Proof of the oval conjecture for proper planar partition surfaces

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Abstract

We prove the ‘oval conjecture’ for planar partition functions, which says that the shift plane and the translation plane defined by a planar partition function form an oval pair of planes in the sense that each non-vertical line of one plane defines a topological oval in the projective closure of the other. The proof uses covering space techniques, and we have to assume that the generating function is proper in order to make those techniques available. As an application, we give a natural geometric construction of a homeomorphism between the Cartesian square of the shift line and its tangent bundle.

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1. Introduction

The surfaces referred to in the title are the graphs of planar partition functions \( f : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \). These are \( C^1 \) maps which define a topological affine plane with a point transitive action of the group \( \mathbb{R}^4 \) in two ways. On one plane, the translation plane \( T_f \) defined by \( f \), the group \( \mathbb{R}^4 \) acts as a transitive translation group. The other plane is called the shift plane \( A_f \) generated by \( f \); the orbit configuration of the \( \mathbb{R}^4 \)-action on the projective closure of \( A_f \) is self-dual, and there is only a two-dimensional group of translations. Both planes have the verticals \( \{x\} \times \mathbb{R}^2, x \in \mathbb{R}^2 \), as lines, and the remaining lines of \( A_f \) are the translates.
of the graph of \( f \), while the remaining lines of \( T_f \) are the translates of all tangent spaces of that graph. The conditions that \( f \) has to satisfy in order to make this possible are given below. Functions \( \mathbb{R}^l \to \mathbb{R}^l \) with analogous properties exist only for \( l \in \{1, 2\} \), but for \( l = 1 \) the question treated in this paper becomes trivial. See [10, Section 74], for a thorough introduction to the subject; compare also [5] for recent progress on the recognition problem for planar partition functions. One of the most beautiful planar partition surfaces, due to Knarr, is considered in detail in [2].

Knarr gave some rather tricky examples of ovals contained in every shift plane and used this to prove their non-existence for \( l > 2 \), see [10, 74.6]. Looking at the pair of planes defined by a function \( f \), one feels that there should be far more obvious ovals, namely, each non-vertical line of the shift plane should be an oval in the translation plane and vice versa; points at infinity have to be added for this to make sense. At least, this is true in the case of the planar partition function \( f(z) = z^2 \), the complex squaring map. In this case, \( A_f \) is the parabola model of the complex affine plane and \( T_f \) is the complex affine plane in standard representation. Now it is a long standing conjecture that the above statement is true for every pair of planes defined by a planar partition function; however, to the present day there did not emerge even the slightest idea on how to prove this, although it has been verified in some concrete cases, see [9] (compare 2.4), [1]. In fact, the analogous question for finite planes has recently been answered in the negative [3].

In [10, 74.17(c)], the oval conjecture is listed together with some other unsolved questions concerning shift planes and their generating functions. Here we shall give a proof of that conjecture under the additional hypothesis that \( f \) is a proper map (continuous at infinity). It is very hard to imagine a planar partition function that is not proper, but despite all efforts, we have not been able to prove that this is impossible. Looking at all the known examples (which usually depend on some parameters) one finds that quite often the conditions on the parameters for the function \( f \) to be planar are precisely the conditions for \( f \) to be proper, so we have another intriguing problem. We shall prove that conversely, a planar partition function \( f \) is proper if its graph is an oval in \( T_f \).

The assumption that \( f \) be proper is used in our proof, together with a theorem of Ehresmann, in order to ensure that a suitable restriction of \( f \) is a covering map of \( \mathbb{R}^2 \setminus \{0\} \). Then we play with the groups of deck transformations of two covering spaces, defined by the complex exponential map and by the map \( f \), respectively.

In Section 6, we shall give some applications of our main result. First, we use it to show that a geometric construction that was done in [4] using concrete verification in special cases actually works for all proper planar partition functions. Moreover, we show that a certain bijection, which can be defined naturally in geometric terms, is always a homeomorphism between the Cartesian square of the graph of such a function and its tangent bundle.

