Asymptotic synchronization of the Colpitts oscillator

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ABSTRACT

In this paper we deal with the observer-based asymptotic synchronization problem for a class of chaotic oscillators. Some results based on a differential algebraic approach are used in order to determine the algebraic observability of unknown variables. The strategy consists of proposing a slave system (observer) which tends to follow asymptotically the master system. The methodology is tested in the real-time asymptotic synchronization of the Colpitts oscillator by means of a proportional reduced order observer (PROO) of free-model type.

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1. Introduction

Chaotic systems synchronization has been investigated since its introduction in the paper [1]. Among the publications dedicated to chaos synchronization, many different approaches can be found [2–4]. We cite the papers [4–7] which propose the use of state observers to synchronize chaotic systems; in references [8–10] use feedback controllers; in [11,12] use nonlinear backstepping control; in papers [13,14] consider synchronization time delayed systems; in works [15,16] consider directional and bidirectional linear coupling; papers [17,18] use nonlinear control; in [10,19] use adaptive control; in [20] apply sliding-mode techniques; in [21] attack the anti-synchronization problem; in [22] a fuzzy sliding-mode controller is applied; in [23] the synchronization in time delay systems is given, and so on.

Synchronization of the chaotic systems problem has received a great deal of attention among scientists in many fields due to its potential applications, such as: secure communications, biological systems, chemical reactions, etc., [4,8,24–26].

As we can note, there exist several methods to solve the synchronization problem since from the control theory perspective in this work, we study the asymptotic synchronization by means of state observers.

The method is based on a master–slave configuration [1]. The main characteristic is that the coupling signal is unidirectional, that is, the signal is transmitted from the master system (transmitter) to the slave system (receiver), the receiver is requested to recover the unknown (or full) state trajectories of the transmitter. By this fact, the terminology transmitter–receiver is also used. Thus, the chaos synchronization problem can be regarded as an observer design procedure, where the coupling signal is viewed as an output and the slave system is the observer [27–29].

The problem of observer design naturally arises in a system approach, as soon as one needs unmeasured internal information from external measurements. In general, it is clear that one cannot use as many sensors as signals of interest characterizing the system behavior for technological constraints, cost reasons, and so on, especially since such signals can come in a quite large number, and they can be of various types: they typically include parameters, time-varying signals characterizing the system (state variables), and unmeasured external disturbances [30,31].

As we know, it is almost impossible to measure all the elements of the state vector in practice (e.g., the unknown state variables, fault signals, etc.). Here arises a basic practical question: would it be possible to reconstruct these unknown
signals? We give an answer to this question by introducing a basic definition related with the estimation (reconstruction) of the states, the algebraic observability property (AOP).

In this work the observability property for a class of systems (oscillators, chaotic systems, and systems with bounded dynamics) is determined by means of a relatively new approach which is related with the differential-algebra framework. This mathematical approach has been recently shown to be a very effective tool for understanding basic questions such as input–output inversions and observer realizations.

The main contributions consist of the following. (i) An observer as a numerical technique to reconstruct unknown variables is designed. Before proposing the observer structure we should verify whether the signal to be estimated satisfies the AOP. Then we design a PROO which is based on the free-model approach. The main advantage of the PROO is that the free-model quality of its structure allows us to reconstruct the unknown variables in spite of model uncertainties. (ii) The suggested approach is implemented in the real-time asymptotic synchronization of the Colpitts oscillator via the PROO.

The paper is organized as follows. In Section 2 the synchronization problem and its solution by means of a reduced order observer are treated. Section 3 presents the procedure for the synchronization in real-time of the Colpitts oscillator [32]. Section 4 illustrates the obtained experimental results and shows the performance of the reduced order observer. Finally, in Section 5 we close the paper with some concluding remarks.

