geq x_2 \):
In this case we have \( u'(q) \leq u'(x_2) = \frac{y_1}{1+y_2} z \), for the one-sided derivatives at \( x = q \) one obtains
\[ h'(q-0) = h'_2(q) = (1+y_2)u'(q) - y_1 z \leq y_1 z - y_1 z = 0 \]
and \( h'(q+0) = h'_1(q) = u'(q) - y_1 z \leq \frac{y_1 z}{1+y_2} - y_1 z < 0. \)

It follows that the function \( h \) is decreasing at \( x = q \) and attains its maximum on \((0,q)\) where \( h(x) = h_2(x) \) and hence the maximum is at \( x^* = x_2 = I(\frac{1+y_2}{y_1} z) \).

Solving the inequality \( u'(q) \leq \frac{y_1}{1+y_2} z \) for \( z \) it follows
\[ z \geq \frac{1+y_2}{y_1} u'(q) = \overline{h}. \] (A.3)

### B Evaluation of the conditional expectations in Eq. (4.5)

The state price density \( H_t \) is the solution of SDE (2.2) and its terminal value \( H_T \) at the horizon \( T \) can be expressed in terms of the value \( H_t \) at time \( t \leq T \) by
\[ H_T = H_t \exp \left( -\left(r + \frac{\kappa^2}{2}\right)(T-t) - \kappa(W_T - W_t) \right) \]
\[ = H_t \exp(a + b\eta). \]

Thereby \( a = -(r + \frac{\kappa^2}{2})(T-t), \ b = -\kappa\sqrt{T-t} \) and \( \eta \) is a standard Gaussian random variable, which is independent of \( H_t \) and \( \mathcal{F}_t \).

Using this representation and the \( \mathcal{F}_t \)-measurability of \( H_t \) the conditional expectations in Eq. (4.5) can be written in the form
\[ \frac{c}{H_t} \mathbb{E} \left[ g(H_t, \eta) | \mathcal{F}_t \right] = \frac{c}{H_t} \psi(H_t) \quad \text{with} \quad \psi(z) = \mathbb{E} \left[ g(z, \eta) \right], \ \text{for} \ z \in (0, \infty) \]
with a measurable function \( g \) and a real constant \( c \). Thereby it is used, that \( \eta \) is independent of \( \mathcal{F}_t \) and that \( H_t \) is \( \mathcal{F}_t \)-measurable. Applying this relation, the three conditional expectations can be evaluated as follows.
1. For the first term of equation (4.5) it holds
\[
E \left[ \frac{H_f}{H_t} I(y_1 H_T) 1_{(H_T < \bar{H})} \right] = \frac{y_1}{H_t} \left( \frac{1}{\sqrt{2\pi}} \int z e^{a \lambda + \frac{1}{2} b^2} \Phi \left( \frac{\ln \left( \frac{t}{h} \right) - a}{b} - b \lambda \right) \right) = \frac{y_1}{H_t} \psi(H_t)
\]
with \( g(z, x) = z^\lambda e^{(a+bx)} 1_{\{z e^{a+bx} < \bar{H} \}} \), where \( \lambda = 1 - \frac{1}{\gamma} \). Computing \( \psi \) we get
\[
\psi(z) = \frac{1}{\sqrt{2\pi}} \int z e^{a \lambda + \frac{1}{2} b^2} e^{-\frac{1}{2} x^2} 1_{\{z e^{a+bx} < \bar{H} \}} dx
\]
\[
= \frac{z^\lambda}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2} (x-b)^2} e^{-\frac{1}{2} z^2} dx
\]
\[
= \frac{z^\lambda}{\sqrt{2\pi}} \int_0^\infty e^{-\frac{1}{2} (x-b)^2} dx
\]
\[
= z^\lambda e^{\Gamma(t)} \left[ 1 - \Phi \left( \frac{\ln \left( \frac{t}{h} \right) - a}{b} - b \lambda \right) \right] = z^\lambda e^{\Gamma(t)} \left[ 1 - \Phi(-d_1(h, z, t)) \right].
\]
Finally we get
\[
E \left[ \frac{H_f}{H_t} I(y_1 H_T) 1_{(H_T < \bar{H})} \right] = \frac{y_1}{H_t} \psi(H_t)
\]
\[
= (y_1 H_t)^{-\frac{1}{\gamma}} e^{\Gamma(t)} \left[ 1 - \Phi(-d_1(h, H_t, t)) \right].
\]