### 2. Planar partition functions and the conditions (PA) and (PL)

#### 2.1. Planar functions

Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be continuous. We denote the graph of \( f \) by

\[
\Gamma_f := \{(x, f(x)) \mid x \in \mathbb{R}^2\} \subseteq \mathbb{R}^4,
\]
and we consider the incidence structure \((A_f, L_f)\) with point set \(\mathbb{R}^4\) and line set \(L_f = \{\Gamma_f, \{0\} \times \mathbb{R}^2\} + \mathbb{R}^4\) consisting of all translates \(\Gamma_f + v, v \in \mathbb{R}^4\), and all verticals. In order that \(A_f\) be a topological affine plane, the following planarity condition is necessary and sufficient, see [10, 74.3].

\[(PL)\] For all \(d \in \mathbb{R}^2 \setminus \{0\}\), the function \(f_d : \mathbb{R}^2 \to \mathbb{R}^2\) defined by
\[
f_d(x) = f(x + d) - f(x)
\]
is bijective. A function satisfying this condition is called **planar**.

### 2.2. Partition functions

We consider a \(C^1\) function \(f : \mathbb{R}^2 \to \mathbb{R}^2\). The derivatives \(D_x f, x \in \mathbb{R}^2\), will be considered as linear maps or as matrices, but also as 2-dimensional vector subspaces of \(\mathbb{R}^4\). In other words, we shall identify the map \(D_x f\) with its graph, which is the tangent vector space of \(\Gamma_f\) in the point \((x, f(x))\). We assume that \(f\) satisfies the following condition.

\[(PA)\] The set
\[
S_f := \{D_x f \mid x \in \mathbb{R}^2\} \cup \{(0) \times \mathbb{R}^2\}
\]
is a partition of \(\mathbb{R}^4\), that is, each non-zero vector belongs to exactly one of these subspaces. In particular, we are assuming that the matrices \(D_x f\) are pairwise distinct.

Taking the translates of all \(L \in S_f\) as lines, we obtain a translation plane
\[
T_f = (\mathbb{R}^4, S_f + \mathbb{R}^4).
\]

We claim that \(T_f\) is a topological translation plane. Since \(f\) is of class \(C^1\), the map \(\alpha : x \mapsto D_x f\) is a continuous bijection of \(\mathbb{R}^2\) onto the set \(S_f' := S_f \setminus \{(0) \times \mathbb{R}^2\}\) of matrices. On the other hand, taking intersections with \(W := \{(0, 1)\} \times \mathbb{R}^2\), we obtain a continuous bijection \(\beta : S_f' \to W\). The domain invariance theorem ([11, 4.7.16]; compare [10, 51.19]) implies that \(\beta \alpha\) is a homeomorphism, hence \(S_f' \approx \mathbb{R}^2\). Now we infer from [10, 64.8 and 64.4], that the subspace \(S_f\) of the Grassmann manifold \(G_{2,4}\) is homeomorphic to the 2-sphere and that the translation plane \(T_f\) defined by \(S_f\) is a topological plane. (For background on topological translation planes, we refer the reader to [10, Section 64].) The fact that two tangent spaces are always complementary implies

**Lemma 2.3.** Suppose that \(f : \mathbb{R}^2 \to \mathbb{R}^2\) is a partition function. For any two distinct elements \(x, y \in \mathbb{R}^2\), the matrix \(D_x f - D_y f\) is invertible. \(\Box\)

We are interested in **planar partition functions**, that is, \(C^1\)-functions that satisfy both conditions (PL) and (PA). We remark that it is an open problem whether (PA) and (PL) are equivalent for \(C^1\)-functions, compare [10, 74.17, 74.19] and [5].