2. Observer design

Let us consider the following nonlinear system:

\[ \dot{x}(t) = f(x, u) \]
\[ y(t) = h(x) \]

(1)

where \( f \in \mathbb{R}^n \) is differentiable and satisfies \( f(0, 0) = 0 \), \( x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n \) is the state vector, \( \bar{x} \in \mathbb{R}^p \) represents the known states (with \( 1 \leq p < n \)), \( y \in \mathbb{R}^q \) denotes the output of the system, \( h : \mathbb{R}^p \to \mathbb{R}^q \) is a continuous function and \( u \in \mathbb{R}^l \) is the known input (\( l \leq p \)).

Now, let us consider the nonlinear system described by (1). We separate (1) into two dynamical systems with states \( \bar{x} \in \mathbb{R}^p \) and \( \eta \in \mathbb{R}^{n-p} \) respectively with \( x = (\bar{x}^T, \eta^T) \). The first system describes the known states and the second represents unknown states, then the system (1) can be rewritten as:

\[ \dot{\bar{x}}(t) = \bar{f}(x, u) \]
\[ \dot{\eta}(t) = \Delta(x, u) \]
\[ y(t) = h(x) \]

(2)

where \( \bar{f} \in \mathbb{R}^p, f^T(x) = (\bar{f}^T(x, u), \Delta^T(x, u)) \), and \( \Delta \in \mathbb{R}^{n-p} \) is an uncertain function. The problem is to estimate the state variable \( \eta(t) = (\eta_{i(p+1)}, \ldots, \eta_{in})^T \). In order to solve this observation problem let us introduce the following property.

**Definition 1** (Algebraic Observability Property-AOP). Let us consider the nonlinear dynamical system (2). A state variable \( \eta_i \in \mathbb{R} \) is said to be algebraically observable if it is algebraic over \( \mathbb{R}(u, y) \), that is to say, \( \eta_i \) satisfies a differential algebraic polynomial in terms of \( u, y \) and some of their first \( r_1, r_2 \in \mathbb{N} \) time derivatives, respectively, i.e.,

\[ \eta_i = \phi_i(u, \dot{u}, \ldots, \dot{u}^{r_1}, y, \dot{y}, \ldots, \dot{y}^{r_2}), \quad i \in \{p + 1, \ldots, n\} \]

(3)

where \( \phi_i : \mathbb{R}^{(r_1+1)p} \times \mathbb{R}^{(r_2+1)q} \to \mathbb{R} \).

If any unknown variable satisfies the AOP, then a numerical technique from the so-called observer can be used to reconstruct the required signal.

The next system represents the dynamics of the unknown states:

\[ \dot{\hat{\eta}}_i(t) = \Delta_i(x, u) \]

(4)

**Lemma 1.** If the following hypotheses are satisfied:

H1: \( \eta_i(t) \) satisfies the AOP (Definition 1), for \( i \in \{p + 1, \ldots, n\} \).
H2: \( \gamma_i \) is a \( C^1 \) real-valued function.
H3: \( \Delta_i \) is bounded, i.e., \( |\Delta| \leq M < \infty \).
H4: For \( t_0 \), sufficiently large, there exists \( K_i > 0 \), such that, \( \limsup_{t \to t_0} \frac{M}{K_i} = 0 \).

Then, the system

\[ \dot{\hat{\eta}}_i = K_i(\eta_i - \hat{\eta}_i) \]

is an asymptotic reduced order observer of free-model type for system (4), where \( \hat{\eta}_i \) denotes the estimate of \( \eta_i \), and \( K_i \in \mathbb{R}^+ \) determines the desired convergence rate of the observer.

---

1 In practice, identification of the variable \( \eta \) depends on the variable choice to be estimated.
2 \( \mathbb{R}(u, y) \) denotes the differential field generated by the field \( \mathbb{R} \), the input \( u \), the measurable output \( y \), and the time derivatives of \( u \) and \( y \).
**Proof.** Let us define the estimation error as \( e(t) = \eta_i(t) - \hat{\eta}_i(t) \). The dynamics of the error is given by

\[
\dot{e}_i(t) = \dot{\eta}_i(t) - \dot{\hat{\eta}}_i(t).
\]

then

\[
\dot{e}_i(t) + K_ie_i(t) = \Delta_i(t).
\]

The solution of (6) is given by,

\[
e_i(t) = \exp(-K_i t) \left[ e_{i0} + \int_0^t \exp(K_i \tau) \Delta_i(\tau) d\tau \right],
\]

(7)

where \( e_{i0} = e_i(0) \) denotes the initial condition.