2. For the second term of equation (4.5) we derive
\[
E \left[ \frac{H_f}{H_t} q 1_{(h \leq H_T < \bar{H})} \right] = \frac{q}{H_t} E \left[ g(H_t, \eta) \right] = \frac{q}{H_t} \psi(H_t),
\]
where \( g(z, x) = z e^{a+bx} 1_{\{h \leq z e^{a+bx} < \bar{H} \}} \). Computing \( \psi \) we get
\[
\psi(z) = \frac{1}{\sqrt{2\pi}} \int z e^{a \lambda + \frac{1}{2} b^2} e^{-\frac{1}{2} x^2} 1_{\{h \leq z e^{a+bx} < \bar{H} \}} dx
\]
\[
= \frac{z}{\sqrt{2\pi}} \int_0^{\ln \left( \frac{h}{b} \right) - a} e^{-\frac{1}{2} (x-b)^2} dx
\]
\[
= \frac{z}{\sqrt{2\pi}} \int_0^{\ln \left( \frac{h}{b} \right) - a} e^{-\frac{1}{2} x^2} dx
\]
\[
= ze^{-r(T-t)} \left[ \Phi \left( \frac{\ln \left( \frac{h}{b} \right) - a}{b} - b \right) - \Phi \left( \frac{\ln \left( \frac{h}{b} \right) - a}{b} - b \right) \right]
\]
\[
= ze^{-r(T-t)} \left[ \Phi(-d_2(h, z, t)) - \Phi(-d_2(h, z, t)) \right],
\]
and we have
\[
E \left[ \frac{H_f}{H_t} q 1_{(h \leq H_T < \bar{H})} \right] = \frac{q}{H_t} \psi(H_t)
\]
\[
= q e^{-r(T-t)} \left[ \Phi(-d_2(h, H_t, t)) - \Phi(-d_2(h, H_t, t)) \right].
\]
3. For the third term we obtain

$$E \left[ \frac{H_t}{H_t} I(\frac{y_1}{1 + y_2} H_t) \mathbb{1}_{\{H_t \geq \overline{y}\}} \right] F_t = \left( \frac{y_1}{1 + y_2} \right)^{-\frac{1}{\gamma}} E \left[ g(H_t, \eta) \right] F_t = \left( \frac{y_1}{1 + y_2} \right)^{-\frac{1}{\gamma}} \psi(H_t)$$

with $g(z, x) = z^\lambda e^{\lambda(a + bx)} \mathbb{1}_{\{ze^{\alpha + bx} \geq \overline{y}\}}$ and $\lambda = 1 - \frac{1}{\gamma}$. Computing $\psi$ we get

$$\psi(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} z^\lambda e^{a\lambda + \frac{1}{2}b^2\lambda^2} e^{-\frac{1}{2}(x-b\lambda)^2} dx$$

$$= \frac{z^\lambda}{\sqrt{2\pi}} e^{a\lambda + \frac{1}{2}b^2\lambda^2} \int_{-\infty}^{\ln(z) - a - b\lambda} e^{-\frac{1}{2}x^2} dx$$

$$= \frac{z^\lambda}{\sqrt{2\pi}} e^{a\lambda + \frac{1}{2}b^2\lambda^2} \int_{-\infty}^{\ln(z) - b\lambda} e^{-\frac{1}{2}x^2} dx$$

$$= z^\lambda e^{\Gamma(t)} \left[ \Phi(\frac{\ln(z)}{b} - a - b\lambda) \right] = z^\lambda e^{\Gamma(t)} \Phi(-d_1(\overline{h}, z, t))].$$

Finally we obtain

$$E \left[ \frac{H_t}{H_t} I(\frac{y_1}{1 + y_2} H_t) \mathbb{1}_{\{\overline{y} \leq H_t\}} \right] F_t = \left( \frac{y_1}{1 + y_2} H_t \right)^{-\frac{1}{\gamma}} e^{\Gamma(t)} \Phi(-d_1(\overline{h}, H_t, t))].$$

C Proof of Lemma 4.10

The proof is analogous to the above proof of Lemma 4.5 if the quantity $q$ is substituted by $q = q(z) = Az^{-\nu}$. The dependence of $q$ on $z$ affects only the solutions of the inequalities given in (A.1),(A.2) and (A.3). Taking into account the dependence of $q$ on $z$ we get the following solutions.

