Evolutionary Games in Wireless Networks

Hamidou Tembine, Eitan Altman, Rachid El-Azouzi, Yezekael Hayel
LIA/CERI, University of Avignon
INRIA Sophia-Antipolis
Corresponding author (Tembine): email: tembine@ieee.org
Updated: September 15, 2009

Abstract

This paper studies both power control and multiple access control in wireless networks. We consider a noncooperative interaction among large population of mobiles that interfere with each other through many local interactions. The first objective of this paper is to extend the evolutionary game framework to allow an arbitrary number of mobiles that are involved in a local interaction. Furthermore the interactions between mobiles are not necessarily reciprocal as the set of mobiles causing interference to a given mobile may differ from the set of those suffering from its interference. We focus our study on (i) the multiple access control in slotted ALOHA based wireless network and (ii) power control in W-CDMA wireless networks. We define and characterize the equilibrium (called Evolutionarily Stable State or Strategy - ESS) for these games and we show how the evolution of dynamics and the equilibrium behavior of ESS are influenced by the characteristics of the wireless channel and pricing characteristics.

Keywords: evolutionary game, evolutionarily stable state, evolutionary game dynamics, slotted Aloha, W-CDMA.
Contents

1 Introduction 3
   1.1 Related Work ........................................... 4
   1.2 Contributions ........................................... 5
   1.3 Organization of the paper ............................ 5

2 Motivating Examples: Symmetry and Reciprocity 6
   2.1 Non-reciprocal pairwise interactions ................. 7
   2.2 Non-reciprocal interactions between groups three players ... 8
   2.3 Interactions between a random number of players .... 9

3 Formulation of evolutionary games with arbitrary number of
   players in local interaction 10
   3.1 Solution concepts and refinement ....................... 12
   3.2 Evolutionary Game Dynamics ............................ 14
      3.2.1 Replicator Dynamics ................................. 14
      3.2.2 Other Evolutionary Game Dynamics ............... 15
      3.2.3 Delayed Evolutionary Game Dynamics ............ 15

4 Slotted ALOHA based access network 15
   4.1 Utility function and ESS ............................... 17
   4.2 Existence and Uniqueness of ESS ....................... 18
   4.3 Spatial Nodes Distribution and ESS .................... 19
      4.3.1 Fixed number of nodes in a local interaction .... 20
      4.3.2 Poisson distribution ............................... 21
   4.4 Optimization issue ..................................... 22
      4.4.1 Fixed number of nodes in a local interaction .... 22
      4.4.2 Poisson distribution ............................... 23
   4.5 Evolutionary Game Dynamics ............................ 24

5 W-CDMA Wireless Networks 26
   5.1 Evolutionary Game dynamics in W-CDMA ................. 31
   5.2 More than two power levels ............................ 33

6 Numerical investigation 34
   6.1 Slotted Aloha based wireless networks ............... 34
      6.1.1 Optimization of total throughput ............... 34
      6.1.2 Dynamics .......................................... 34
1 Introduction

The evolutionary game formalism is a central mathematical tool developed by biologists for predicting population dynamics in the context of interactions between populations. This formalism identifies and studies two concepts: the ESS (Evolutionarily Stable Strategy), and the Evolutionary Game Dynamics. The ESS, first defined in 1972 by the biologist Maynard Smith [27], is characterized by a property of robustness against invaders (mutations). More specifically, (i) if an ESS is reached, then the proportions of each population do not change in time. (ii) at ESS, the populations are immune from being invaded by other small populations. This notion is stronger than Nash equilibrium in which it is only requested that a single user would not benefit by a change (mutation) of its behavior. The ESS concept helps to understand mixed strategies in games with symmetric payoffs. A mixed strategy can be interpreted as a composition of the population. An ESS can be also interpreted as a Nash equilibrium of the one-shot game but a (symmetric) Nash equilibrium cannot be an ESS. As is shown in [34], ESS has strong refinement properties of equilibria such as proper equilibrium, perfect equilibrium etc. Although ESS has been defined in the context of biological systems, it is highly relevant to engineering as well [35]. In the biological context, the replicator dynamics is a model for the change of the size of the population(s) as biologist observe, where as in engineering, we can go beyond characterizing and modeling existing evolution. The evolution of protocols can be engineered by providing guidelines or regulations for the way to upgrade existing ones and in determining parameters related to deployment of new protocols and services.

There have been a lot of work on non-cooperative modeling of power control and multiple access control using game theory [1, 19]. There are two advantages in doing so within the framework of evolutionary games:

- it provides the stronger concept of equilibria, the ESS, which allows us to identify robustness against deviations of more than one mobile, and
it allows us to apply the generic convergence theory of replicator dynamics, and stability results that we introduce in future sections.

1.1 Related Work

Several previous papers have already studied evolutionary games with pairwise local interactions in the context of wireless networks. Bonneau et al. have introduced evolutionary games in the context of unslotted Aloha in [7]. They have identified conditions for the existence of non-trivial ESS and have computed them explicitly. In [30], the authors considered an evolutionary multiple access game and studied delay effect under various models of evolutionary game dynamics with asymmetric delay [31]. In [32], the authors extended this model by including a regret cost, incurred when no user transmits, and studied the impact of that cost on the proportion of mobiles that transmit at equilibrium. In the last three papers, the delay is shown to have negative impact on the stability of the system. For other applications of evolutionary games to modeling network problems, such as congestion control [3, 18, 20, 37], energy management [2], and resource sharing mechanisms in P2P [25]. The authors in [41] study selfish routing problem using the replicator dynamics as a model for the dynamic behaviour of selfish agents in networks. For the symmetric case they have given a tight bound for the time of convergence of this dynamics that is polynomial in the degree of approximation and logarithmic in network parameters. For the multicommodity case, they derived an upper bound which is linear in the network. The same authors have studied in [38] routing policies using evolutionary game dynamics via a fluid limit. They have shown that oscillation effects can be avoided, however, when using smooth adaption policies that do not always switch to better alternatives but only with a probability depending on the advantage in the latency and derived dynamics that converge to a fixed point corresponding to a Nash equilibrium for the underlying routing game under the condition that update periods are not too large. In [39] the authors present a routing protocol applying evolutionary game theory in the context of wireless mesh networks. The authors consider a traffic model with infinite number of agents; each is responsible for an infinitesimal load and routing strategy of every agent is being revised continuously by sampling another path using input from the physical layer. The effect of time delays on the stability of the network is not analyzed in [38, 39, 40, 41, 42]. In order to take into consideration non-reciprocal interactions (see next Section) and to
cover dense networks case, an extension of the standard evolutionary game framework into variable number of locally interacting players is needed. We present such an interaction model in next Section.

1.2 Contributions

The contribution of this work can be summarized in three points: The first objective of this paper is to extend the evolutionary game framework to allow an arbitrary (possibly random) number of players that are involved in a local interaction (possibly non-reciprocal interaction). The second objective of this paper is to apply the extended model to multiple access control games studied in [32] which we extend to more than two interacting nodes. In the context of Medium Access Games, we study the impact of the node distribution in the game area on the equilibrium stable strategies of the evolutionary game. Considering this kind of games, we use the notion of expected utility as this game is not symmetric, indeed the number of players with which a given one interacts may vary from one to another; and also non-reciprocity property. We consider the following parameters in the access game: transmission cost, collision cost and regret cost. We analyze the impact of these parameters on the probability of successful transmission and give some optimization issues. The third objective of this paper is to apply evolutionary game models to study the interaction of numerous mobiles in competition in a W-CDMA wireless environment. Specifically, we focus our study in a power control game in a dense wireless network where each user transmits using orthogonal codes like in W-CDMA. The utility function of each mobile is based on carrier (signal)-to-interference ratio and pricing scheme proportional to transmitted power. We provide and predict the evolution of population between two types of behaviors : aggressive (high power) and peaceful (low power). We identify cases in which at ESS, only one population prevails (ESS in pure strategies) and others, in which an equilibrium between several population types is obtained. We also provide the conditions of the uniqueness of ESS. Furthermore, we study different pricing for controlling the evolution of population.

1.3 Organization of the paper

The rest of this paper is organized as follows. In next section, we give some motivating examples that illustrate the limitation of evolutionary games,
developed in the biology literature, in the area of computer networks. Section 3 describes a general model of population games with random number of interacting players at each local interaction and gives the solution concept of ESS and evolutionary game dynamics. We then study in Section 4 a generalized multiple access game with a random number of players. Section 5 presents an evolutionary uplink power control in W-CDMA networks. In Section 6, we present numerical examples on effect of time delays on the convergence and stability of the replicator dynamics to the ESS. Section 7 concludes the paper.

