On asymptotic behavior of Battle–Lemarié scaling functions and wavelets

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Abstract

We show that the ‘centered’ Battle–Lemarié scaling function and wavelet of order $n$ converge in $L^q(2 \leq q \leq \infty)$, uniformly in particular, to the Shannon scaling function and wavelet as $n$ tends to infinity.

Keywords: B-spline; Multiresolution analysis; Battle–Lemarié wavelet; Shannon wavelet; Euler–Frobenius polynomial

1. Introduction

The Battle–Lemarié scaling function is obtained by applying the orthogonalization trick to the $B$-spline functions. In order to get the symmetry about the origin, we will take the centered $B$-spline of order $n$ as

\begin{align}
B_1(x) &:= \chi_{[-1/2, 1/2]}(x), \\
B_n(x) &:= B_{n-1} \ast B_1(x), \quad n = 2, 3, \ldots.
\end{align}

(1.1)

The Fourier transform of $B_n$ then has the form

\begin{equation}
\hat{B}_n(w) = \left( \frac{\sin w/4}{w/2} \right)^n = (\cos w/4)^n \hat{B}_n(w/2).
\end{equation}

(1.2)

We note that

\begin{align}
\phi_n(w) &:= \sum_{k \in \mathbb{Z}} |\hat{B}_n(w + 2\pi k)|^2 \\
&= (\cos w/4)^{2n} \phi_n(w/2) + (\sin w/4)^{2n} \phi_n(w/2 + \pi).
\end{align}

(1.3)
and apply the orthonormalization trick to $B_n$ to get the Battle–Lemarié scaling function $\varphi_n$ of order $n$ defined by
\[
\hat{\varphi}_n(w) := \frac{\hat{B}_n(w)}{\sqrt{\Phi_n(w)}} = m_n(w/2)\hat{\varphi}_n(w/2),
\]
where
\[
m_n(w) = (\cos w/2)^n \frac{\Phi_n(w)}{\sqrt{\Phi_n(2w)}}.
\]

The filter $m_n$ is $2\pi$-periodic if $n$ is even and $4\pi$-periodic if $n$ is odd. We note that $m_n$ is a CQF filter in the sense that
\[
|m_n(w)|^2 + |m_n(w + \pi)|^2 = 1.
\]
The corresponding wavelet is given by
\[
\hat{\psi}_n(2w) = e^{-iw}M_n(w)\hat{\varphi}_n(w),
\]
where
\[
M_n(w) = |(\sin w/2)^n \frac{\Phi_n(w + \pi)}{\sqrt{\Phi_n(2w)}}| = |m_n(w + \pi)|.
\]

Note that $M_n$ is $2\pi$-periodic. Therefore, if $n$ is even, the function $\varphi_n$ defines an orthonormal scaling function for a multiresolution analysis. If $n$ is odd, $\varphi_n$ does not define a scaling function of a multiresolution analysis, but they have the same asymptotic behavior as will be seen in the main theorem in this work. See [1–3] for the standard Battle–Lemarié wavelet. In this short note, we show that the Battle–Lemarié scaling function $\varphi_n$ and its corresponding wavelet $\psi_n$ converge, in $L^q(\mathbb{R})$ ($2 \leq q \leq \infty$), uniformly in particular, to the Shannon scaling function $\varphi_{SH}$ and Shannon wavelet $\psi_{SH}$ as $n$ tends to infinity, where
\[
\hat{\varphi}_{SH}(w) := \chi_{[-\pi,\pi]}(w)
\]
and
\[
\hat{\psi}_{SH}(w) := e^{-iw/2}\chi_{[-2\pi, -\pi] \cup [\pi, 2\pi]}(w).
\]

It is known that the centered $B$-spline $B_n$ tends to the Gaussian distribution as $n \to \infty$ [4,5]. For the asymptotic behavior of Daubechies filters and scaling functions, see [6–8]. The idea of the proof also appears in [9,10] for the analogous asymptotic behaviors of other family of wavelets.

2. Main result

We need the following property of the Euler–Frobenius polynomials.