**Example 2.4** (Examples of Planar Partition Functions). (a) The standard example is the complex squaring map \(f(z) = z^2\), i.e., \(f(s, t) = (s^2 - t^2, 2st)\). In this case, \(A_f\) is the parabola model of the complex affine plane, which is isomorphic to the complex plane via \((z, w) \mapsto (z, w - z^2)\), and \(T_f\) is the complex plane itself.
(b) A large class of examples was constructed by Polster [9, 4.2]. Given two functions \( a, b : \mathbb{R} \to \mathbb{R} \) whose first derivatives are increasing and bijective, he defines

\[
f(s, t) = (a(s) - b(t), 2st).
\]

These functions satisfy (PA) and (PL). Example 2.4 (a) is the special case \( a(s) = b(s) = s^2 \).

### 3. Ovals

#### 3.1. Topological ovals

By definition, an oval in a projective plane is a set \( O \) of points meeting every line in at most two points, and such that every point \( p \in O \) is on exactly one line \( T = T_p \) such that \( T \cap O = \{p\} \). Lines meeting \( O \) in 0, 1, or 2 points are called exterior lines, tangents, and secants, respectively; they form subsets \( L_i \subseteq L \) for \( i \in \{0, 1, 2\} \).

If the plane is compact and connected, then the oval is called a topological oval if the map which sends a line \( L \in L \setminus L_0 \) to \( L \cap O \) is a homeomorphism onto the symmetric square \( O \ast O \), i.e., the set of all unordered pairs \( \{x, y\} \), \( x, y \in O \) (\( x = y \) allowed) endowed with the quotient topology derived from the map \( O \times O \to O \ast O \) given by \((x, y) \to \{x, y\}\). This is a kind of differentiability property, with the lines of the plane playing the role of the linear maps. Indeed, it follows that, given distinct points \( x_n, y_n \in O \) for every \( n \in \mathbb{N} \), convergence of both sequences \( x_n \) and \( y_n \) to the same point \( p \in O \) implies convergence of the secants \( x_n \lor y_n \) to the tangent \( T_p \). The property of being topological is equivalent to compactness of \( O \), see [6] or [10, 55.11]. Note also that for compact ovals in 4-dimensional planes, the set \( L_0 \) is always empty, see [6] or [10, 55.14].

We remark here that there is a tendency with some authors to define topological ovals as compact ovals. This terminology obscures the main result of [6], namely, that compact ovals have the differentiability property mentioned above.

The planes that we shall consider are the projective closures of \( A_f \) and of \( T_f \). The affine planes have the same point set, \( \mathbb{R}^4 \), and their closures have one point at infinity in common, namely the point \( \infty \) corresponding to the parallel class of vertical lines. Our ovals will consist of certain subsets of \( \mathbb{R}^4 \) (namely, non-vertical lines of one of the planes), augmented by the point \( \infty \). The lines are homeomorphic to \( \mathbb{R}^2 \), hence [10, 55.18] asserts that in order to show that we obtain topological ovals, we need only verify the incidence properties of ovals, not the continuity property.

**Definition 3.2.** By an oval pair of affine planes we mean a pair of affine planes with the same point set and having one parallel class of lines in common (called the verticals and defining the point \( \infty \) at infinity) such that each non-vertical line of either plane, augmented by the point \( \infty \), becomes an oval in the projective closure of the other plane.

Clearly, each of these ovals has the respective line at infinity as a tangent. If the planes are locally compact, connected, and of dimension \( 2l \leq 4 \), then their lines are homeomorphic to \( \mathbb{R}^l \) by [10, 53.5 and 53.15], the projective closures are compact and connected, and the ovals that we obtain are topological ovals by the result stated at the end of Section 3.1.
Example 3.3 (Examples of Oval Pairs Abound). Take any locally compact finite-dimensional connected Laguerre plane and form the derived affine planes at two parallel points ([8]; compare also [12, 2.2]). In the particular case of the complex Laguerre plane, we obtain the pair of planes $A_f, T_f$ of Example 2.4(a).

We state our main result:

**Theorem 3.4.** Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a proper $C^1$ map. If $f$ is a planar partition function, that is, if $f$ satisfies (PL) and (PA), then the shift plane $A_f$ and the translation plane $T_f$ defined by $f$ form an oval pair of planes.