2 Motivating Examples: Symmetry and Reciprocity

The classical evolutionary game formalism is a central mathematical tool developed by biologists for predicting population dynamics in the context of interaction between populations. In order to make use of the wealth of tools and theory developed in the biology literature, many works in the area of computer networks [31, 2, 7] ignore cases where local interactions between populations involve more than two individuals. This restriction limits the modeling power of evolutionary games which are not useful in a network operating at heavy load, such as wireless networks with high density. This motivated us in this paper to consider a random number of users interacting locally.

Consider a large population of players. Each individual needs occasionally to take some action. When doing so, it interacts with the actions of some $K$ (possibly random number of) other individuals. We shall consider throughout the paper symmetric games in the sense that any individual faces the same type of game. All players have the same actions available, and same expected utility. We note however that the actual realizations need not be symmetric. In particular, (i) the number of players with which a given player interacts may vary from one player to another. (ii) We do not even need the reciprocity property: if player A interacts with player B, we do not require the converse to hold. We provide some examples of multiple access games to illustrate this non-reciprocity.

For example, we consider local interactions between transmitters; for each transmitter there corresponds a receiver. We shall say that a transmitter A
is subject to an interaction (interference) from transmitter B if the transmission from B overlaps that from A, and provided that the receiver of the transmission from A is within interference range of transmitter B.

2.1 Non-reciprocal pairwise interactions

Consider the example depicted in Figure 1. It contains 4 sources (circles) and 3 destinations (squares). A transmission of a source $i$ within a distance $r$ of the receiver $R$, causes interference to a transmission from a source $j \neq i$ to receiver $R$. We see that Source A and Source C cause no interference to any other transmission but the transmission from A suffers from interference from source B, and the one from C suffers from the transmission of the top most source (called D). Source B and D interfere with each other at their common destination. Thus each of the four sources suffers interference from a single other source, but except for nodes B and D, the interference is not reciprocal.

![Figure 1: Non-reciprocal pairwise interactions](image)

It is easy to see that in this non-reciprocal pairwise interaction, the following configurations leads to equilibria:
• A and D transmit at the same time slot. There are 2 successful transmissions if the others stay quiet.

• A and C transmit at the same time slot. There are 2 successful transmissions if the others stay quiet.

• B and C transmit at the same time slot. There are 2 successful transmissions if the others stay quiet.

• A, D and C transmit at the same time slot. There is 1 successful transmission (only from A) if B stay quiet.

• A, B and C transmit at the same time slot. There is 1 successful transmission (only from C) if D stay quiet.

• A, D transmit at the same time slot. There is 1 successful transmission (only from A) if B stay quiet.

• B, D transmit at the same time slot. There is no successful transmissions in the system.

2.2 Non-reciprocal interactions between groups three players

In Figure 2 there are four sources and only two destinations. Node A does not cause any interference to the other nodes but suffers interference from nodes B and D. Nodes B, C, D interfere with each other. This is a situation in which each mobile is involved in interference from two other mobiles but again the interference is not reciprocal.

The following configurations leads to equilibria:

• A and C transmit at the same time slot. There are 2 successful transmissions if the others stay quiet. This is a global optimum.

• Only B transmits. There is one successful transmissions if the others stay quiet.

• Only D transmits. There is one successful transmissions if the others stay quiet.

• B and D transmit at the same time slot. There is no successful transmissions.
2.3 Interactions between a random number of players

In this example the number of interfering nodes is not fixed. A suffers interference from 2 nodes, B and D suffer interference from a single other node and C does not suffer (and does not cause) interference. The following configurations leads to equilibria:

- A and C transmit at the same time slot. There are 2 successful transmissions if the others stay quiet. This is a global optimum.

- B and C transmit at the same time slot. There are 2 successful transmissions if the mobiles A and D stay quiet. This is a global optimum.

- D and C transmit at the same time slot. There are 2 successful transmissions if the others mobiles A and B stay quiet. This is a global optimum.

Note that if D or B moves from its position. The mobile A can play the role of C in this analysis and a new game between B or D and C will be played.

All examples exhibit asymmetric realizations and non-reciprocity. We next show how such a situation can still be considered as symmetric (due
Figure 3: Interactions between a random number of players

to the fact that we consider distributions of nodes rather than realizations). Assume that the location of the transmitters follow a Poisson distribution with parameter $\lambda$ over the two dimensional plane. Consider an arbitrary user A. Let $r$ be the interference range. Then the number of transmitters within the interference range of the receiver of A has a Poisson distribution with parameter $\lambda \pi r^2/2$. Since this holds for any node, the game is considered to be symmetric. The reason that the distribution is taken into account rather than the realization is that we shall assume that the actions of players will be taken before knowing the realization.

3 Formulation of evolutionary games with arbitrary number of players in local interaction

In this section, we extend the evolutionary game framework to allow an arbitrary (possibly random) number of players that are involved in a local interaction. First, we describe in this part notations of our model.
• There is one population of users. The number of users in the population is large.

• We assume that there are finitely many pure strategies or actions. Each member of the population chooses from the same set of strategies \( \mathcal{A} = \{1, 2, \ldots, N\} \).

• Let
\[
M := \{(x_1, \ldots, x_N) \mid x_j \geq 0, \sum_{j=1}^{N} x_j = 1\}
\]
be the set of probability distributions over the \( N \) pure actions. \( M \) can be interpreted as the set of mixed strategies. It is also interpreted as the set of distributions of strategies among the population, where \( x_j \) represents the proportion of users choosing the strategy \( j \). A distribution \( x \) is sometime called the "state" or "profile" of the population.

• The number of users interfering with a given randomly selected user is a random variable \( K \) in the set \( \{0, 1, \ldots\} \). In the bounded case, we will denote by \( k_{\text{max}} \) the maximum number of interacting users simultaneously with a user. This value depends on the node density and the transmission range. When making a choice of a strategy, a player knows the distribution of \( K \) but not its realization.

• A player does not know how many players would interact with it.

• The payoff function of all players depends of the player’s own behavior and the behavior of the other players. The expected payoff of a user playing strategy \( j \) when the state of the population is \( x \), is given by
\[
f(j, x) = \sum_{k \geq 0} \mathbb{P}(K = k) u_k(j, x),
\]
where \( u_k(j, x) \) is the payoff of a user playing strategy \( j \) when the state of the population is \( x \) and given that the number of users interfering with a given randomly selected user is \( k \). Although the payoffs are symmetric, the actual interference or interactions between two players that use the same strategy need not be the same, allowing for non-reciprocal behavior. The reason is that the latter is a property of
the random realization whereas the actual payoff already averages over
the randomness related to the interactions, the number of interfering
players, the topology etc.

- The game is played many times.

3.1 Solution concepts and refinement

Suppose that, initially, the population profile is \( x \in M \). The average payoff is

\[
f(x, x) := \sum_{j=1}^{N} x_j f(j, x).
\]

Now suppose that a small group of mutants enters the population playing
according to a different profile \( \text{mut} \in M \). If we call \( \epsilon \in (0, 1) \) the size of the
subpopulation of mutants after normalization, then the population profile
after mutation will be \( \epsilon \text{mut} + (1 - \epsilon)x \). After mutation, the average payoff
of non-mutants will be given by \( f(x, \epsilon \text{mut} + (1 - \epsilon)x) \) where

\[
f(x, y) := \sum_{j=1}^{N} x_j f(j, y).
\]

Note that \( f \) need not to be linear in the second variable. Analogously, the
average payoff of a mutant is \( f(\text{mut}, \epsilon \text{mut} + (1 - \epsilon)x) \).

**Evolutionarily Stable Strategy or State** A strategy \( x \in M \) is an ESS
if for any \( \text{mut} \neq x \), there exists some \( \epsilon_{\text{mut}} \in (0, 1) \), which may depend on
\( \text{mut} \), such that for all \( \epsilon \in (0, \epsilon_{\text{mut}}) \) one has

\[
f(x, \epsilon \text{mut} + (1 - \epsilon)x) > f(\text{mut}, \epsilon \text{mut} + (1 - \epsilon)x)
\]

which can be rewritten as

\[
\sum_{j=1}^{N} (x_j - \text{mut}_j) f(j, \epsilon \text{mut} + (1 - \epsilon)x) > 0.
\]

That is, \( x \) is ESS if, after mutation, non-mutants are more successful than
mutants. In other words, mutants cannot invade the population and will
eventually get extinct.
**Neutrally Stable Strategy:** We say that $x$ is a neutrally stable strategy if the inequality (1) is non-strict.