Then, using Cauchy–Schwartz and triangle inequalities in the expression (7), we have,

\[
0 \leq |e_i(t)| \leq \exp(-K_i t) |e_{i0}| + \exp(-K_i t) \int_0^t \exp(K_i \tau) |\Delta_i(\tau)| d\tau.
\]

If H3 is satisfied, we obtain,

\[
0 \leq |e_i(t)| \leq \exp(-K_i t) |e_{i0}| + M \int_0^t \exp(K_i (\tau - t)) d\tau
\]

\[
= \exp(-K_i t) |e_{i0}| + \frac{M}{K_i} [1 - \exp(-K_i t)].
\]

If \( t \to t_0 \), for \( t_0 \) sufficiently large, then

\[
0 \leq \limsup_{t \to t_0} |e_i(t)|
\]

\[
\leq |e_{i0}| \limsup_{t \to t_0} |\exp(-K_i t)| + \limsup_{t \to t_0} \left\{ \frac{M}{K_i} [1 - \exp(-K_i t)] \right\}
\]

\[
= \limsup_{t \to t_0} \frac{M}{K_i}.
\]

Finally, by taking into account H4, we have

\[
0 \leq \limsup_{t \to t_0} |e_i(t)| \leq \limsup_{t \to t_0} \frac{M}{K_i} = 0.
\]

Then,

\[
\limsup_{t \to t_0} |e_i(t)| = 0.
\]

Therefore, (5) is an asymptotic reduced order observer for (4).

Sometimes the output time derivatives (which are unknown), appear in the algebraic equation of the state variable, then it is necessary to use an auxiliary variable to avoid using them.

**Corollary 1.** The dynamic system (5) along with

\[
\dot{\gamma} = \psi(x, u, \gamma), \quad \text{with} \ \gamma_0 = \gamma(0) \quad \text{and} \quad \gamma \in C^1
\]

consider a proportional asymptotic reduced order observer for system (4), where \( \gamma \) is a change of variable which depends on the estimated state \( \hat{\eta} \), and the state variables.

### 3. Synchronization of the Colpitts oscillator

In 1994, chaotic oscillations in the Colpitts circuit with a generic transistor 2N2222A were reported [33]. In the work [34], the author refers to the multi-oscillations phenomenon in the RF bipolar oscillators, which are parasitic oscillations or not desired that coexist with a main oscillation. Due to these parasitic oscillations the resultant signal in steady state is severely affected, in this sense, those circuits have few applications. The paper [33] shows the parasitic oscillations are not present in the Colpitts oscillator, and moreover, the chaotic behavior of the circuit is presented, in fact, the Colpitts oscillator is widely used in electronic devices and communication systems. Fig. 1(a) shows the circuit configuration [32], as well as some values for which the Colpitts oscillator has chaotic behavior.
3.1. Modeling

In this work we consider the classical configuration of the Colpitts oscillator. The circuit contains a bipolar junction transistor (BJT) as the gain element, two resistors, and a resonant network consisting of an inductor and two capacitors (Fig. 1(a)).

By taking into account the qualitative theory in nonlinear dynamics, we select a minimum model for the circuit. The idea is to consider as simple a circuit model as possible that describes the essential features exhibited by the real Colpitts oscillator.

For the modeling of such circuit the following assumptions are considered:

1. The emitter current is generated by means of $I_0$.
2. The passive and active elements are ideal.
3. The transistor is modeled as a nonlinear resistor $R_E$ controlled by voltage and a linear input controlled by current (Fig. 1(b)), indeed,

(a) We model $V-I$ characteristic of $R_E$ with an exponential function, as follows

$$I_E = I_S \exp \left( \frac{V_{BE}}{V_T} \right) - 1$$

where $I_S$ is the inverse saturation current and $V_T \simeq 26 \text{ mV}$ at room temperature. (8)

(b) We assume that $\alpha_F = 1$, where $\alpha_F$ is the common–base forward short–circuit current gain. This corresponds to neglecting the base current.