The proof will be given in the next section. As we remarked in the Introduction, it seems desirable to prove the theorem without assuming that $f$ be proper, but this appears to be hard. Compare also Section 5.

4. Proof of Theorem 3.4

What we want to show amounts to the following.

**Proposition 4.1.** Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a proper planar partition function, and consider a line $L = D_x f + w, x \in \mathbb{R}^2, w \in \mathbb{R}^4$, of the plane $T_f$. The intersection $L \cap \Gamma_f$ consists of one single point $p$ if $L$ is the tangent space of $\Gamma_f$ at $p$ (in the analytic sense). In all other cases, the intersection consists of exactly two points.

**Proof of Theorem 3.4.** In both planes, the line at infinity is a tangent for the sets that are claimed to be ovals. Moreover, Proposition 4.1 implies that no line intersects such a set in more than two points. It remains to be shown that the sets have unique tangents at all points $p$. No vertical line can be a tangent, hence we may assume that $p$ is an affine point. For $p \in \Gamma_f + v$, the proposition says explicitly that the differential tangent space at $p$ is the only tangent in the geometric sense. Conversely, suppose that $L$ is a line of $T_f$ and that $\Gamma_f + v$ and $\Gamma_f + w$ are two distinct translates of $\Gamma_f$ that intersect $L$ only in the point $p$. Then $\Gamma_f$ has parallel tangent spaces in the points $p-v$ and $p-w$, contrary to the definition of a planar function. □

**Proof of Proposition 4.1.** (1) By applying a translation, we may arrange that $(x, f(x)) = (0, 0)$. Subsequently, we apply the transformation $(u, v) \mapsto (u, v - D_0 f(u))$ to both planes. This shows that we may assume that $D_0 f$ is the zero map and, hence, that

$$L = \mathbb{R}^2 \times \{v\}$$

for some $v$. Our task is to show that $L$ intersects $\Gamma_f$ only in $(0, 0)$ if $v = 0$, and that the intersection consists of exactly two points in all other cases. In other words, we want to prove that

$$f \text{ induces a } 2:1\text{-map } \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}. \quad (*)$$

(2) First we restrict $f$ to $U := \mathbb{R}^2 \setminus f^{-1}(0)$. Setting $y = 0$ in Lemma 2.3, we see that $D_x f$ is invertible for each $x \neq 0$, and the restriction of $f$ is a local diffeomorphism. Moreover, $f$ is a proper map by assumption; this property carries over to the restriction. A theorem of Ehresmann, see [7, 8.12], asserts that such a map is a locally trivial fibration.
and, more precisely, a surjective covering map. In the sequel, we shall freely use results about covering maps, see, for instance, [11].

(3) Since \( f \) is a local diffeomorphism at all points except 0, we conclude that \( f^{-1}(0) \setminus \{0\} \) is discrete. It follows that the number \( k \) of sheets of our covering map cannot be infinite, since the only infinite covering of \( \mathbb{R}^2 \setminus \{0\} \) is the universal covering by \( \mathbb{R}^2 \). Moreover, we conclude that the fundamental group \( \pi_1U \) is a non-cyclic free group unless \( f^{-1}(0) = \{0\} \). Now the infinite cyclic group \( \pi_1(\mathbb{R}^2 \setminus \{0\}) \cong \mathbb{Z} \) contains an isomorphic copy of \( \pi_1U \) (the image under the homeomorphism induced by \( f \)). Thus we have shown that the map considered in (\( * \)) exists and is a \( k \)-fold covering map, for some \( k \in \mathbb{N} \). Our goal is to show that \( k = 2 \).

(4) Of course, the unique \( k \)-fold covering map \( \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\} \) is well known: we may identify the topological space \( \mathbb{R}^2 \setminus \{0\} \) with the multiplicative group \( \mathbb{C}^\times = \mathbb{C} \setminus \{0\} \) in such a way that \( f \) corresponds to the power map \( z \mapsto z^k \). Note however, that this identification destroys the additive structure, hence we have to be careful not to use it directly in conjunction with the planarity condition (PL).