Note that an evolutionarily stable strategy is a neutrally stable strategy which is a symmetric (Nash) equilibrium of the one-shot game i.e the strategy is the same for each player and no player has incentive to unilaterally change his/her action. A population profile $x$ is an equilibrium if the variational inequality

$$\sum_{j \in A} (x_j - \text{mut}_j) f(j, x) \geq 0$$

holds for all mut $\in M$. Now we consider the following assumption:

**Assumption (H):** Given any distribution of the integer random variable $K$, the payoff function

$$x \mapsto D(x) = [f(1, x), f(2, x), \ldots, f(N, x)]$$

is continuous on $M$.

Note that (H) is satisfied for any integer random variable with finite support. Let $\text{proj}_M$ be the projection into the set $M$ defined as

$$\text{proj}_M(y) = \arg \min_{x \in M} \|y - x\|_2$$

where $\|y\|_2 = \sqrt{y_1^2 + \ldots + y_N^2}$ is the Euclidian norm of $y$. Since $M$ is a non-empty, convex and compact subset of $\mathbb{R}^N$, the projection $\text{proj}_M(y)$ is unique (there is a unique minimizer on $M$ of the function $x \mapsto \|y - x\|_2$).

**Theorem 1.** Under assumption (H), the evolutionary game with random number of interacting users has at least one equilibrium.

**Proof.** We show that there exists a probability vector $x \in M$ such that the inequality

$$\sum_{j \in A} (x_j - \text{mut}_j) f(j, x) \geq 0$$

holds for all mut $\in M$. Let $\theta > 0$. The problem is equivalent to the existence of solution of the variational inequality problem:

$$\langle x - \text{mut}, \theta D(x) \rangle = \sum_{j \in A} (x_j - \text{mut}_j) \theta f(j, x) \geq 0.$$
The term $\langle x - \text{mut}, \theta D(x) \rangle$ can be rewritten as

$$\langle x - \text{mut}, (x + \theta D(x)) - x \rangle.$$ 

Thus, the equilibrium is a solution of $\langle x - \text{mut}, (x + \theta D(x)) - x \rangle \leq 0, \forall \text{mut}$. This implies that

$$x = \text{proj}_M(x + \theta D(x)).$$

It is known that $\text{proj}_M$ is a 1-Lipschitz function (hence continuous). Since $D$ is a continuous function, the composition $\text{proj}_M(Id + \theta D)$ is continuous and $M$ is a non-empty, convex and compact subset of $\mathbb{R}^N$. Using Brouwer fixed point theorem, the map $\text{proj}_M(Id + \theta D)$ has at least one fixed point $x^*$ in $M$. $x^*$ is our desired equilibrium.

This result can be extended into the lower semi-continuity function case.

### 3.2 Evolutionary Game Dynamics

Evolutionary game theory considers a dynamic scenario where players are interacting with others players and adapting their choices based on the fitness they receive. A strategy having higher fitness than others tends to gain ground: this is formulated through rules describing the dynamics (such as the replicator dynamics or others) of the sizes of populations (of strategies).

#### 3.2.1 Replicator Dynamics

Replicator dynamics is one of the most studied dynamics in evolutionary game theory. It has been introduced by Taylor and Jonker [29]. The replicator dynamics has been used for describing the evolution of road traffic congestion in which the fitness is determined by the strategies chosen by all drivers [9, 21]. It has also been studied in the context of the association problem in wireless networks in [24]. We introduce the replicator dynamics which describes the evolution in the population of the various strategies. In the replicator dynamics, the share of a strategy $j$ in the population grows at a rate proportional to the difference between the payoff of that strategy and the average payoff of the population. The replicator dynamic equation is given by

$$\dot{x}_j(t) = \mu x_j(t) \left[ f(j, x(t)) - \sum_{l=1}^{N} x_l(t) f(l, x(t)) \right].$$

(2)
where $\mu$ is some positive constant. The parameter $\mu$ can be used to tune the rate of convergence and it may be interpreted as the rate that a player of the population participates in a (local interaction) game. In biology, it can represent the probability that an animal finds a resource available.

### 3.2.2 Other Evolutionary Game Dynamics

There is a large number of population dynamics other than the replicator dynamics which have been used in the context of non-cooperative games. Examples are the excess payoff dynamics, the fictitious play dynamics, gradient methods [17], Smith dynamics, G-function based dynamics [35]. Much literature can be found in the extensive survey on evolutionary game dynamics in [11] and in the book of Sandholm on Population Games and Evolutionary Dynamics [22].

#### 3.2.3 Delayed Evolutionary Game Dynamics

The fitness for a player at a given time is determined by the action $i$ taken by the player at that time, as well as by the actions of the population it interacts with, that was taken $\tau_i$ units ago. More precisely, if an anonymous user $1$ chooses the strategy $j$ at time $t$ when the population profile is $x$ then user $1$ will receive the payoff $f(j, x(t))$ only $\tau_k$ times later. Thus, the payoff at time $t$ is given by $f(j, x(t - \tau_j))$. In the replicator dynamics with time delays, the share of a strategy $j$ in the population grows at a rate proportional to the difference between the payoff of that strategy delayed by an average time delay $\tau_j$ and the average delayed payoff of the population. The replicator dynamic equation is then given by

$$
\dot{x}_j(t) = \mu x_j(t) \left[ f(j, x(t - \tau_j)) - \sum_{l=1}^{N} x_l f(l, x(t - \tau_l)) \right].
$$

The parameters $\tau_j$ and $\mu$ don’t change the evolutionary stable strategies set but have a big influence on the stability of the system.

### 4 Slotted ALOHA based access network

In this section, we consider an Aloha system in which mobiles make transmission decisions in an effort to maximize their utility. We assume that mobiles
are randomly placed over a plane. All mobiles use the same fixed transmission range of $r$. The channel is ideal for transmission and all errors are due to collision. A mobile decides to transmit a packet or not to transmit to a receiver when they are within transmission range of each other (see Figure 4). Interference occurs as in the Aloha protocol: if more than one neighbors of a receiver transmit a packet at the same time then there is a collision. Let $\mu$ be the probability that a mobile $i$ has its receiver $R(i)$ within its range. When a mobile $i$ transmits to $R(i)$, all mobiles within a circle of radius $R$ centered at $R(i)$ cause interference to the node $i$ for its transmission to $R(i)$. This means that more than one transmission within a distance $R$ of the receiver in the same slot cause a collision and the loss of mobile’s $i$ packet at $R(i)$.

![Figure 4: Many local interactions in Evolving Aloha based Access network. Users are represented in circle, receivers in square.](image)

Each mobile has two possible strategies: either to transmit ($T$) or to stay quiet ($S$). If mobile $i$ transmits a packet, it incurs a transmission cost of $\delta \geq 0$. The packet transmission is successful if the other users don’t transmit (stays quiet) in that given time slot, otherwise there is a collision and the corresponding cost is $\Delta \geq 0$. If there is no collision, user $i$ gets a reward of $V$ from the successful packet transmission. We suppose that the reward $V$ is greater than the cost of transmission $\delta$. When all users stay quiet, they have to pay a regret cost $\kappa$. If $\kappa = 0$ the game is called *degenerate multiple access*
4.1 Utility function and ESS

Let $\mathcal{A} := \{T, S\}$ be the set of strategies. An equivalent interpretation of strategies is obtained by assuming that individuals choose pure strategies and then the probability distribution represents the fraction of individuals in the population that choose each strategy. We denote by $s$ (resp. $1 - s$) the population share of strategy $T$ (resp. $S$). The payoff obtained by a node with $k$ others interfering nodes when it plays $T$ is

$$u_k(T, s) = (-\Delta - \delta) (1 - \eta_k) + (V - \delta) \eta_k$$

where $\eta_k := (1 - s)^k$, and the node-mutant receives

$$u_k(S, s) = -\kappa (1 - s)^k$$

when it stays quiet. The expected payoff of an anonymous transmitter node-mutant is given by

$$f(T, s) = \mu \sum_{k \geq 0} P(K = k) u_k(T, s)$$

$$= \mu \left( -(\Delta + \delta) + (V + \Delta) \sum_{k \geq 0} P(K = k) (1 - s)^k \right)$$

$$= -\mu(\Delta + \delta) + \mu(V + \Delta) G_K(1 - s).$$

1The one-shot game with $n$ nodes has $2^n - 1$ Nash equilibria and a unique ESS.
where $G_K$ is the generating function of $K$. Analogously, we have

$$f(S, s) := \mu \sum_{k \geq 0} \mathbb{P}(K = k) u_k(S, s)$$

$$= -\mu \kappa \sum_{k \geq 0} (1 - s)^k \mathbb{P}(K = k).$$

4.2 Existence and Uniqueness of ESS

We introduce two alternative information scenario that have an impact on the decision making. Thus, we will study three different scenarios as follows:

- **[Case 1]** The mobile does not know whether there are zero or more other mobiles in a given local interaction game about to be played.