(c) The parasitic dynamics of the transistor are omitted.

The Colpitts circuit is described by a system of three nonlinear differential equations, as follows:

$$
\begin{align*}
C_1 \dot{V}_{C_1} &= -f(V_{C_2}) + I_L \\
C_2 \dot{V}_{C_2} &= I_L - I_0 \\
\dot{I}_L &= -V_{C_1} - V_{C_2} - R I_L + V_{CC}
\end{align*}
$$

(9)

where $f(\cdot)$ is the driving-point characteristic of the nonlinear resistor. This can be expressed in the form $I_E = f(V_{C_2}) = f(-V_{BE})$. In particular, we have $f(V_{C_2}) = I_S \exp \left( -V_{C_2}/V_T \right)$.

3.2. Parameter normalization

We introduce the dimensionless state variables $(x_1, x_2, x_3)$, and we choose the operating point of (9) to be the origin of the new coordinate system. In particular, we normalize voltages, currents and time with respect to $V_{ref} = V_T$, $I_{ref} = I_0$ and $t_{ref} = 1/w_0$, respectively, where $w_0 = 1/\sqrt{L_1C_2/(C_1 + C_2)}$, is the resonant frequency of the unloaded $L-C$ tank circuit. Then, the state equations for the Colpitts oscillator can be rewritten in the next form [35]:

$$
\begin{align*}
\dot{x}_1 &= -a \exp(-x_2) + ax_3 + a \\
\dot{x}_2 &= bx_3 \\
\dot{x}_3 &= -cx_1 - cx_2 - dx_3
\end{align*}
$$

(10)

where, $a = b \frac{C_2}{L_1}$, $b = \frac{I_0}{w_0 C_2 V_T}$, $c = \frac{V_T}{w_0 I_0}$, $d = \frac{R}{L w_0}$. 

![Fig. 1. Colpitts oscillator. (a) Circuit configuration. (b) Model of the bipolar junction transistor (BJT). The circuit parameters are: $L = 100 \mu \text{H}; C_1 = C_2 = 47 \text{ nF}; R = 45 \Omega; I_0 = 5 \text{ mA}$.](image-url)
3.3. Observer for the Colpitts oscillator (slave system)

Let us consider the normalized system of the Colpitts oscillator. Throughout this paper we assume that the output system is \( y = x_2 \) and the unknown variables are \( \eta = (x_1, x_3)^T \). Therefore, the slave system consists of two estimation structures to achieve synchronization with the master system. Such structures are obtained as follows. Firstly, verify that the master system (Colpitts oscillator) fulfills the AOP, and then, by using (5), construct the observer for the unknown states.

The AOP for \( x_3 \) is given by,

\[
\dot{x}_3 = \frac{\dot{y}}{b} = \phi_3(y).
\]  

(11)

For \( x_1 \), we have

\[
x_1 = -\frac{1}{c} \left[ \frac{1}{b} \dot{y} + \frac{d}{b} \ddot{y} + cy \right] \Rightarrow x_1 = \phi_1(y, \dot{y}, \ddot{y}).
\]  

(12)

Then, both unknown states of the master system satisfies the AOP, therefore, we can construct the observers based on (5) and Corollary 1.