(i) $q < x_1$:

The inequality $u'(q) = (Az^{-\nu})^{-\gamma} = A^{-\gamma} z^{\gamma\nu} > y_1 z$ is fulfilled

(a) for $\nu < \frac{1}{\gamma}$ if $z < \left( \frac{1}{y_1 A^\gamma} \right)^{\frac{1}{1-\gamma\nu}} = \overline{h}$,

(b) for $\nu > \frac{1}{\gamma}$ if $z > \overline{h}$

(ii) $x_1 \leq q < x_2$:

The inequalities $\frac{y_1}{1 + y_2} z < u'(q) = (Az^{-\nu})^{-\gamma} = A^{-\gamma} z^{\gamma\nu} \leq y_1 z$ are fulfilled

(a) for $\nu < \frac{1}{\gamma}$ if $\overline{h} \leq z < H = \left( \frac{1+y_2}{y_1 A^\gamma} \right)^{\frac{1}{1-\gamma\nu}}$,

(b) for $\nu > \frac{1}{\gamma}$ if $\overline{h} \leq z < \overline{h}$
(iii) $\mathbf{q} \geq \mathbf{x}_2$:

The inequality $\frac{w_1}{1+y_2} z \geq u'(q) = (Az^{-\nu})^{-\gamma} = A^{-\gamma}z^{-\gamma}$ is fulfilled

(a) for $\nu < \frac{1}{\gamma}$ if $z \geq \bar{h}$,

(b) for $\nu > \frac{1}{\gamma}$ if $z \leq \bar{h}$.

D Evaluation of the conditional expectations in Eq. (4.10)

We refer to Appendix B containing the evaluation of the conditional expectations in Eq. (4.5). Using the same arguments the three conditional expectations can be computed as follows.

1. For the first term of Eq. (4.10) we get

$$
E \left[ \frac{H_T}{H_t} I(y_1 H_T) 1_{(H_T < \bar{h})} | \mathcal{F}_t \right] = \frac{-y_1^{-\gamma}}{H_t} E \left[ g(H_t, \eta) | \mathcal{F}_t \right] = \frac{-y_1^{-\gamma}}{H_t} \psi(H_t)
$$

with $g(z, x) = z^\lambda e^{\lambda (a+bx)} 1_{\{x \in (z, \bar{h})\}}$, where $\lambda = 1 - \frac{1}{\gamma}$. Computing $\psi$ we get

$$
\psi(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} z^\lambda e^{a \lambda + b \lambda x} e^{-\frac{1}{2}x^2} 1_{\{x \in (z, \bar{h})\}} dx
$$

$$
= \frac{z^\lambda}{\sqrt{2\pi}} e^{a \lambda + \frac{1}{2} b^2 \lambda^2} \int_{\ln\left(\frac{h}{\bar{h}}\right)/b}^{\infty} e^{-\frac{1}{2}(x-b\lambda)^2} dx
$$

$$
= \frac{z^\lambda}{\sqrt{2\pi}} e^{a \lambda + \frac{1}{2} b^2 \lambda^2} \int_{\ln\left(\frac{h}{\bar{h}}\right)/b}^{-b\lambda} e^{-\frac{1}{2}x^2} dx
$$

$$
= z^\lambda e^{\Gamma_1(t)} \left[ 1 - \Phi\left(\frac{\ln\left(\frac{h}{\bar{h}}\right) - a}{b} - b\lambda\right)\right] = z^\lambda e^{\Gamma_1(t)} [1 - \Phi(-d_1(h, z, t))].
$$

Finally we get

$$
E \left[ \frac{H_T}{H_t} I(y_1 H_T) 1_{(H_T < \bar{h})} | \mathcal{F}_t \right] = \frac{-y_1^{-\gamma}}{H_t} \psi(H_t)
$$

$$
= (y_1 H_t)^{-\frac{1}{\gamma}} e^{\Gamma_1(t)} [1 - \Phi(-d_1(h, H_t, t))].
$$