- **[Case 2]** The mobile knows if there are a transmitter at the range of its receiver, consequently he transmits with probability one in case no other potential interferers are present.

- **[Case 3]** ”massively dense” : The mobile is never alone to transmit during a slot. That means there is at least one other mobile that is involved in the local interaction game.

We denote $\alpha := \frac{\Delta + \delta}{V + \Delta + \kappa}$, which represents the ratio between the collision cost $-\Delta - \delta$ (cost when there is a collision during a transmission) and the difference between global cost perceived by a mobile $-\Delta - \delta - \kappa$ (collision and regret) and the benefit $V - \delta$ (reward minus transmission cost). When the collision cost $\Delta$ becomes high, the value $\alpha$ converges to one and when the reward or regret cost becomes high, the value $\alpha$ is close to zero.

A transmitter does not know if there are other transmitters at the range of its receiver. Then, even when it is the only transmitter, it has to decide to transmit or not.

**Theorem 2.**

- **Case 1:** If $\mathbb{P}(K = 0) < \alpha$, then the game has a unique ESS $s^*_1$ given by $s^*_1 = \phi^{-1}(\alpha)$ where

$$\phi : s \mapsto \sum_{k \geq 0} \mathbb{P}(K = k) (1 - s)^k.$$
• **Case 2:** An anonymous user without interfering user receives the fitness $V - \delta$. If $P(K = 0) < \frac{\Delta + \delta}{V + \Delta}$, then the game has a unique ESS $s^*_2$ given by

$$s^*_2 = \phi^{-1} \left( \frac{\Delta + \delta + \kappa P(K = 0)}{V + \Delta + \kappa} \right)$$

where

$$\phi : s \mapsto \sum_{k \geq 0} P(K = k) (1 - s)^k.$$ 

• **Case 3:** The game has always an unique ESS which is solution of the following equation

$$\sum_{k \geq 1} P(K = k)(1 - s)^k = \alpha$$

**Proof.** A strictly mixed equilibrium $s$ is characterized by $f(T, s) = f(S, s)$ i.e $\phi(s) = \alpha$. The function $\phi$ is continuous and strictly decreasing monotone on $(0, 1)$ with $\phi(1) = P(K = 0)$ and $\phi(0) = 1$. Then the equation $\phi(s) = \frac{\Delta + \delta}{V + \Delta + \kappa}$ has a unique solution in the interval $(P(K = 0), 1)$. Thus we have

$$f(s, y) - f(mut, y) = \mu (V + \Delta + \kappa) (s - mut) (\phi(y) - \phi(s)).$$

Since $s - \epsilon mut - (1 - \epsilon)s = \epsilon (s - mut)$, for $y = \epsilon mut + (1 - \epsilon)s$ one has

$$\sum_{j \in \{T, S\}} (x_j - mut) f(j, y) > 0$$

(because $\phi$ is strictly decreasing continuous function) for all $mut \neq s$. This completes the proof. 

### 4.3 Spatial Nodes Distribution and ESS

In this part, we study two cases of spacial nodes distribution. In the first one, we assume that the number of nodes in local interaction, is fixed and second one, we assume that nodes is are distributed over a plan following the Poisson distribution. These allow us to compute explicitly the ESS and propose some optimization issues in Slotted aloha based wireless networks.
4.3.1 Fixed number of nodes in a local interaction

In this part, we suppose that the population of nodes is composed with many local interaction between \( n \geq 2 \) nodes. Let \( \mathcal{A} := \{T, S\} \) the set of strategies and assume that the strategy \( T \) has a delay \( \tau_T \) and the strategy \( S \) has the delay \( \tau_S \). The payoff of a player using the action \( a_i \in \mathcal{A} \) against the other players when they use the multi-strategy \( a_{-i} = (a_1, \ldots, a_i-1, a_{i+1}, \ldots, a_n) \) is given by \( U_i(a) \). Each user plays the \( n \)-player following game \( \Gamma_n = (\mathcal{N}, \mathcal{A}, (U_i)_{i \in \mathcal{N}}) \) where (a) \( \mathcal{N} \) is the set of users (nodes) with the cardinal of \( \mathcal{N} \) is \( n \), (b) \( \mathcal{A} \) the set of pure actions (the same for every user), (c)for every user \( i \) in \( \mathcal{N} \), the payoff function \( U_i : \mathcal{A}^n \rightarrow \mathbb{R} \) is given by

\[
U_i(a) = \begin{cases} 
V - \delta & \text{if } a_i = T \text{ and } a_j = S, \quad \forall \ j \neq i \\
0 & \text{if } a_i = S \text{ and } \{ j \in \mathcal{N} \mid a_j = T \} \geq 1 \\
-\Delta - \delta & \text{if } a_i = T \text{ and } \{ j \in \mathcal{N} \mid a_j = T \} \geq 2 \\
-\kappa & \text{if } a_j = S, \quad \forall \ j \in \mathcal{N}
\end{cases}
\]

Let \( s \) be the proportion of nodes in the population using the strategy \( T \). Then \( x = (s, 1 - s) \) is the state of the population. Let \( \Delta(\mathcal{A}) := \{sT + (1 - s)S \mid 0 \leq s \leq 1\} \) the set of mixed strategies. The average payoff is

\[
f(s, s) = \mu s \left[ (\Delta - \delta) \left(1 - (1 - s)^{n-1}\right) \right] + (V - \delta) \left(1 - s\right)^{n-1} - \mu \kappa (1 - s)^n.
\]

It is not difficult to see that the one-shot game \( \Gamma_n \) has \( 2^n - 1 \) Nash equilibrium, \( n \) of them are optimal in Pareto sense\(^2\):

- If only one node transmit and the others stay quiet, then the node which transmit gets the payoff \( V - \delta \) and the others receive nothing and has no cost. This configuration is an equilibrium.

- There are exactly \( n \) pure equilibria and all these pure equilibria are Pareto optimal.

- \( k \) (\( 1 \leq k < n - 1 \)) of the \( n \) nodes choose to stay quiet and the \( n - k \) others are active and play the optimal mixed strategy in the game \( \Gamma_{n-k} := \left(1 - \alpha^{\frac{1}{n-k-1}}, \alpha^{\frac{1}{n-k-1}}\right) \) where \( \alpha := \frac{\Delta + \delta}{V + \Delta + \kappa} \). Thus, there are exactly \( \sum_{k=1}^{n-2}(\frac{n}{k}) = 2^n - (n + 2) \) partially mixed Nash equilibria.

\(^2\)An allocation of payoffs is Pareto optimal or Pareto efficient if there is no other allocation that makes every node at least as well off and at least one node strictly better off.
• The game has a unique strictly mixed Nash equilibrium given by \((1 - \alpha^{\frac{1}{n-1}}, \alpha^{\frac{1}{n-1}})\)

• the allocation of payoff obtained in these (partially or completely) mixed strategy are not Pareto optimal.

Note that the first interference scenario described in the previous section holds here because the number of interferes is fixed and is equal to \(n - 1\). Then, from theorem 2 with the function \(\phi(s) = (1 - s)^{n-1}\), the ESS exists and is uniquely defined by \(s^* = 1 - \alpha^{\frac{1}{n-1}}\). This result generalizes the ESS in the two-player case that we have shown in [32].

4.3.2 Poisson distribution

We consider that nodes are distributed over a plan following a Poisson distribution with density \(\lambda\). The probability that a node has \(k\) neighbors is given by the following distribution.

Cases 1 and 2:

\[
P(K = k) = \frac{(\lambda \pi r^2)^k}{k!} e^{-\lambda \pi r^2}, \quad k \geq 0.
\]

Case 3:

\[
P(K = k) = \frac{(\lambda \pi r^2)^{k-1}}{(k-1)!} e^{-\lambda \pi r^2}, \quad k \geq 1.
\]

Considering those node distributions and from previous theorems, the unique ESS \(s^*\) for all cases, is solution of the following equation:

\[
\begin{cases}
  e^{-\lambda \pi r^2 s_1} = \alpha & \text{for case 1} \\
  e^{-\lambda \pi r^2 s_2} = \alpha + \frac{\kappa \mathbb{P}(K=0)}{V + \Delta + \kappa} & \text{for case 2} \\
  (1 - s_3) e^{-\lambda \pi r^2 s_3} = \alpha & \text{for case 3}
\end{cases}
\]

Thus we obtain the following equilibria in the different scenario:

\[
s_1^* = \log \left( \alpha^{-\frac{1}{\lambda \pi r^2}} \right),
\]

\[
s_2^* = \log \left( \left( \alpha + \frac{\kappa \mathbb{P}(K=0)}{V + \Delta + \kappa} \right)^{-\frac{1}{\lambda \pi r^2}} \right),
\]

and \(s_3^* = 1 - \frac{\text{LambertW}(\lambda \pi r^2 \alpha e^{\lambda \pi r^2})}{\lambda \pi r^2}\),

where \(\text{LambertW}(s)\) is the LambertW function which is the inverse function of \(f(w) = we^w\).
4.4 Optimization issue

In this subsection, we discuss some optimization issue which can be attained by changing the cost parameters. There we look for the probability of success that can be achieved in a local interaction depending on the nodes distribution and also cost parameters.