(5) The group of deck transformations of the covering map \( f \) is a cyclic group \( \Delta \) of order \( k \), with generator \( \delta \) satisfying \( f \circ \delta = f \). (The map \( \delta \) is multiplication by a primitive \( k \)th root of unity. We used the symbol \( \circ \) once to indicate that the second factor in a composition comes first, and we shall suppress the symbol henceforth.) There is another covering map relevant to us: the infinite universal covering \( q : \mathbb{C} \to \mathbb{C}^\times \) given by the complex exponential map. Its group \( \Omega \) of deck transformations is generated by the map \( \omega : z \mapsto z + 2\pi i \).

(6) Since \( \mathbb{C} \) is simply connected, the map \( q \) lifts over the covering \( f \), that is, there is a map \( h : \mathbb{C} \to \mathbb{C}^\times \) such that \( fh = q \). All other lifts of \( q \) are obtained from \( h \), using the deck transformations, as

\[
 h_i := \delta^{-1}h_i, \quad i = 1, \ldots, k.
\]

(7) Now the map \( h \omega \) is a lift of \( q \), hence \( h \omega = \delta^a h \), and

\[
 h_i \omega = h_{j+a},
\]

where \( j + a \) is taken modulo \( k \). In other words, right multiplication by \( \omega \) induces a cyclic permutation of the lifts \( h_i \).

(8) Consider a fixed vector \( d \in \mathbb{R}^2 \setminus \{0\} \). The planarity condition (PL) asserts the existence of a unique pair of elements \( x, x' \in \mathbb{C} \) such that \( f(x) = f(x') \) and \( x' - x = d \) (namely, \( x = f_d^{-1}(0) \) and \( x' = x + d \)). From (3) it follows that \( x, x' \in \mathbb{C}^\times \). Choose an element \( z \in \mathbb{C} \) such that \( q(z) = f(x) \). Then

\[
 x, x' \in f^{-1}(q(z)) = \{h_1(z), \ldots, h_k(z)\}.
\]
Hence, there are numbers $i, j \leq k$ such that
\[ d = x' - x = h_i(z) - h_j(z). \]

(9) The numbers $i, j$ depend on $d$, of course, but also on our choice of $z \in q^{-1}(f(x))$. A different choice has the form $\tilde{z} = \omega^m(z)$. Then (7) shows that
\[ x' = h_j\omega^{-m}(\tilde{z}) = h_{i-ma}(\tilde{z}), \]
and similarly for $x$, hence the new pair of indices is $(\tilde{i}, \tilde{j}) = (i - ma, j - ma)$, and we have shown that $d$ determines $(i, j)$ up to a cyclic permutation.

(10) We shall end the proof by showing that $h_i - h_j$ is a submersion and, in particular, an open map. Then the set of all $d \in \mathbb{R}^2 \setminus \{0\}$ that can be represented as $d = h_i(z) - h_j(z)$ is open. By connectedness of $\mathbb{R}^2 \setminus \{0\}$ and by (9), there is then only one class of index pairs $(i, j)$ up to cyclic permutation, and it follows that $k = 2$.

(11) Locally, we have $h_i = f^{-1}q$, hence
\[ D_z(h_i - h_j) = ((D_{h_i(x)}f)^{-1} - (D_{h_j(x)}f)^{-1})D_zq. \]

Here, the inverted matrices are distinct elements $A, B \in S_f \setminus \{0\}$, and
\[ A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1} \]
is regular by Lemma 2.3. □

**Corollary 4.2.** Every proper planar partition function $f : \mathbb{R}^2 \to \mathbb{R}^2$ is topologically equivalent to the complex squaring map $g(z) = z^2$.

