Examples of cyclically-interval non-colorable bipartite graphs

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Abstract. For an undirected, simple, finite, connected graph $G$, we denote by $V(G)$ and $E(G)$ the sets of its vertices and edges, respectively. A function $\varphi : E(G) \to \{1, 2, \ldots, t\}$ is called a proper edge $t$-coloring of a graph $G$ if adjacent edges are colored differently and each of $t$ colors is used. An arbitrary nonempty subset of consecutive integers is called an interval. If $\varphi$ is a proper edge $t$-coloring of a graph $G$ and $x \in V(G)$, then $S_G(x, \varphi)$ denotes the set of colors of edges of $G$ which are incident with $x$. A proper edge $t$-coloring $\varphi$ of a graph $G$ is called a cyclically-interval $t$-coloring if for any $x \in V(G)$ at least one of the following two conditions holds: a) $S_G(x, \varphi)$ is an interval, b) $\{1, 2, \ldots, t\} \setminus S_G(x, \varphi)$ is an interval. For any $t \in \mathbb{N}$, let $\mathcal{M}_t$ be the set of graphs for which there exists a cyclically-interval $t$-coloring, and let

$$\mathcal{M} \equiv \bigcup_{t \geq 1} \mathcal{M}_t.$$  

Examples of bipartite graphs that do not belong to the class $\mathcal{M}$ are constructed.

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1 Introduction

We consider undirected, simple, finite, and connected graphs. For a graph $G$ we denote by $V(G)$ and $E(G)$ the sets of its vertices and edges, respectively. For a graph $G$, we denote by $\Delta(G)$ and $\chi'(G)$ the maximum degree of a vertex of $G$ and the chromatic index of $G$ [14], respectively. The terms and concepts which are not defined can be found in [17].

For an arbitrary finite set $A$, we denote by $|A|$ the number of elements of $A$. The set of positive integers is denoted by $\mathbb{N}$. An arbitrary nonempty subset of consecutive integers is called an interval. An interval with the minimum element $p$ and the maximum element $q$ is denoted by $[p, q]$.

For any $t \in \mathbb{N}$ and arbitrary integers $i_1, i_2$ satisfying the conditions $i_1 \in [1, t]$, $i_2 \in [1, t]$, we define [8,9] the sets $\text{intcyc}_1((i_1, i_2), t)$, $\text{intcyc}_1((i_1, i_2), t)$, $\text{intcyc}_2((i_1, i_2), t)$, $\text{intcyc}_2((i_1, i_2), t)$ as follows:

$$\text{intcyc}_1((i_1, i_2), t) \equiv [\min\{i_1, i_2\}, \max\{i_1, i_2\}],$$

$$\text{intcyc}_2((i_1, i_2), t) \equiv \text{intcyc}_1((i_1, i_2), t) \setminus \{i_1\} \cup \{i_2\},$$

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\intcyc_{2}(i_{1}, i_{2}), t) \equiv [1, t] \setminus \intcyc_{1}([i_{1}, i_{2}), t],
\intcyc_{2}([i_{1}, i_{2}), t) \equiv [1, t] \setminus \intcyc_{1}((i_{1}, i_{2}), t).

If \( t \in \mathbb{N} \) and \( Q \) is a non-empty subset of the set \( \mathbb{N} \), then \( Q \) is called a \( t \)-cyclic interval if there exist integers \( i_{1}, i_{2}, j_{0} \) satisfying the conditions \( i_{1} \in [1, t], i_{2} \in [1, t], j_{0} \in \{1, 2\} \), \( Q = \intcyc_{j_{0}}([i_{1}, i_{2}), t] \).

A function \( \varphi : E(G) \to [1, t] \) is called a proper edge \( t \)-coloring of a graph \( G \) if adjacent edges are colored differently and each of \( t \) colors is used.

For a graph \( G \) and a positive integer \( t \), where \( \chi'(G) \leq t \leq |E(G)| \), we denote by \( \alpha(G, t) \) the set of all proper edge \( t \)-colorings of \( G \). Let us set
\[
\alpha(G) \equiv \bigcup_{t=\chi'(G)}^{\alpha(G, t)} \alpha(G, t).
\]

If \( G \) is a graph, \( \varphi \in \alpha(G) \), and \( x \in V(G) \), then the set \( \{\varphi(e) / e \in E(G), e \text{ is incident with } x\} \) is denoted by \( S_{G}(x, \varphi) \).

A proper edge \( t \)-coloring \( \varphi \) of a graph \( G \) is called a cyclically-interval \( t \)-coloring of \( G \), if for any \( x \in V(G) \) at least one of the following two conditions holds: a) \( S_{G}(x, \varphi) \) is an interval, b) \( [1, t] \setminus S_{G}(x, \varphi) \) is an interval.