**Proposition 2.1** ([11]). Let $n$ be any positive integer and let $E_{2n-1}$ be the Euler–Frobenius polynomial of degree $2n - 2$ defined by
\[
E_{2n-1}(z) := (2n - 1)! \sum_{k=0}^{2n-2} B_{2n}(-n + k + 1)z^k.
\]
Then the $2n - 2$ roots, $\{\lambda_{n,j} : j = 1, \ldots, 2n - 2\}$, of $E_{2n-1}$ have the properties that
\[
\lambda_{n,2n-2} < \lambda_{n,2n-3} \cdots < \lambda_{n,n} < -1 < \lambda_{n,n-1} < \cdots < \lambda_{n,1} < 0;
\]
\[
\lambda_{n,j}\lambda_{n,2n-1-j} = 1, \quad (j = 1, 2, \ldots, n - 1)
\]
and

\[ \Phi_n(w) = \frac{e^{iw(n-1)}}{(2n-1)!} E_{2n-1}(e^{-iw}) = \frac{1}{(2n-1)!} \prod_{k=1}^{n-1} \frac{1 - 2\lambda_{n,k} \cos w + \lambda_{n,k}^2}{|\lambda_{n,k}|}. \]

Therefore, \( \Phi_n(w + \pi) \leq \Phi_n(w) \) on \( [-\pi/2, \pi/2] \) and \( \Phi_n(w) \leq \Phi_n(w + \pi) \) on \( [-\pi, -\pi/2] \cup (\pi/2, \pi] \).

The \( 2\pi \)-periodic filters for the Shannon scaling function and wavelet are given, respectively, as

\[
m^0_{SH}(w) := \begin{cases} 1, & |w| \leq \pi/2; \\
0, & \pi/2 < |w| \leq \pi. \end{cases}
\]

and

\[
m^H_{SH}(w) := \begin{cases} 0, & |w| < \pi/2; \\
1, & \pi/2 \leq |w| \leq \pi. \end{cases}
\]

We also define a \( 4\pi \)-periodic filter \( m^1_{SH} \in L^2([-2\pi, 2\pi]) \) by

\[
m^1_{SH}(w) := \begin{cases} 1, & |w| \leq \pi/2; \\
0, & \pi/2 < |w| \leq \pi; \\
-1, & \pi < |w| < 3\pi/2; \\
0, & 3\pi/2 \leq |w| \leq 2\pi. \end{cases}
\]

Notice that

\[ \hat{\phi}_{SH}(w) := \chi_{[-\pi,\pi]}(w) = \prod_{j=1}^{\infty} m^0_{SH}(w/2^j) = \prod_{j=1}^{\infty} m^1_{SH}(w/2^j). \]

**Lemma 2.2.** As \( n \) tends to \( \infty \),

(a) \( m_{2n}(w) \) converges to \( m^0_{SH}(w) \) for every \( w \in [-\pi, \pi] \setminus \{ \pm \pi/2 \} \);

(b) \( m_{2n+1}(w) \) converges to \( m^1_{SH}(w) \) for every \( w \in [-2\pi, 2\pi] \setminus \{ \pm \pi/2, \pm 3\pi/2 \} \);

and so, \( M_n(w) \) converges to \( m^H_{SH}(w) \) for every \( w \in [-\pi, \pi] \setminus \{ \pm \pi/2 \} \).

**Proof.** For \( w \in (-3\pi/2, -\pi/2) \cup (\pi/2, 3\pi/2) \), \( \Phi_n(w) \leq \Phi_n(w + \pi) \) by Proposition 2.1. By use of (1.3), we see that

\[
|m_n(w)|^2 = \frac{(\cos w/2)^{2n} \Phi_n(w)}{\Phi_n(2w)} = \frac{(\cos w/2)^{2n}}{(\sin w/2)^{2n}} \left( \frac{\sin w/2)^{2n} \Phi_n(w)}{(\cos w/2)^{2n} \Phi_n(w) + (\sin w/2)^{2n} \Phi_n(w + \pi)} \right) \leq \frac{1}{(\tan w/2)^{2n}} \left( \frac{\sin w/2)^{2n} \Phi_n(w)}{(\sin w/2)^{2n} \Phi_n(w + \pi)} \right) \leq \frac{1}{(\tan w/2)^{2n}} \to 0 \quad \text{as} \quad n \to \infty.
\]

Now, let \( w \in (-2\pi, -3\pi/2) \cup (-\pi/2, \pi/2) \cup (3\pi/2, 2\pi) \). Note that

\[
|m_n(w)|^2 + |m_n(w + \pi)|^2 = 1.
\]