2. For the second term of equation (4.10) we derive

$$
E \left[ \frac{H_T}{H_t} A H_T^{-\nu} 1_{\{H_T < \bar{h}\}} | \mathcal{F}_t \right] = \frac{A}{H_t^\nu} E \left[ g(H_t, \eta) | \mathcal{F}_t \right] = \frac{A}{H_t^\nu} \psi(H_t)
$$
with \( g(z, x) = z^\lambda e^{(a+bx)} \mathbf{1}_{\{\|z\| < 1\}} \), where \( \lambda = 1 - \nu \). Then we get

\[
\psi(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} z^\lambda e^{a\lambda + bx^2} e^{-\frac{1}{2}x^2} 1_{\{|z| < 1\}} dx
\]

\[
= \frac{z^\lambda}{\sqrt{2\pi}} e^{a\lambda + \frac{1}{2}b^2}\lambda^2 \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-b\lambda)^2} dx
\]

\[
= \frac{z^\lambda}{\sqrt{2\pi}} e^{a\lambda + \frac{1}{2}b^2}\lambda^2 \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx
\]

\[
= z^\lambda e^{\Gamma_2(t)} \left[ \Phi\left( \frac{\ln(z)}{b} - a - b\lambda \right) - \Phi\left( \frac{\ln(\frac{z}{b})}{a} - b\lambda \right) \right]
\]

and we have

\[
E \left[ \frac{H_t}{H_t} A H_T^{-\nu} 1_{\{H_T < \eta\}} | \mathcal{F}_t \right] = \frac{A}{H_t} \psi(H_t)
\]

\[
= AH_T^{-\nu} e^{\Gamma_2(t)} \left[ \Phi(-d_2(h, H_t, t)) - \Phi(-d_2(\eta, H_t, t)) \right].
\]

3. For the third term we get

\[
E \left[ \frac{H_t}{H_t} I\left( \frac{y_1}{1 + y_2 H_T} \right) 1_{\{H_T \geq \eta\}} | \mathcal{F}_t \right] = \left( \frac{y_1}{1 + y_2} \right)^{\frac{1}{\gamma}} E \left[ g(H_T, \eta) | \mathcal{F}_t \right] = \left( \frac{y_1}{1 + y_2} \right)^{\frac{1}{\gamma}} \psi(H_t)
\]

with \( g(z, x) = z^\lambda e^{(a+bx)} 1_{\{z \geq 1\}} \), where \( \lambda = 1 - \frac{1}{\gamma} \). Computing \( \psi \) we get

\[
\psi(z) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} z^\lambda e^{a\lambda + bx^2} e^{-\frac{1}{2}x^2} 1_{\{z \geq 1\}} dx
\]

\[
= \frac{z^\lambda}{\sqrt{2\pi}} e^{a\lambda + \frac{1}{2}b^2}\lambda^2 \int_{-\infty}^{\infty} e^{-\frac{1}{2}(x-b\lambda)^2} dx
\]

\[
= \frac{z^\lambda}{\sqrt{2\pi}} e^{a\lambda + \frac{1}{2}b^2}\lambda^2 \int_{-\infty}^{\infty} e^{-\frac{1}{2}x^2} dx
\]

\[
= z^\lambda e^{\Gamma_1(t)} \left[ \Phi\left( \frac{\ln(z)}{b} - a - b\lambda \right) \right] = z^\lambda e^{\Gamma_1(t)} [\Phi(-d_1(\eta, z, t))].
\]

Finally we obtain

\[
E \left[ \frac{H_t}{H_t} I\left( \frac{y_1}{1 + y_2} H_T \right) 1_{\{H_T \leq \eta\}} | \mathcal{F}_t \right] = \left( \frac{y_1}{1 + y_2} H_t \right)^{\frac{1}{\gamma}} e^{\Gamma_1(t)} [\Phi(-d_1(\eta, H_t, t))].
\]
E Evaluation of the derivative $F_z(z, t)$ in Eq. (4.7)