4.4.1 Fixed number of nodes in a local interaction

We assume here that every mobile has the same number of interfering users, that is \( n - 1 \). At the equilibrium point, the probability of success \( P_{\text{succ}}(n) \) of a node is given by \( s^*(1 - s^*)^{n-1} \). The total probability to have a successful transmission in a local interaction, which we denote later as \( \beta \) the total throughput of the system, is given by

\[
\beta(\alpha, n) = n\mu P_{\text{succ}} = n\mu s^*(1 - s^*)^{n-1},
\]

\[
= n\mu(1 - \alpha \frac{1}{n^{n-1}})\alpha,
\]

where \( \mu \) is the probability that a mobile has a receiver in its range. The total throughput \( \beta(\alpha, n) \) goes to \(-\mu_\alpha \log(\alpha)\) when the number of nodes \( n \) goes to infinity, i.e.

\[
\lim_{n \to \infty} \beta(\alpha, n) = -\mu_\alpha \log(\alpha).
\]

Hence, when \( n \) is very large, the total throughput is maximized when the cost ratio is \( \alpha = 1/e \). Then the total throughput \( \beta(\frac{1}{e}, n) \) tends to the value \( \mu/e \) when the number of nodes tends to infinity.

For a fixed number of nodes \( n \), the optimal total throughput is obtained when \( \alpha^* = (1 - \frac{1}{n})^{n-1} \) and the corresponding total throughput converges to the value \( \frac{\mu}{e} \) when the number of nodes tends to infinity, i.e.

\[
\lim_{n \to \infty} \beta(\alpha^*, n) = \lim_{n \to \infty} \mu(1 - \frac{1}{n})^{n-1} = \frac{\mu}{e}.
\]

The optimal total throughput with an infinite number of nodes is \( \frac{\mu}{e} \), which is the product between the probability \( \mu \) for a node to have a receiver in its range and \( \frac{1}{e} \) the maximum throughput of a slotted aloha system with infinite number of nodes [8].
4.4.2 Poisson distribution

We look for the average total throughput that can be achieved in a local interaction depending on distribution parameters and also cost parameters. We consider the Poisson distribution with parameters $\lambda$ and $r$. The average total throughput is given by:

$$\beta(\alpha, \lambda) = \mu \sum_{k \geq 0} k \mathbb{P}(K = k) P_{\text{succ}}(k),$$

where $P_{\text{succ}}(k)$ is the probability of success for one node when the number of nodes is $k$, which depends on the scenario considered. Then, the total throughput that can be achieved in a local interaction is given by different equation depending on the scenario considered. In the case 1 we have:

$$\beta(\alpha, \lambda) = \mu s_1^* \sum_{k \geq 0} k \mathbb{P}(K = k)(1 - s_1^*)^k$$

$$= \mu s_1^* \sum_{k \geq 0} \frac{k(\lambda \pi r^2)^k}{k!} (1 - s_1^*)^k$$

$$\approx \mu s_1^*(1 - s_1^*)\lambda \pi r^2 \alpha,$$

In the case 2, we have:

$$\beta(\tilde{\alpha}, \lambda) \approx \mu s_2^*(1 - s_2^*)\lambda \pi r^2 (\alpha + \frac{\kappa \mathbb{P}(K = 0)}{V + \Delta + \kappa}).$$

We derive immediately the following result:

**Proposition 1.** The maximum total throughput under Poisson distribution is attained when $\alpha = e^h(\lambda, r)$ in the case 1 (resp. $\alpha = e^h(\lambda, r) - \frac{\kappa \mathbb{P}(K = 0)}{V + \Delta + \kappa}$ in the case 2) where $h$ is one of the two functions defined by

$$(\lambda, r) \in \mathbb{R}^2_+ \mapsto \frac{-(1 + 2\lambda \pi r^2) \pm \sqrt{1 + 4(\lambda \pi r^2)^2}}{2}.$$

In the case 3, we have:

$$\beta(\alpha, \lambda) = \mu s_3^* \sum_{k \geq 1} k P(K = k)(1 - s_3^*)^k$$

$$= \mu s_3^* \sum_{k \geq 1} \frac{(\lambda \pi r^2)^{k-1}}{(k - 1)!} (1 - s_3^*)^k \approx \mu \alpha s_3^*(1 + \lambda \pi r^2(1 - s_3^*)).$$

In the following proposition, we drive the optimal throughput in case 3.

23
Proposition 2. In the case 3, there exists a unique $\alpha_3^*$ in which the total throughput is maximized. The $\alpha_3^*$ is given by $\alpha_3^* = (1 - s)e^{-\lambda \pi r^2 s}$ where $s$ is the unique solution in $[0, 1]$ of the following equation:

$$1 + \gamma - s(2 + 5\gamma + \gamma^2) + s^2(4\gamma + 2\gamma^2) - \gamma^2 s^3 = 0$$

Proof  The derivative of the function $H := \frac{\partial \beta}{\partial s}$ is given by

$$H(s) = (1 + \gamma - s(2 + 5\gamma + \gamma^2) + s^2(4\gamma + 2\gamma^2) - s^3\gamma^2)e^{-\gamma s}.$$ 

We prove that the above function is strictly decreasing on $[0, 1]$. For that, it is sufficient to study the following function

$$G(s) = 1 + \gamma - s(2 + 5\gamma + \gamma^2) + s^2(4\gamma + 2\gamma^2) - s^3\gamma^2.$$ 

We have $\frac{\partial G(s)}{\partial s}$ is given by

$$\frac{\partial G(s)}{\partial s} = -(2 + 5\gamma + \gamma^2) + 2s(4\gamma + 2\gamma^2) - 3s^2\gamma^2.$$

It is easy to show that the above function is always negative. Since $H(0) = 1 + \gamma > 0$ and $H(1) = -e^{-\gamma} < 0$ then the function $H$ is positive for $s \in [0, \bar{s})$ and is negative for $s \in (\bar{s}, 1]$ where $\bar{s}$ is the solution of the equation $G(s) = 0$. Since $s^*$ is decreasing function on $\alpha$, we conclude that function $P_{\text{succ}}$ is positive if $s \in [0, \bar{s})$ and is negative $s \in (\bar{s}, 1]$. Since the optimal of function $P_{\text{succ}}$ is attained at $\alpha = (1 - \bar{s})e^{-\lambda \pi r^2 \bar{s}}$.

4.5 Evolutionary Game Dynamics

Evolutionary game dynamics give a tool for observing the evolution of strategies in the population in time. The most famous one is the replicator, based on replication by imitation, which we consider in this subsection for observing the evolution of the strategies $T$ and $S$ in the population of nodes.

Proposition 3. The ESS given in theorem 2 is asymptotically stable in the replicator dynamics without delays for all non-trivial initial state ($s_0 \notin \{0, 1\}$).
Proof. The replicator dynamics is given by

$$\frac{d}{dt}s(t) = \mu(V + \Delta + \kappa)s(t)(1 - s(t))(\phi(s(t)) - \alpha).$$

The function $\phi$ is decreasing on $(0, 1)$ implies that the derivative of the function $s(1 - s)(\phi(s) - \alpha)$ at the ESS is negative. Hence, the ESS is asymptotically stable.