**Proof.** The proof of Proposition 4.1 has shown that there exist $a, b \in \mathbb{R}^2$ such that the restriction $f : \mathbb{R}^2 \setminus \{a\} \to \mathbb{R}^2 \setminus \{b\}$ is equivalent to the squaring map restricted to $\mathbb{R}^2 \setminus \{0\}$, and we may assume that $a = b = 0$. This means that there is a homeomorphism $\varphi : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ such that $\varphi f = g\varphi$. The homeomorphism $\varphi$ extends to a homeomorphism of the 2-sphere $S_2 = \mathbb{R}^2 \cup \{\infty\}$. (For example, this follows from the fact that $S_2$ is the Freudenthal compactification of $\mathbb{R}^2 \setminus \{0\}$.) We may assume that $\varphi(0) = 0$; if not, then $\varphi(0) = \infty$ and we replace $\varphi$ by $\sigma\varphi$, where $\sigma$ denotes the inversion map, which commutes with $g$. By continuity, the equation $\varphi f = g\varphi$ holds on all of $\mathbb{R}^2$. □

5. Necessity of $f$ being proper

As we remarked in the Introduction, we would like to prove that every planar function is proper. Our attempts were, however, so fruitless that we even doubt whether this is true, however unlikely it may seem. In particular, we tried to exploit the fact that the maps $f_d$ appearing in condition (PL) are homeomorphisms (and hence proper), as well as the topology on the point and line spaces of the projective closure of $A_f$ and the way the group $\mathbb{R}^3$ acts on these spaces. The trouble arises from the fact that the values of the function $f$ have no geometric meaning in the plane $A_f$, because the horizontal fibers of the Cartesian decomposition $\mathbb{R}^4 = \mathbb{R}^2 \times \mathbb{R}^2$ are not lines. At least we can show that failure to prove this does not reduce the number of examples of oval pairs of planes provided by Theorem 3.4. Indeed, we have the following result.
Proposition 5.1. Let \( f : \mathbb{R}^2 \to \mathbb{R}^2 \) be a planar partition function and assume that \( O = \Gamma_f \cup \{\infty\} \) is an oval in the projective closure of the translation plane \( T_f \). Then \( f \) is a proper map.

Proof. From the fact that \( \Gamma_f \) is homeomorphic to \( \mathbb{R}^2 \) it follows that \( O \) is a compact and, hence, topological oval, compare Section 3.1. We may assume that both factors \( X = \mathbb{R}^2 \times \{0\} \) and \( Y = \{0\} \times \mathbb{R}^2 \) of \( \mathbb{R}^4 \) are lines, with points at infinity \( 0 \) and \( \infty \). The map \( f \) can be described as follows: \( (0, f(x)) = (((\infty \vee (x, 0)) \cap \Gamma_f) \vee 0) \wedge Y \). For \( x \) tending to infinity in the one-point compactification of \( \mathbb{R}^2 \), the lines \( \infty \vee (x, 0) \) tend to the line \( W \) at infinity, which is a tangent of \( O \). The affine intersection points with \( O \) then tend to the point of tangency of \( W \), i.e., to \( \infty \). This implies that \( f(x) \) tends to infinity. \( \square \)

6. Applications

6.1. Non-commutative geometries

The following fact Proposition 6.2 was needed in the article [4] in order to construct non-commutative geometries in the sense of André from planar partition functions. Since a general proof was not available, the authors verified the assertion for those concrete examples that they intended to study.

The lines of the geometries under construction were obtained as follows. Given a planar partition function \( f : \mathbb{R}^2 \to \mathbb{R}^2 \), a vector \( v \in \mathbb{R}^4 \setminus \{0\} \times \mathbb{R}^2 \), and a real number \( t \neq 0 \), the planarity condition (PL) shows that there is a unique pair of points \( p(t), p'(t) \in \Gamma_f \) such that \( p'(t) - p(t) = tv \). The problem is to show that, as \( t \to 0 \), the points \( p(t) \) tend to a point \( q = q(v) \). If this is the case, then \( p'(t) \) converges to the same point, and \( q(v) \) together with the set of all \( p(t) \) and \( p'(t) \) forms a continuous curve, the socket curve associated with \( v \). The point \( q(v) \) is called the base point of the socket curve. Then a non-commutative join operation is obtained: the authors prove that each point \( r \in \Gamma_f \setminus \{q\} \) is contained in a unique socket curve with base point \( q \), which is denoted by \( q \sqcup r \). We can now see that for proper planar partition functions, the existence of base points is a general fact:

Proposition 6.2. Given a proper planar partition function \( f \) on \( \mathbb{R}^2 \) and a vector \( v \in \mathbb{R}^4 \setminus \{0\} \times \mathbb{R}^2 \), the points \( p(t) \) and \( p'(t) \) defined above for non-zero \( t \in \mathbb{R} \) tend to a limit \( q(v) \in \Gamma_f \) as \( t \to 0 \).

Proof. For all \( t \neq 0 \), the points \( 0, tv \in \mathbb{R}^4 \) belong to the same line \( L \) of the translation plane \( T_f \). We have shown that \( L \) is a topological oval in the projective closure of the shift plane \( A_f \), hence the lines \( 0 \vee tv = \Gamma_f + w(t) \) of \( A_f \) converge to the tangent \( \Gamma_f + w(0) \) of \( L \) in the point 0. This means that \( w(t) \to w(0) \) as \( t \to 0 \). Now we have \( \Gamma_f = (0 \vee tv) - w(t) \), hence \( p(t) = -w(t) \) and \( p'(t) = tv - w(t) \). It follows that both \( p(t) \) and \( p'(t) \) converge to the point \( -w(0) \). \( \square \)

The preceding result is only a special aspect of the following, more general relationship.

6.3. The space of chords and the tangent bundle of a planar partition surface

The space of chords of \( \Gamma_f \) is simply the Cartesian square \( \Gamma_f \times \Gamma_f \) with the product topology, but we think of a pair \( (q, r) \) as an oriented chord of the graph with end points
The tangent bundle is the set $T \Gamma_f \subseteq \mathbb{R}^4 \times \mathbb{R}^4$ consisting of all pairs $(p, v)$ such that $p = (x, f(x)) \in \Gamma_f$ and $v$ belongs to the tangent vector space $T_p \Gamma_f = D_x f$.

There is a map $\varphi : \Gamma_f \times \Gamma_f \rightarrow T \Gamma_f$, defined as follows. If $(q, r)$ is a chord such that $q \neq r$, then $r - q \in \mathbb{R}^4 \setminus \{0\} \times \mathbb{R}^2$, hence the partition property (PA) shows that there is a unique tangent vector space $T_p \Gamma_f$ containing $r - q$, and we define $\varphi(q, r) := (p, r - q)$. In case of equality, we put $\varphi(q, q) := (q, 0)$, the zero tangent vector at $q$. It follows from the planarity condition (PL) that the map $\varphi$ is bijective.

**Theorem 6.4.** The space of chords $\Gamma_f \times \Gamma_f$ of a planar partition surface $\Gamma_f$, $f$ proper, is homeomorphic to the tangent bundle $T \Gamma_f$ via the map $\varphi$ defined in Section 6.3.

**Proof.** In order to find the point $p \in \Gamma_f$ whose tangent space contains a non-zero difference $r - q \neq 0$, we simply have to take the line $L$ of $T_f$ joining $r - q$ to 0; then $L = D_x f$ for a unique $x \in \mathbb{R}^2$, and $p = (x, f(x))$. Now joining is continuous, and the map sending $x$ to $D_x f$ is continuous because $f$ is of class $C^1$. That map is a bijection $\alpha : \mathbb{R}^2 \rightarrow S_f \setminus \{(0) \times \mathbb{R}^2\} \approx \mathbb{R}^2$, hence it is a homeomorphism (we are using the domain invariance theorem). Thus, each of the maps $(q, r) \mapsto L \mapsto x \mapsto p$ is continuous.