We rewrite Equation (4.7) as follows

\[
F_z(z, t) = -\frac{1}{\gamma z} \left[ F(z, t) - q e^{-r(T-t)} \left( \Phi(-d_2(z, t)) - \Phi(-d_2(T, z, t)) \right) \right] + \frac{1}{z \kappa \sqrt{T-t}} (V_1 + V_2),
\]

where

\[
V_1 = -\frac{e^{\Gamma(t)}}{(y_1 z)^{\frac{3}{2}}} \left[ \varphi(d_1(z, t)) - (1 + y_2)^\frac{1}{2} \varphi(d_1(T, z, t)) \right]
\]

and

\[
V_2 = q e^{-r(T-t)} \left[ \varphi(d_2(z, t)) - \varphi(d_2(T, z, t)) \right].
\]

For simplicity we make the following notation:

\[
a = \kappa \sqrt{T-t}, \quad d_1(z, t) = \bar{d}_1, \quad d_2(z, t) = \bar{d}_2, \quad d_1(T, z, t) = \bar{d}_1, \quad d_2(T, z, t) = \bar{d}_2.
\]

Since $d_1 = d_2 + \frac{a}{\gamma}$, this implies that $\varphi(d_1) = \varphi(d_2)e^{-\frac{1 + 2}{2} \gamma} e^{-\frac{a}{\gamma} \bar{d}_2}$. Moreover, it holds

\[
-\frac{\ln(1 + y_2)}{a} \bar{d}_1 = -\frac{\ln(1 + y_2)}{a} \bar{d}_2 + \ln(1 + y_2)^{\frac{1}{2}}.
\]

As consequence we obtain

\[
V_1 = -\frac{e^{\Gamma(t)}}{(y_1 z)^{\frac{3}{2}}} \varphi(d_1) \left[ 1 - (1 + y_2)^{\frac{1}{2}} e^{-\frac{1}{2} \ln(1 + y_2)} \varphi(d_2) \right] = \frac{e^{\Gamma(t) - \frac{1 + 2}{2} \gamma} e^{-\frac{a}{\gamma} \bar{d}_2}}{(y_1 z)^{\frac{3}{2}}} \varphi(d_2) \left[ 1 - e^{-\frac{1}{2} \ln(1 + y_2)} \varphi(d_2) \right] = -\frac{\ln(1 + y_2)}{a} \bar{d}_2 + \ln(1 + y_2)^{\frac{1}{2}}.
\]

From the other hand we have

\[
\Gamma(t) - \frac{1}{2} \frac{a^2}{\gamma^2} - \frac{a}{\gamma} \bar{d}_2 = \Gamma(t) - \frac{1}{2} \frac{a^2}{\gamma^2} - \frac{1}{\gamma} (r - \frac{\kappa^2}{2}) (T-t) + \ln \left( \frac{h}{z} \right)^{-\frac{1}{2}},
\]

which implies that

\[
V_1 = -\frac{e^{-r(T-t)} (h/z)^{-\frac{1}{2}}}{(y_1 z)^{\frac{3}{2}}} \varphi(d_2) \left[ 1 - e^{-\frac{1}{2} \ln(1 + y_2)} \varphi(d_2) \right] = -\frac{q e^{-r(T-t)} \varphi(d_2)}{1 - e^{-\frac{1}{2} \ln(1 + y_2)} \varphi(d_2)}
\]

Hence, the claim is proved.
F Evaluation of the derivative \( F_z(z, t) \) in Eq. (4.12)

We rewrite Equation (4.12) as follows

\[
F_z(z, t) = -\frac{1}{\gamma z} \left[ F(z, t) - (1 - \gamma \nu) \frac{A e^{\Gamma_2(t)}}{z^\nu} \left( \Phi(-d_2(h, z, t)) - \Phi(-d_2(h, z, t)) \right) \right] + \frac{1}{z \kappa \sqrt{T-t}} (V_1 + V_2),
\]

where

\[
V_1 = -\frac{e^{\Gamma_1(t)}}{(y_1 z)^\frac{1}{\gamma}} \left[ \varphi(d_1(h, z, t)) - (1 + y_2)^\frac{1}{\gamma} \varphi(d_1(h, z, t)) \right],
\]

and

\[
V_2 = \frac{A e^{\Gamma_2(t)}}{z^\nu} \left[ \varphi(d_2(h, z, t)) - \varphi(d_2(h, z, t)) \right].
\]