Hence \( \lim_{n \to \infty} |m_n(w)| = 1 \), since \( \lim_{n \to \infty} |m_n(w + \pi)| = 0 \) by (2.5). Since \( m_{2n}(w) \) is \( 2\pi \)-periodic and positive by the definition of \( m_{2n} \), \( \lim_{n \to \infty} m_{2n}(w) = 1 \). Therefore, (a) is satisfied. For (b), note that \( m_{2n+1} \) is \( 4\pi \)-periodic. If \( w \in (-\pi/2, \pi/2) \), then \( m_{2n+1}(w) \) is positive. Hence \( \lim_{n \to \infty} m_{2n+1}(w) = 1 \). If \( w \in (-2\pi, -3\pi/2) \cup (3\pi/2, 2\pi) \), then \( m_{2n+1}(w) \) is negative. Therefore \( \lim_{n \to \infty} m_{2n+1}(w) = -1 \). \( \square \)
Lemma 2.3. For all $n$,

$$|m_n(w) - 1| \leq \begin{cases} 2, & \text{for all } w; \\ 2|w|/\pi, & \text{if } |w| \leq \pi/2. \end{cases}$$

Proof. We note that

$$|m_n(w) - 1| \leq |m_n(w)| + 1 \leq 2.$$

For $|w| \leq \pi/2$ and for $n \geq 1$, $|\tan w/2|^{2n} \leq |\tan w/2| \leq 2|w|/\pi$. Therefore, we have for $|w| \leq \pi/2$,

$$|m_n(w) - 1| = \left| \frac{\Phi_n(w)}{\Phi_n(2w)} (\cos w/2)^n - 1 \right|$$

$$= \left| \frac{\Phi_n(w)\cos w/2^n - \Phi_n(2w)}{\sqrt{\Phi_n(2w)}(\sqrt{\Phi_n(w)}(\cos w/2^n + \sqrt{\Phi_n(2w)})} \right|$$

$$\leq \frac{(\sin w/2)^{2n} \Phi_n(w + \pi)}{\Phi_n(2w)}$$

$$= \frac{(\sin w/2)^{2n} (\cos w/2)^{2n} \Phi_n(w + \pi)}{(\cos w/2)^{2n} \Phi_n(2w)}$$

$$= \frac{(\tan w/2)^{2n} (\cos w/2)^{2n} \Phi_n(w + \pi)}{(\cos w/2)^{2n} \Phi_n(w) + (\sin w/2)^{2n} \Phi_n(w + \pi)}$$

$$\leq \frac{2}{\pi |w|},$$

where we used the fact that $\Phi_n(w + \pi) \leq \Phi_n(w)$ on $[-\pi/2, \pi/2]$. 

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Lemma 2.4. (a) For each fixed $w$, $\hat{\psi}_n(w) = \prod_{j=1}^{\infty} m_n(w/2^j)$ converges uniformly on $n$.

(b) $\hat{\psi}_n(w) \rightarrow \hat{\psi}_{SH}(w)$ pointwise a.e. as $n \rightarrow \infty$.

(c) $\hat{\psi}_n(w) \rightarrow \hat{\psi}_{SH}(w)$ pointwise a.e. as $n \rightarrow \infty$.

Proof. (a) Fix $w$ and choose $j_0$ so that $|w/2^{j_0}| \leq \pi/2$. By Lemma 2.3,

$$\sum_{j=1}^{\infty} |m_n\left(\frac{w}{2^j}\right) - 1| = \sum_{j=1}^{j_0} |m_n\left(\frac{w}{2^j}\right) - 1| + \sum_{j=j_0+1}^{\infty} |m_n\left(\frac{w}{2^j}\right) - 1|$$

$$\leq 2j_0 + \sum_{j=j_0+1}^{\infty} \frac{2 |w|}{\pi 2^j} = 2j_0 + \frac{2 |w|}{\pi 2^{j_0}},$$

uniformly on $n$. Therefore, the product $\varphi_n(w)$ converges uniformly on $n$.

(b) Fix $w \notin \cup_{j=1}^{\infty} 2^j(\pm \pi + 2\pi \mathbb{Z})$ and let $\epsilon > 0$. By (a), we can choose $j_1$ (independent of $n$) so that

$$\left| \hat{\psi}_n(w) - \prod_{j=1}^{j_1} m_n\left(\frac{w}{2^j}\right) \right| < \epsilon,$$

and

$$\left| \hat{\psi}_{SH}(w) - \prod_{j=1}^{j_1} m_{SH}^i\left(\frac{w}{2^j}\right) \right| < \epsilon,$$
for \( i = 0, 1 \). Therefore, we have

\[
|\hat{\phi}_n(w) - \hat{\phi}_{SH}(w)| \leq \left| \hat{\phi}_n(w) - \prod_{j=1}^{j} m_n\left(\frac{w}{2^j}\right) \right| + \left| \prod_{j=1}^{j} m_n\left(\frac{w}{2^j}\right) - \prod_{j=1}^{j} m_{SH}\left(\frac{w}{2^j}\right) \right| \\
+ \left| \prod_{j=1}^{j} m_{SH}\left(\frac{w}{2^j}\right) - \hat{\phi}_{SH}(w) \right| \\
< 2\epsilon + \left| \prod_{j=1}^{j} m_n\left(\frac{w}{2^j}\right) - \prod_{j=1}^{j} m_{SH}\left(\frac{w}{2^j}\right) \right|.
\]