It remains to prove continuity of $\varphi$ at a chord $(s, s)$; then continuity of the inverse will follow, again by domain invariance. The restriction of $\varphi$ to the set of all such chords is continuous, hence it suffices to consider a sequence of chords $(q_n, r_n)$ converging to $(s, s)$ such that $q_n \neq r_n$ for all $n$. In this situation, we use the fact that $\Gamma_f$ is a topological oval in the translation plane $T_f$. The secants $q_n \vee r_n = D_{s_n} f + q_n$ of this oval converge to the tangent in the point $s = (x, f(x))$, that is, to the line $D_x f + s = T_s \Gamma_f + s$. Using continuity of $\alpha^{-1}$, we infer that $s_n \rightarrow s$, hence $s_n := (x_n, f(x_n)) \rightarrow s$. By assumption, $r_n - q_n \rightarrow 0$, and we have shown that $\varphi(q_n, r_n) = (s_n, r_n - q_n) \rightarrow \varphi(s, s) = (s, 0)$. □

7. A partial converse

Throughout this section, we assume that $T = (\mathbb{R}^4, \mathcal{L})$ is a topological translation plane such that the vertical subspace $\{0\} \times \mathbb{R}^2$ is a line; its point at infinity will be denoted by $\infty$. By a parabolic oval in $T$, we mean an oval in the projective closure such that the line at infinity is a tangent. Usually, the point of tangency is assumed to be $\infty$; this implies that the oval is the graph $\Gamma_f$ of a function $f : \mathbb{R}^2 \rightarrow \mathbb{R}^2$, augmented by the point $\infty$. A parabolic oval is topological if, and only if, the function $f$ is continuous. This follows from the affine characterization of topological ovals, [10, 55.18].

**Proposition 7.1.** Let $\Gamma_f$ be a topological parabolic oval in a topological translation plane $T = (\mathbb{R}^4, \mathcal{L})$. Then $f$ is of class $C^1$, and $f$ is a partition function such that $T = T_f$. The tangent vector space $T_p \Gamma_f$ in a point $p$ is parallel to the geometric tangent $T_p$ of the oval.

**Proof.** Consider a point $x \in \mathbb{R}^2$ and define $E_i(t) := x + te_i$, where $e_1 = (1, 0)$, $e_2 = (0, 1)$ are standard basis vectors, $i \in \{1, 2\}$, and $t \in \mathbb{R}$. Vectors in $\mathbb{R}^4$ will be written as pairs $(u, v)$, where $u, v \in \mathbb{R}^2$. By $V_i$ we denote the affine hyperplane of $\mathbb{R}^4$ defined by $u_i = x_i$. We want to differentiate the function $f_j \circ E_i$ at $t = 0$, where $f_j$ stands for the $j$-component of $f$. 
Let \( L_i(t) \in \mathcal{L} \) be the line joining \( p = (x, f(x)) \) to \( p(t) = (E_i(t), f \circ E_i(t)) \). The given oval is topological, hence the secant \( L_i(t) \) converges, as \( t \to 0 \), to the tangent \( L(0) \) of the oval in the point \( p \). Let \( l_i(t) = L_i(t) \cap V_i \); then \( l_i(t) \) is the 1-dimensional affine subspace joining \( p \) to \( p(t) \), and we infer that \( l_i(t) \to l_i(0) = L(0) \cap V_i \). Using this fact it is easily seen that the derivative of \( f_j \circ E_i \) at \( t = 0 \) equals the \((j, i)\)-entry of the \((2 \times 2)\)-matrix describing the line \( L(0) \). In other words, that matrix is the Jacobi matrix of \( f \) at \( x \). Since \( L(0) \) is the tangent of the topological oval \( \Gamma_f \) in the point \( p \), the Jacobi matrix depends continuously on \( x \), and it follows that \( f \) is of class \( C^1 \). \( \square \)

**Remark 7.2.** In order to obtain a complete converse of Theorem 3.4, we would have to prove in addition that \( f \) is planar. This is true in all known cases and can be proved under mild additional assumptions, compare [10, 74.19] and [5], but a general proof is not known.

**References**