We denote by \( \alpha = \frac{1}{1-\gamma \nu} \) and we save the same notation as in the last section. We have \( d_1 = d_1 + \frac{\ln(1+y_2)^\alpha}{\alpha} \), this implies that \( \varphi(d_1) = \varphi(d_1)e^{-\frac{(\ln(1+y_2)^\alpha)}{\alpha} \frac{\ln(1+y_2)^\alpha}{d_1}} \) and as consequence we get

\[
V_1 = -\frac{e^{\Gamma_1(t)} \varphi(d_1)}{(y_1 z)^\frac{1}{\gamma}} \left[ 1 - (1 + y_2)^\frac{1}{\gamma} e^{-\frac{1}{2} \frac{(\ln(1+y_2)^\alpha)}{\alpha} \frac{\ln(1+y_2)^\alpha}{d_1}} \right].
\]

From other hand, we have

\[
V_2 = \frac{A e^{\Gamma_2(t)} \varphi(d_1)}{z^\nu} \left[ 1 - e^{-\frac{1}{2} \frac{(\ln(1+y_2)^\alpha)}{\alpha} \frac{\ln(1+y_2)^\alpha}{d_1}} \right].
\]

Since \( d_2 = d_1 + (\sigma - \frac{\xi}{\gamma}) \sqrt{T-t} \), this implies \( \varphi(d_2) = \varphi(d_1) e^{-\frac{1}{2} (\sigma - \frac{\xi}{\gamma})^2 (T-t) e^{-\frac{1}{2} (\sigma - \frac{\xi}{\gamma}) \sqrt{T-t} d_1}} \).

As consequence \( V_2 \) can be written as

\[
V_2 = \frac{A e^{\Gamma_2(t) \varphi(d_1)}}{z^\nu} e^{-\frac{1}{2} (\sigma - \frac{\xi}{\gamma})^2 (T-t) - (\sigma - \frac{\xi}{\gamma}) \sqrt{T-t} d_1} \left[ 1 - e^{-\frac{1}{2} \frac{(\ln(1+y_2)^\alpha)}{\alpha} \frac{\ln(1+y_2)^\alpha}{d_1}} \right].
\]

Let us compute

\[
\Gamma_0(t) := \Gamma_2(t) - \frac{1}{2} \left( \sigma - \frac{\kappa}{\gamma} \right)^2 (T-t) - \left( \sigma - \frac{\kappa}{\gamma} \right) \sqrt{T-t} d_1
\]

\[
= \Gamma_2(t) - \frac{1}{2} \left( \sigma - \frac{\kappa}{\gamma} \right)^2 (T-t)
\]

\[
- \left( \sigma - \frac{\kappa}{\gamma} \right) \sqrt{T-t} \left[ \frac{\frac{\ln h}{\kappa} + (r - \frac{\kappa^2}{2}) (T-t)}{\kappa \sqrt{T-t}} + \frac{\kappa}{\gamma} \sqrt{T-t} \right]
\]