For any \( t \in \mathbb{N} \), we denote by \( \mathcal{M}_{t} \) the set of graphs for which there exists a cyclically-interval \( t \)-coloring. Let
\[
\mathcal{M} \equiv \bigcup_{t \geq 1} \mathcal{M}_{t}.
\]

For an arbitrary tree \( D \), it was shown in \[8\] that \( D \in \mathcal{M} \), and, moreover, all possible values of \( t \) were found for which \( D \in \mathcal{M}_{t} \). For an arbitrary simple cycle \( C \), it was shown in \[7, 10\] that \( C \in \mathcal{M} \), and, moreover, all possible values of \( t \) were found for which \( C \in \mathcal{M}_{t} \). Some interesting results on this and related topics were obtained in \[1, 3, 4, 11, 13, 15, 16\].

In this paper, the examples of bipartite graphs that do not belong to the class \( \mathcal{M} \) are constructed.

For any integer \( m \geq 2 \), set:
\[
V_{0, m} \equiv \{x_{0}\}, \quad V_{1, m} \equiv \{x_{i,j} / 1 \leq i < j \leq m\},
V_{2, m} \equiv \{y_{p,q} / 1 \leq p \leq m, 1 \leq q \leq m\},
E'_{m} \equiv \{(x_{0}, y_{p,q}) / 1 \leq p \leq m, 1 \leq q \leq m\}.
\]

For any integers \( i, j, m \) satisfying the inequalities \( m \geq 2 \), \( 1 \leq i < j \leq m \), set:
\[
E''_{i,j,m} \equiv \{(x_{i,j}, y_{j,q}) / 1 \leq q \leq m\} \cup \{(x_{i,j}, y_{j,q}) / 1 \leq q \leq m\}.
\]

For any integer \( m \geq 2 \), let us define a graph \( G(m) \) by the following way:
\[
G(m) \equiv \left( \bigcup_{k=0}^{2} V_{k,m}, E'_{m} \cup \bigcup_{1 \leq i < j \leq m} E''_{i,j,m} \right).
\]
It is not difficult to see that for any integer $m \geq 2$, $G(m)$ is a bipartite graph with $\Delta(G(m)) = \chi'(G(m)) = m^2$, $|V(G(m))| = \frac{3m^2 - m}{2} + 1$, $|E(G(m))| = m^3$.

**Theorem 1.** For any integer $m \geq 8$, $G(m) \notin \mathcal{M}$.

**Proof.** Assume the contrary. It means that there exist integers $m_0, t_0, k_0$, satisfying the conditions $m_0 \geq 8$, $m_0^2 \leq t_0 \leq m_0^3$, $t_0 = m_0^2 + k_0$, $0 \leq k_0 \leq m_3^3 - m_0^2$, $G(m_0) \in \mathcal{M}_{t_0}$.

Let $\varphi_0$ be a cyclically-interval $t_0$-coloring of the graph $G(m_0)$. Without loss of generality, we can suppose that $S_{G(m_0)}(x_0, \varphi_0) = [1, m_0^2]$. Let us consider the edges $e'$ and $e''$ of the graph $G(m_0)$, which are incident with the vertex $x_0$ and satisfy the equalities $\varphi_0(e') = 1$, $\varphi_0(e'') = \lfloor \frac{m_0^2}{2} \rfloor$.

Suppose that $e' = (x_0, y)$, $e'' = (x_0, y')$. Clearly, there exists a vertex $\bar{x} \in V_{1,m_0}$ in the graph $G(m_0)$ which is adjacent to the vertices $y'$ and $y''$. It is not difficult to see that $S_{G(m_0)}(y', \varphi_0) \cup S_{G(m_0)}(\bar{x}, \varphi_0) \cup S_{G(m_0)}(y'', \varphi_0)$ is a $t_0$-cyclic interval.

Clearly, the inequalities $m_0^2 + k_0 - 4m_0 + 4 > 4m_0 - 2$ and $4m_0 - 1 \leq \lfloor \frac{m_0^2}{2} \rfloor \leq m_0^2 + k_0 - 4m_0 + 3$ are true. Consequently, $\lfloor \frac{m_0^2}{2} \rfloor \leq \intcyc_2((4m_0 - 2, m_0^2 + k_0 - 4m_0 + 4), m_0^2 + k_0)$. But it is incompatible with the evident relations $\lfloor \frac{m_0^2}{2} \rfloor \leq S_{G(m_0)}(y'', \varphi_0)$ and $S_{G(m_0)}(y', \varphi_0) \cup S_{G(m_0)}(\bar{x}, \varphi_0) \cup S_{G(m_0)}(y'', \varphi_0) \subseteq \intcyc_2((4m_0 - 2, m_0^2 + k_0 - 4m_0 + 4), m_0^2 + k_0)$. Contradiction.

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**References**


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