Now, we study the effect of the time delays on the convergence of replicator dynamics to the evolutionarily stable strategies in which each pure strategy is associated with its own delay. Let $\tau_T$ (resp. $\tau_S$) be the time delay of the strategy $(T)$ (resp. $(S)$). The replicator dynamic equation given in (2) becomes

$$\dot{s}(t) = \mu s(t)(1 - s(t)) \left[ f(T, s(t - \tau_T)) - f(S, s(t - \tau_S)) \right] \quad (5)$$

where

$$f(T, s(t)) := -\mu(\Delta + \delta)(1 - (1 - s(t))^{n-1}) + \mu(V - \delta)(1 - s(t))^{n-1},$$

$$f(S, s(t)) := -\mu\kappa(1 - s(t))^{n-1}.$$

In order to study the asymptotically stability of the replicator dynamics (5) around the unique ESS $s^*_1 = 1 - \left(\frac{\Delta + \delta}{V + \Delta + \kappa}\right)^{\frac{1}{n-1}}$, we linearize (5) at $s^*_1$. We obtain the following linear delay differential equation

$$\dot{z}(t) = -\mu(n - 1)s^*_1(1 - s^*_1)^{n-1} \times$$

$$\times \left( (V + \Delta)z(t - \tau_T) + \kappa z(t - \tau_S) \right) \quad (6)$$

where $z(t) = s(t) - s^*_1$. The following theorem give sufficient conditions of stability of (6) at zero.

**Theorem 3.** Suppose at least one of the following conditions holds

- $(V + \Delta)\tau_T + \kappa\tau_S < \frac{1}{(n-1)s(1-s)^{n-1}\mu}$
- $V + \Delta > \kappa$ and $(V + \Delta)\tau_T < \frac{V + \Delta - \kappa}{(n-1)s(1-s)^{n-1}\mu(V + \Delta + \kappa)}$
- $V + \Delta < \kappa$ and $\kappa\tau_S < \frac{-V - \Delta + \kappa}{(n-1)s(1-s)^{n-1}\mu(V + \Delta + \kappa)}$

Then the ESS $s$ is asymptotically stable.
A proof of the theorem 3 can be obtained using theorem 3 in [31] applying to equation (6).

We are looking for a necessary and sufficient condition of stability of the differential equation 6. For finding this, we need the following lemma.

Lemma 1 ([30]). The trivial solution of the linear delay differential equation
\[ \dot{z}(t) = -az(t - \tau), \quad \tau, a > 0 \]
is asymptotically stable if and only if \( 2a\tau < \pi \).

Given this lemma, a necessary and sufficient condition of stability of (6) at zero when delays are symmetric is given in the following theorem.

Theorem 4 (symmetric delay). Suppose that \( \tau_T = \tau_S = \tau \), then, the ESS \( s_1^* \) is asymptotically stable if and only if
\[ \tau < \frac{\pi}{2(n-1)\mu s_1^*(1-s_1^*)^{n-1}(V + \Delta + \kappa)} \]

Proof  By applying symmetric delay \( \tau_T = \tau_S = \tau \), in (6), one has,
\[ \dot{z}(t) = -\mu(n-1)s_1^*(1-s_1^*)^{n-1}(V + \Delta + s_1^* - \kappa) \times (z(t - \tau)) \]

We then apply the Lemma 1 for the parameter \( a = \mu(n-1)s_1^*(1-s_1^*)^{n-1}(V + \Delta + \kappa) > 0. \)

5 W-CDMA Wireless Networks

In this section, we apply evolutionary games to non-cooperative power control in wireless networks. Specifically, we focus our study in an uplink power control in W-CDMA wireless systems. In this section, the random number of interfering mobiles with a given randomly selected mobile is induced by the geographical position of the mobiles compared to the base stations.

We study in this section competitive decentralized power control in a wireless network where the mobiles uses, as uplink MAC protocol, the W-CDMA technique to transmit to a base station. We assume that there is a
large population of mobiles which are randomly placed over a plane following a Poisson process with density $\lambda$. We consider a random number of mobiles interacting locally. When a mobile $i$ transmits to its receiver $R(i)$, all mobiles within a circle of radius $R$ centered at the receiver $R(i)$ cause interference to the transmission from node $i$ to receiver $R(i)$ as illustrated in figure 6.

We assume that a mobile is within a circle of a receiver with probability $\mu$. We define a random variable $\mathcal{R}$ which will be used to represent the distance between a mobile and a receiver. Let $\varsigma(r)$ be the probability density function (pdf) for $\mathcal{R}$. Then we have

$$\mu = \int_0^R \varsigma(r)dr.$$ 

**Remark 1.** If we assume that the receivers or access points are randomly distributed following a poisson process with density $\nu$, the probability density function is expressed by $\varsigma(r) = \nu e^{-\nu r}$.

For uplink transmissions, a mobile has to choose between High (H) power level named $P_H$ and Low (L) power level named $P_L$. Let $s$ be the population share strategy $H$. Hence, the signal $P_r$ received at the receiver from a mobile is given by $P_r = gP_l(r)$, where $g$ is the gain antenna, $P_l \in \{P_L, P_H\}$ the power level used by the mobile and $r$ the distance from the mobile to the base station. For the attenuation, the most common function is $l(t) = \frac{1}{t^\alpha}$, with $\alpha$ ranging from 3 to 6. Note that such $l(t)$ explodes at $t = 0$, and thus in particular is not correct for a small distance $r$ and large intensity $\lambda$. Then, it makes sense to assume attenuation to be a bounded function in the vicinity of the antenna. Hence the last function becomes $l(t) = \max(t, r_0)^{-\alpha}$.
First we note that the number of transmission within a circle of radius $r_0$ centered at the receiver is $\lambda \pi r_0^2$. Then the interference caused by all mobiles in that circle is $I_0(s) = \frac{\lambda \pi g(sP_H + (1-s)P_L)}{r_0^2}$.

Now we consider a thin ring $A_j$ with the inner radius $r_j = jdr$ and the outer radius $r_j = r_0 + jdr$. The signal power received at the receiver from any node in $A_j$ is $P_{r_i} = \frac{gP_i}{r_i^\alpha}$. Hence the interference caused by all mobiles in $A_j$ is given by

$$I_j(s) = \begin{cases} 2g\lambda\pi r_j dr \left( \frac{gP_H + (1-s)P_L}{r_j^\alpha} \right) & \text{if } r_j < R, \\ 2\mu g\lambda\pi r_j dr \left( \frac{gP_H + (1-s)P_L}{r_j^\alpha} \right) & \text{if } r_j \geq R. \end{cases}$$

Hence, the total interference contributed by all nodes at the receiver is

$$I(s) = I_0(s) + 2g\lambda\pi(sP_H + (1-s)P_L) \times \left[ \int_{r_0}^{R} \frac{1}{r^{\alpha-1}} dr + \mu \int_{R}^{\infty} \frac{1}{r^{\alpha-1}} dr \right] = g\lambda\pi(sP_H + (1-s)P_L) \times \left( \frac{\alpha}{\alpha - 2} r_0^{-(\alpha-2)} - 2(1-\mu)R^{-(\alpha-2)} \right).$$

Hence the signal to interference ratio $SINR_i$ is given by

$$SINR_i(P_i, s, r) = \begin{cases} \frac{gP_i/\sigma}{\sigma + \beta I(s)} & \text{if } r \leq r_0, \\ \frac{gP_i/\sigma}{\sigma + \beta I(s)} & \text{if } r \geq r_0, \end{cases}$$

where $\sigma$ is the power of the thermal background noise and $\beta$ is the inverse of the processing gain of the system. This parameter weights the effect of interference, depending on the orthogonality between codes used during simultaneous transmissions. In the sequel, we compute the mobile’s utility (fitness) depending on his decision but also on the decision of his interferers. We assume the user’s utility (fitness) choosing power level $P_i$ is expressed by

$$f(P_i, s) = w \int_{0}^{R} \log(1 + SINR(P_i, s, r)) \zeta(r) dr - \eta P_i.$$

The pricing function $P_i$ define the instantaneous ”price” a mobile pays for using a specific amount of power that causes interference in the system, $w$ and
\( \eta \) are cost parameters. The parameter \( \eta \) can be the power cost consumption for sending packets.

We are now looking at the existence and uniqueness of the ESS. For this, we need the following result.

**Lemma 2.** For all density function \( \varsigma \) defined on \([0, R]\), the function \( h : [0, 1] \to \mathbb{R} \) defined as \( s \mapsto \int_0^R \log \left( \frac{1 + \text{SINR}(P_H, s, r)}{1 + \text{SINR}(P_L, s, r)} \right) \varsigma(r) \, dr \) is continuous and strictly monotone.