\[
= \Gamma_2(t) - \left( \sigma - \frac{\kappa}{\gamma} \right) (T-t) \left( \frac{1}{2} \left( \sigma - \frac{\kappa}{\gamma} \right) + \left( r - \frac{\kappa^2}{2} \right) + \frac{\kappa}{\gamma} \right) - \left( \frac{\sigma}{\kappa} - \frac{1}{\gamma} \right) \ln \frac{h}{z}.
\]
After some algebra we get
\[
\Gamma_0(t) = \Gamma_1(t) - \left(\frac{\sigma}{\kappa} - \frac{1}{\gamma}\right) \ln \frac{h}{z} = \Gamma_1(t) + \ln \left(\frac{h}{z}^{-(\frac{\sigma}{\kappa} - \frac{1}{\gamma})}\right),
\]
and as consequence \(e^{\Gamma_0(t)} = e^{\Gamma_1(t)} \left(\frac{h}{z}^{-(\frac{\sigma}{\kappa} - \frac{1}{\gamma})}\right)\) which we substitute into \(V_2\) to get
\[
V_2 = \frac{A e^{\Gamma_1(t)} \varphi(d_t)}{z \nu z - (\frac{\sigma}{\kappa} - \frac{1}{\gamma})} \left[1 - e^{-\frac{1}{2} \left(\ln(1 + y_2)\right)^2} e^{-\frac{\ln(1 + y_2)}{a} d_2}\right]
= \frac{A e^{\Gamma_1(t)} \varphi(d_t)}{z \nu z - (\frac{\sigma}{\kappa} - \frac{1}{\gamma})} \left(\frac{1}{y_1}\right)^{(\frac{\sigma}{\kappa} - \frac{1}{\gamma})} \left[1 - e^{-\frac{1}{2} \left(\ln(1 + y_2)\right)^2} e^{-\frac{\ln(1 + y_2)}{a} d_2}\right]
= \frac{e^{\Gamma_1(t)} \varphi(d_t)}{(y_1 z)^{\frac{1}{\gamma}}} \left[1 - e^{-\frac{1}{2} \left(\ln(1 + y_2)\right)^2} e^{-\frac{\ln(1 + y_2)}{a} d_2}\right]
\]
since we have \(-\left(\frac{\sigma}{\kappa} - \frac{1}{\gamma}\right)\alpha = \frac{1}{\gamma}\). Moreover, we have
\[
-\frac{\ln(1 + y_2)^\alpha}{a} d_2 = -\frac{\ln(1 + y_2)^\alpha}{a} d_2 = -\frac{\ln(1 + y_2)^\alpha}{a} d_1 - \ln(1 + y_2)^\alpha \left(\frac{\sigma}{\kappa} - \frac{1}{\gamma}\right)
= -\frac{\ln(1 + y_2)^\alpha}{a} d_1 - \ln(1 + y_2)^\alpha \left(\frac{\sigma}{\kappa} - \frac{1}{\gamma}\right)
= -\frac{\ln(1 + y_2)^\alpha}{a} d_1 - \ln(1 + y_2)^\alpha \left(\frac{\sigma}{\kappa} - \frac{1}{\gamma}\right).
\]
Hence, we obtain \(e^{-\frac{\ln(1 + y_2)^\alpha}{a} d_2} = (1 + y_2)^{\frac{1}{\gamma}} e^{-\frac{\ln(1 + y_2)^\alpha}{a} d_1}\) which we substitute into \(V_2\) to obtain
\[
V_2 = \frac{e^{\Gamma_1(t)} \varphi(d_t)}{(y_1 z)^{\frac{1}{\gamma}}} \left[1 - (1 + y_2)^{\frac{1}{\gamma}} e^{-\frac{1}{2} \left(\ln(1 + y_2)\right)^2} e^{-\frac{\ln(1 + y_2)}{a} d_1}\right].
\]
Finally we deduce that \(V_1 + V_2 = 0\).

**G  Proof of Proposition 4.15**

Following the argumentation of of the proof of Lemma 4.10 we get the following solutions of the stated inequalities.

(i) \(q < x_1\): The inequality \(u'(q) = (Az^{-\nu})^{-\gamma} = A^{-\gamma} Az^{-\nu} > y_1 z\) is fulfilled for all \(z > 0\) if \(y_1 A^{\gamma} \in (0, 1)\). Otherwise the inequality is never fulfilled.

(ii) \(x_1 \leq q < x_2\): The inequalities \(\frac{y_1}{1 + y_2} z < u'(q) = (Az^{-\nu})^{-\gamma} = A^{-\gamma} Az^{-\nu} \leq y_1 z\) are fulfilled for all \(z > 0\) if
\(y_1 A^\gamma \in [1, 1 + y_2]\). Otherwise the inequality is never fulfilled.

(iii) \(q \geq x_2\):

The inequality \(\frac{y_1}{1+y_2} z \geq u'(q) = (Az^{-\nu})^{-\gamma} = A^{-\gamma} z^{-\nu}\) is fulfilled for all \(z > 0\) if \(y_1 A^\gamma \in [1 + y_2, \infty)\). Otherwise the inequality is never fulfilled.