We choose \( i := i(n) = 0 \) (\( n = \text{even} \)), \( 1 \) (\( n = \text{odd} \)). Note that \( w/2^j \not\equiv \pm\pi/2 + 2\pi\mathbb{Z} \) for any \( j \geq 1 \). Since \( m_{2n}(w/2^j) \rightarrow m_{SH}^0(w/2^j) \) and \( m_{2n+1}(w/2^j) \rightarrow m_{SH}^1(w/2^j) \) as \( n \rightarrow \infty \) as shown in Lemma 2.2, we can choose \( n_0 \in \mathbb{N} \) so that

\[
\left| \prod_{j=1}^{j} m_n\left(\frac{w}{2^j}\right) - \prod_{j=1}^{j} m_{SH}\left(\frac{w}{2^j}\right) \right| < \epsilon \quad \text{for} \quad n \geq n_0.
\]

Therefore, \( \hat{\phi}_n(w) \rightarrow \hat{\phi}_{SH}(w) \) pointwise as \( n \rightarrow \infty \) for \( w \not\in \bigcup_{j=1}^{\infty} \{ \pm\pi/2 + 2\pi\mathbb{Z} \} \).

(c) The proof follows from (b) in view of the definition of \( \hat{\psi}_n \) in (1.7). It is also proved in [3] with a different proof. \( \square \)

Now, we state and prove our main result.

**Theorem 2.5.** (a) For \( 1 \leq p < \infty \), \( \| \hat{\phi}_n - \hat{\phi}_{SH} \|_{L^p(\mathbb{R})} \rightarrow 0 \) and \( \| \hat{\psi}_n - \hat{\psi}_{SH} \|_{L^p(\mathbb{R})} \rightarrow 0 \) as \( n \rightarrow \infty \).

(b) For \( 2 \leq q \leq \infty \), \( \| \phi_n - \phi_{SH} \|_{L^q(\mathbb{R})} \rightarrow 0 \) and \( \| \psi_n - \psi_{SH} \|_{L^q(\mathbb{R})} \rightarrow 0 \), as \( n \rightarrow \infty \).

In particular, \( \phi_n \rightarrow \phi_{SH} \) and \( \psi_n \rightarrow \psi_{SH} \) uniformly on \( \mathbb{R} \) as \( n \rightarrow \infty \).

**Proof.** We define an auxiliary \( 2\pi \)-periodic continuous function \( M \), via

\[
M(w) = \begin{cases} 
1, & |w| \leq \frac{\pi}{2}; \\
2^{3/2}(\cos w/2)^3, & \frac{\pi}{2} < |w| \leq \pi,
\end{cases}
\]
and let $\hat{\phi}(w) := \prod_{j=1}^{\infty} M(w/2^j)$. It is obvious that

$$0 \leq |m_n(w)| \leq M(w), \quad n = 3, 4, \ldots$$

and that $\hat{\phi}(w)$ has the decay $|\hat{\phi}(w)| \leq C(1 + |w|)^{-3/2}$ by Theorem 5.5 of [11]. We have

$$|\hat{\psi}_n(w)| = \prod_{j=1}^{\infty} |m_n \left( \frac{w}{2^j} \right)|$$

$$\leq \prod_{j=1}^{\infty} |M \left( \frac{w}{2^j} \right)| = |\hat{\phi}(w)| \leq C(1 + |w|)^{-3/2}.$$ 

Therefore (a) follows from Lemma 2.4 by the dominated convergence theorem. (b) follows from (a) by Hausdorff–Young inequality:

$$\|f\|_{L^q(\mathbb{R})} \leq \|\hat{f}\|_{L^p(\mathbb{R})}, \quad \text{for } 1 \leq p \leq 2,$$

where $q$ is the exponent conjugate to $p$. □

**Remark 1.** We illustrate the convergence of the Battle–Lemarié scaling functions and wavelets (for $n = 4$ and 10) to the Shannon scaling function and wavelet in Fig. 1.

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**References**