**Proof** The function

\[
s \mapsto \log \left( \frac{1 + \text{SINR}(P_H, s, r)}{1 + \text{SINR}(P_L, s, r)} \right) \varsigma(r)
\]

is continuous and integrable in \( r \) on the interval \([0, R]\). The function \( h \) is continuous. Using derivative properties of integral with parameter, we can see that the derivative function of \( h \) is the function \( h' : [0, 1] \to \mathbb{R} \) defined as

\[
s \mapsto \int_0^R \frac{\partial}{\partial s} \left[ \log \left( \frac{1 + \text{SINR}(P_H, s, r)}{1 + \text{SINR}(P_L, s, r)} \right) \right] \varsigma(r).
\]

We show that the term

\[
\frac{\partial}{\partial s} \left[ \log \left( \frac{1 + \text{SINR}(P_H, s, r)}{1 + \text{SINR}(P_L, s, r)} \right) \right]
\]

is negative. Let \( W(s) := \frac{1 + \text{SINR}(P_H, s, r)}{1 + \text{SINR}(P_L, s, r)} \). The function \( W \) can be rewritten as \( W(s) = 1 + \frac{g(P_H - P_L)}{\sigma + I(s) + gP_L} \) where \( I(s) = (s(P_H - P_L) + P_L)c(r) \) and \( c(r) = \lambda \pi g \left[ \frac{\alpha}{\alpha - 2} r_0^{-(\alpha - 2)} - 2(1 - \mu)R^{-(\alpha - 2)} \right] \) if \( r \geq r_0 \) and \( \frac{\lambda \pi g}{r_0^{\alpha - 1}} \) otherwise. Since \( W \) satisfies \( W(s) > 1 \) and \( W'(s) = -c(r)\beta(P_H - P_L) \frac{g(P_H - P_L)}{\sigma + I(s) + gP_L} < 0 \). Hence,

\[
\frac{\partial}{\partial s} \left[ \log \left( \frac{1 + \text{SINR}(P_H, s, r)}{1 + \text{SINR}(P_L, s, r)} \right) \right] = \frac{\partial}{\partial s} (\log W(s)) = \frac{W'(s)}{W(s)} < 0
\]

i.e \( h'(s) < 0 \). We conclude that \( h \) is strictly decreasing. \( \blacksquare \)
Using this lemma, we have the following proposition which gives pure strategies depending on the parameters.

**Proposition 4.** For all density function \( \zeta \), the pure strategy \( P_H \) dominates the strategy \( P_L \) if and only if
\[
\frac{\eta}{w}(P_H - P_L) < \int_0^R \log \left( \frac{1 + \text{SINR}(P_H, P_H, r)}{1 + \text{SINR}(P_L, P_H, r)} \right) \zeta(r) \, dr = h(1) .
\]
For all density function \( \zeta \), the pure strategy \( P_L \) dominates the strategy \( P_H \) if and only if
\[
\frac{\eta}{w}(P_H - P_L) > \int_0^R \log \left( \frac{1 + \text{SINR}(P_H, P_H, r)}{1 + \text{SINR}(P_L, P_H, r)} \right) \zeta(r) \, dr = h(0) .
\]

**Proof** We decompose the existence of the ESS in several cases. (a) \( P_H \) is preferred to \( P_L \): The higher power level dominates the lower if and only if \( f(P_L, P_H) > f(P_H, P_L) \) and \( f(P_H, P_L) > f(P_L, P_L) \). These two inequalities implies that
\[
\frac{\eta}{w}(P_H - P_L) < \int_0^R \log \left( \frac{1 + \text{SINR}(P_H, P_H, r)}{1 + \text{SINR}(P_L, P_H, r)} \right) \zeta(r) \, dr .
\]
(b) \( P_L \) is preferred to \( P_H \): Analogously, the lower power dominates the higher power if and only if \( f(P_L, P_H) > f(P_H, P_L) \) and \( f(P_H, P_L) > f(P_H, P_L) \) i.e
\[
\frac{\eta}{w}(P_H - P_L) > \int_0^R \log \left( \frac{1 + \text{SINR}(P_H, P_H, r)}{1 + \text{SINR}(P_L, P_H, r)} \right) \zeta(r) \, dr .
\]

The following result gives sufficient conditions for existence and uniqueness of ESS in the W-CDMA uplink power control.

**Proposition 5.** For all density function \( \zeta \), if \( h(1) < \frac{\eta}{w}(P_H - P_L) < h(0) \), then there exists an unique ESS \( s^* \) which is given by
\[
s^* = h^{-1} \left( \frac{\eta}{w}(P_H - P_L) \right) .
\]

**Proof** Suppose that the parameters \( w, \eta, P_H \) and \( P_L \) satisfy the following inequality \( h(1) < \frac{\eta}{w}(P_H - P_L) < h(0) \). Then the game has no dominant strategy. A mixed equilibrium is characterized by \( f(P_H, s) = f(P_L, s) \). It is easy to see that this last equation is equivalent to \( h(s) = \frac{\eta}{w}(P_H - P_L) \). From the lemma 2, we have that the equation \( h(s) = \frac{\eta}{w}(P_H - P_L) \) has an unique solution given by \( s^* = h^{-1} \left( \frac{\eta}{w}(P_H - P_L) \right) \). We now prove that this mixed equilibrium is an ESS. To prove this result, we compare \( s^* f(P_H, \text{mut}) + (1 - s^*) f(P_L, \text{mut}) \) and \( \text{mut} f(P_H, \text{mut}) + (1 - \text{mut}) f(P_L, \text{mut}) \) for all \( \text{mut} \neq s^* \). The difference between two values is exactly \( w(s^* - \text{mut})(h(\text{mut}) - h(s^*)) \). According to lemma 2, \( h \) is a decreasing function. Hence, \( (s^* - \text{mut})(h(\text{mut}) - h(s^*)) \) is strictly positive for all strategy \( \text{mut} \) different from \( s^* \). We conclude that the mixed equilibrium \( (s^*, 1 - s^*) \) is an ESS.
From the last proposition, we can use the pricing parameter $\eta$ as a design tool for creating an incentive for the user to adjust their power control. We observe that the ESS $s^*$ decreases when $\eta$ increases. That means the mobiles become less aggressive when pricing function increases and the system can limit aggressive requests for SINR.

5.1 Evolutionary Game dynamics in W-CDMA

In this subsection, we use the replicator dynamics for observing the evolution of the strategies $P_H$ and $P_L$ in the population of nodes. We study the effect of the time delays on the convergence of replicator dynamics to the evolutionarily stable strategies in which each pure strategy is associated with its own delay. Let $\tau_H$ (resp. $\tau_L$) be the time delay of the strategy $P_H$ (resp. $P_L$). The delayed replicator dynamic equation given in (2) becomes

$$
\dot{s}(t) = \mu s(t)(1 - s(t)) \left[ f(P_H, s(t - \tau_H)) - f(P_L, s(t - \tau_L)) \right]
$$

Proposition 6. The ESS $s^* = h^{-1}\left( \frac{\eta}{w}(P_H - P_L) \right)$ is asymptotically stable under the replicator dynamics without time delays for all non-trivial initial state.

Proof. The replicator dynamics without time delays is given by

$$
\frac{d}{dt} s(t) = \mu w s(t)(1 - s(t)) \left( h(s(t)) - \frac{\eta(P_H - P_L)}{w} \right).
$$

The function $h$ is decreasing on $(0, 1)$ implies that the derivative of the function $s(1 - s)(h(s) - \frac{\eta(P_H - P_L)}{w})$ at the ESS $s^* = h^{-1}\left( \frac{\eta}{w}(P_H - P_L) \right)$ is negative. Hence, the system is asymptotically stable at $s^* = h^{-1}\left( \frac{\eta}{w}(P_H - P_L) \right)$. 

In order to study the asymptotically stability of W-CDMA network under the delayed replicator dynamics (5) around the unique ESS $s^* = h^{-1}\left( \frac{\eta}{w}(P_H - P_L) \right)$ we linearize (8) at $s^*$. Thus, we obtain the following delayed differential equation:

$$
\begin{align*}
\dot{y}(t) &= \mu s^*(1 - s^*) \left( y(t - \tau_L) \frac{\partial}{\partial s} f(P_i, s)_{|s=s^*} - y(t - \tau_H) \frac{\partial}{\partial s} f(P_L, s)_{|s=s^*} \right)
\end{align*}
$$

(9)
where $\tau_H$ (resp. $\tau_L$) is the time delay of $P_H$ (resp. $P_L$). The following theorem gives sufficient conditions of stability of (9) at zero.