Provided an optimal solution exists Lemma 4.10 gives the following form of the optimal terminal wealth which depends on the value of \(y_1 A^\gamma\)

\[
\xi^* = \begin{cases} 
I(y_1 H_T) & \text{if } y_1 A^\gamma \in (0, 1) \\
AH_T^{-\nu} = Q & \text{if } y_1 A^\gamma \in [1, 1+y_2) \\
I\left(\frac{y_1}{1+y_2} H_T\right) & \text{if } y_1 A^\gamma \in [1 + y_2, \infty). 
\end{cases}
\]

In order to check the existence of an optimal solution one has to check whether there exist real numbers \(y_1 > 0\) and \(y_2 \geq 0\) such that it holds

\[
E[H_T \xi^*] = x \\
E\left[(u(\xi^*) - u(Q))^-\right] \leq \varepsilon.
\]

In the first case the parameter \(y_1\) can be chosen such that the budget constraint is fulfilled with equality, i.e., \(E[H_T \xi^*] = E[H_T I(y_1 H_T)] = x\) which gives \(y_1 = y^N\). It can be observed that \(\xi^*\) coincides with the optimal terminal wealth of the problem without risk constraint. If the risk constraint is fulfilled (not necessarily with equality) then it is the optimal solution since there is no other terminal wealth exceeding the expected utility of \(\xi^*\).

In the second case \(\xi^*\) coincides with the benchmark \(Q\). Using \(\nu = \frac{1}{\gamma}, I(z) = z^{-\frac{1}{\gamma}} = z^{-\nu}\) and defining \(\bar{y} = A^\nu\) we get

\[
\xi^* = AH_T^{-\nu} = (\bar{y} H_T)^{-\nu} = I(\bar{y} H_T).
\]

For the budget constraint this implies

\[
x = E[H_T \xi^*] = E[H_T I(\bar{y} H_T)]
\]

For the Expected Utility Loss we get \(E\left[(u(\xi^*) - u(Q))^-\right] = E\left[(u(Q) - u(Q))^-\right] = 0 < \varepsilon\).

If the parameters of our model which are involved in \(\bar{y}\) are such that the budget constraint is fulfilled with equality then the optimal terminal wealth is \(\xi^* = I(\bar{y} H_T)\) since the risk constraint is always fulfilled. Moreover it coincides with the optimal terminal wealth of the problem without risk constraint.

For other parameters there is no optimal solution.

In the third case the optimal terminal wealth is \(\xi^* = I\left(\frac{y_1}{1+y_2} H_T\right)\) if it satisfies

\[
E[H_T \xi^*] = x \\
E\left[(u(\xi^*) - u(Q))^-\right] \leq \varepsilon.
\]
For the first equation we get
\[ E[H_T \xi^*] = E[H_T I \left( \frac{y_1}{1 + y_2} H_T \right)]. \]

From the considerations of the problem without risk constraint it is known, that the equation is fulfilled for \( \frac{y_1}{1 + y_2} = y^N \). For the risk constraint we use that in the considered case it holds \( \xi^* \leq Q = AH_T^{-\nu} \) (see case (iii) of the proof of Lemma 4.10 which holds for \( y_1 A^\gamma \in [1 + y_2, \infty) \)). From this property it follows for the Expected Utility Loss
\[
E[(u(\xi^*) - u(Q))] = E[u(Q) - u(\xi^*)] = E[u(AH_T^{-\nu}) - u(I(y^N H_T))].
\]

If the numbers \( y_1 \) and \( y_2 \) are such that \( \frac{y_1}{1 + y_2} = y^N \) and \( y_1 A^\gamma \in [1 + y_2, \infty) \) then the Expected Utility Loss depends on the parameters of the financial market (as \( \mu, r, \sigma, x \)) via \( A, \nu, H_T, y^N \) and the parameter \( \gamma \) of the utility function via \( I \) but not directly on \( y_1 \) and \( y_2 \). So, if to a given \( \varepsilon \) and a \( y^N \) following from the solution of the first equation the risk constraint is fulfilled then the optimal terminal wealth is
\[ \xi^* = I \left( \frac{y_1}{1 + y_2} H_T \right) = I(y^N H_T). \]

Hence, it coincides with the optimal terminal wealth of the problem without risk constraint.

On the other hand, if the risk constraint is not fulfilled, then there is no admissible solution and consequently no optimal solution.

The given form of \( y^N \) follows directly from the representation of \( y^N \) given in Example 3.2 for \( \nu = \frac{1}{\gamma} \). Moreover in this case we have \( \theta^N = \frac{2\gamma}{\sigma} = \frac{1}{\gamma \nu} = 1 \).