Observer for \( x_3 \):

\[
\dot{\hat{x}}_3 = K_3(x_3 - \hat{x}_3) = \frac{K_3}{b} \dot{y} - K_3 \hat{x}_3
\]

\[
\dot{\hat{x}}_3 = K_3 \dot{y} - K_3 \hat{x}_3.
\]

If we define \( \gamma_3 = -\frac{K_3}{b} y + \hat{x}_3 \), then

\[
\dot{\gamma}_3 = -\frac{K_2}{b} y - K_3 \gamma_3
\]

\[
\dot{\hat{x}}_3 = \frac{K_3}{b} y + \gamma_3.
\]  

(13)

Observer for \( x_1 \):

\[
\dot{\hat{x}}_1 = K_1(x_1 - \hat{x}_1) = -\frac{K_1}{c} \dot{y} - \frac{K_1}{c} \ddot{y} - K_1 y - K_1 \hat{x}_1.
\]

Now, we consider the change of variable \( x_4 = \dot{y} \) and we design an observer for this new variable according to (5),

\[
\dot{\hat{x}}_4 = -K_4 [\gamma_4 + K_4 y]
\]

\[
\dot{\hat{x}}_4 = \gamma_4 + K_4 y.
\]

Then,

\[
\dot{\hat{x}}_1 = -\frac{K_1}{c} \dot{\hat{x}}_4 - K_1 \frac{\ddot{x}_4}{c} - K_1 y - K_1 \hat{x}_1
\]

\[
\dot{\hat{x}}_1 + \frac{K_1}{c} \dot{\hat{x}}_4 = -\frac{K_1}{c} \dot{x}_4 - K_1 y - K_1 \hat{x}_1.
\]

If we define \( \gamma_5 = \hat{x}_1 + \frac{K_1}{c} \dot{\hat{x}}_4 \), then

\[
\dot{\gamma}_5 = -\frac{K_1}{c} \ddot{x}_4 - K_1 y + \frac{K^2}{c} \dot{x}_4 - K_1 \gamma_5
\]

\[
= [K_1 - d] \frac{K_1}{c} \dot{x}_4 - K_1 y - K_1 \gamma_5
\]

\[
= [K_1 - d] \frac{K_1}{c} [\gamma_4 + K_4 y] - K_1 y - K_1 \gamma_5.
\]

Finally, the observer scheme for \( x_1 \) is given by

\[
\dot{\hat{x}}_4 = -K_4 [\gamma_4 + K_4 y]
\]

\[
\dot{\hat{x}}_5 = [K_1 - d] \frac{K_1}{c} [\gamma_4 + K_4 y] - K_1 y - K_1 \gamma_5
\]

\[
\dot{\hat{x}}_1 = -\frac{K_1}{c} [\gamma_4 + K_4 y] + \gamma_5.
\]  

(14)

Therefore, (13) and (14) constitute the slave system.
4. Experimental results

We verified the real time performance of the proposed observers by using the WINCON platform. To achieve the synchronization in real time, in WINCON were implemented the schemes (13) and (14) in the master–slave configuration. The circuit parameters are: $L = 100 \, \mu H; C_1 = C_2 = 47 \, nF, R = 45 \, \Omega, I_0 = 5 \, mA$. Using the circuit parameters we obtain $a = b = 6.2723, c = 0.0797, \text{ and } d = 0.6898$. In Fig. 2 the real Colpitts circuit (master system) is shown.

The performance index (quadratic synchronization error) of the corresponding synchronization process is calculated as [36].

$$f(t) = \frac{1}{t + 0.001} \int_0^t |e(t)|^2 \, dt, \quad Q_0 = I$$

where $e(t)$ denotes the synchronization error.
Fig. 3(a)–(b) show the obtained results for the initial conditions \( \hat{x}_1 = 1.506 \) and \( \hat{x}_3 = -2.498 \) in the schemes (13) and (14), respectively. As we can note, the synchronization results achieved with the reduced order observer are good. Fig. 3(c) presents the phase portrait, where clearly the chaotic behavior of the Colpitts oscillator is observed. Finally, Fig. 3(d) illustrates the performance index, which has a tendency to decrease.

5. Conclusions

In this paper we study the asymptotic synchronization problem in chaotic systems based on the observer theory. We also show the real-time synchronization in the Colpitts oscillator by using a reduced order observer of free-model type, which has asymptotic convergence. Some experimental results show the effectiveness of the proposed methodology.

References