**Theorem 5.** Let $P_D = P_H - P_L$. Suppose at least one of the following conditions holds

- $\mu \tau_H < \Phi_1 + \Phi_2$
- $\Phi_1 \tau_H + |\Phi_2| \tau_L < \Phi_1 + \Phi_2$

Then the ESS $s^*$ is asymptotically stable. Moreover if $\tau_H = \tau_L = \tau$, the ESS is asymptotically stable if and only if

$$\tau < \frac{\pi}{2\mu h^{-1}(\frac{2}{w}P_D)(1-h^{-1}(\frac{2}{w}P_D))(\Phi_1 + |\Phi_2|)}.$$  

**Proof.** In order to derive the sufficient condition of stability, we need to compute value of $\frac{\partial}{\partial s} f(P_i, s)$ at $s = s^*$. Applying rule of Lebesgue integration of function with several parameter, one has,

$$\frac{\partial}{\partial s} f(P_i, s)_{|s=s^*} = w \int_0^R \frac{\partial}{\partial s} \text{SINR}(P_i, s, r)_{|s=s^*} \varsigma(r) dr$$

Define $T$ as

$$T(r) = \begin{cases} 1 \frac{1}{r^\alpha} & \text{if } r \leq r_0, \\ \frac{1}{r^\alpha} & \text{if } r \geq r_0. \end{cases}$$

Since

$$\text{SINR}(P_i, s, r) = \begin{cases} \frac{gP_i/\alpha}{\sigma + \beta I(s)} & \text{if } r \leq r_0, \\ \frac{gP_i/r_0^\alpha}{\sigma + \beta I(s)} & \text{if } r \geq r_0, \end{cases}$$

with

$$I(s) = (s(P_H - P_L) + P_L)c(r),$$

$$c(r) = \begin{cases} \lambda \pi g \left[ \frac{\alpha}{\alpha - 2} \frac{\alpha - 2}{2} (\alpha - 2) - 2(1 - \mu)R^{-(\alpha - 2)} \right] & \text{if } r \geq r_0, \\ \frac{\lambda \pi g}{r_0^{\alpha - 2}} & \text{otherwise}. \end{cases}$$

Then,

$$\frac{\partial}{\partial s} \text{SINR}(P_i, s, r)_{|s=s^*} = -\frac{qP_i T(r)c(r) \beta (P_H - P_L)}{(\sigma + \beta I(s^*))^2} < 0.$$  

32
Let
\[
\Phi_1 := -\frac{\partial}{\partial s} f(P_H, s)_{s=s^*} = -w \int_0^R \frac{[\frac{\partial}{\partial s} \text{SINR}(P_H, s, r)]_{s=s^*}}{1 + \text{SINR}(P_H, s^*, r)} \varsigma(r) dr
\]
and
\[
\Phi_2 := W \int_0^R \frac{T(r)c(r)\varsigma(r)}{(\sigma + \beta I(s^*))^2(1 + \text{SINR}(P_L, s^*, r))} dr,
\]
where \( W = -w g P_H \beta (P_H - P_L) \). One has, \( \Phi_1 + \Phi_2 > 0 \). A sufficiency condition of stability of ESS using the stability of the trivial solution of the DDE (9) is then given by \( \mu \tau_H < \Phi_1 + \Phi_2 \frac{\Phi_1 + \Phi_2}{|\Phi_1 + \Phi_2|} \). Note that this stability condition is independent of \( \tau_L \). The others results are derived as in theorem 3.

\section{5.2 More than two power levels}

For continuously differentiable evolutionary game dynamics, a local stability and asymptotic stability areas of equilibria and ESS (when it's exists) can be established by linearizing the delayed differential equations (DDE) at the rest point. When an interior rest point is under the non-delayed replicator, a necessary and sufficient condition of stability under the delayed replicator dynamics is that all roots of the characteristic equation is given
\[
\det \left( \lambda I - K \sum_{b \in A} B^b e^{-\tau_b \lambda} \right) = 0.
\]
(10)
negative real parts. \( I \) is the identity matrix with the same size as the matrix \( B^b \). The matrix \( B^b \) is obtained the Jacobian of system of the rest point. This transcendental equation in \( \lambda \) is in general difficult to solve. If \( x \) is stable under the non-delayed replicator dynamics then a sufficiency condition of stability under the delayed replicator dynamics is obtained for small norm of matrix \( K \sum_{b \in A} B^b \), for example, for \( K \sum_{b \in A} \tau_b \| B^b \|_\infty < 1 \) where \( \| B^b \|_\infty = \max_{i,j} |B^b_{ij}| \).
6 Numerical investigation

In this section, we first investigate the impact of the different parameters and costs on the ESS and on the convergence of the replicator dynamics. We also discuss the impact of pricing function of the capacity of W-CDMA wireless networks.

6.1 Slotted Aloha based wireless networks

We first show the impact of density of nodes on the probability of success at ESS equilibrium with different values of $\alpha$, which is the ratio between the collision cost and the global reward (benefit minus global cost) of a user. In all examples we consider a probability $\mu = 0.8$ for each node of having a receiver in its range.

6.1.1 Optimization of total throughput

In figure 7 we observe the total throughput $\beta(\alpha, n)$ depending on the number of nodes $n$ and with different values of $\alpha$. For $\alpha = 1/3$, we observe that the total throughput is increasing in that case with the number of nodes which it seems non intuitive. The reason is that the number of transmitted mobiles at the ESS, i.e. $s^*$, is exponentially decreasing with $n$. Another important result is that it may have a finite number of interferers that maximize the total throughput like in figure 7 with $\alpha = 0.2$. When the ratio $\alpha$ is small, the total throughput is decreasing in $n$ as shown in figure 7 for $\alpha = 0.05$. Then, depending on the cost parameters, we have different behavior of the total throughput in function of the density of nodes.

In figure 8, we represent the probability of success $\beta(\alpha, n)$ as function of $\alpha$ for several values of $n$. We observe that the probability of success at optimal value $\alpha^* = (1 - \frac{1}{n})^{n-1}$, is increasing with $n$ and tends to the value $1/e$. We observe the same behavior when the nodes are randomly distributed over a plan following a Poisson distribution.

6.1.2 Dynamics

Now, we study the effect of the time delays on the convergence of replicator dynamics to the evolutionary stable strategies in which each pure strategy is associated with its own delay. In Figure 9, we plot the evolution of the
fraction of transmitters for different values of delays when the random variable $K$ is a Dirac $\delta_{\{n-1\}}$. We took $n = 4$, $\Delta = \delta = \frac{1}{4}$ and $V = 1$. The initial condition is 0.02 and the delays $\tau_T$ and $\tau_S$ between 0.02 and 7. For the small delays: $(\tau_T, \tau_S) = (0.02, 0.02)$ and $(\tau_T, \tau_S) = (3, 2)$, the system is stable. For the delay $\tau_H = 7$ and $\tau_S = 5$, the system is unstable and the proportion of transmitters in the cell oscillates around the ESS.

In the figure 10 we describe numerical application of our evolutionary game model with Poisson distribution. We took $k_{\text{max}} = 4$, $\Delta = \delta = \kappa = \frac{1}{4}$, $\lambda = 1$ and $V = 1$. The initial condition in all these figures is 0.02. In the figure 10 we compare the evolution of the fraction of transmitters varying the parameter of density $\lambda$ between 0.1 and 5 for the case 1, 2 and 3 respectively. We observe that we have stability for all cases.

Now, we study the effect of the time delays on the convergence of replicator dynamics to the evolutionarily stable strategies in which each pure strategy is associated with its own delay. The fraction of transmitters in the population is represented in figure 11 for $\lambda = 0.5$ and $r = 1$. The delays $\tau_T$ and $\tau_S$ are between 0.02 and 7. The system is stable for $\tau_T = \tau_S = 0.02$ or $\tau_T = 3$, $\tau_S = 2$. For $\tau_T = 7$ and $\tau_S = 5$ the system is unstable. We display an oscillatory behavior of the population as function of time. The trajectory are seen to converge to periodic ones. All turn out to confirm stability conditions that we obtained in theorem 3. In the figure 12 we compare evolution of the fraction of transmitters varying the parameter of density $\lambda$ between 0.1 and 5 for the case 1, 2 and 3 respectively. We observe that we have stability for all cases.
0.1 and 5 for the case 1, 2 and 3 respectively. In this figure, the time delays are respectively 3 and 2. Note that in this figure the equilibrium point is decreasing function in the density parameter \( \lambda \). Indeed, when the density of nodes increases, the number of mobiles share a receiver increases. To avoid collision, the nodes decrease the probability of transmission. We observe also that for \( \lambda = 5 \), we have stability but the convergence speed is slow than for \( \lambda = 0.1 \).

6.2 W-CDMA Wireless Networks

In the numerical examples below, we show how the pricing function can optimize the overall network throughput. We first investigate the impact of the different parameters and pricing on the ESS and the convergence of the replicator dynamic. We also discuss the impact of the pricing function on the system capacity.

6.2.1 Optimization of average throughput

We first show the impact of density of nodes and pricing on the ESS and the average throughput. We assume that the base stations which are randomly placed over a plane following a Poisson process with density \( \nu \), i.e, \( \zeta(r) = \nu e^{-\nu r} \). We recall that the rate of a mobile using power level \( P_i \) at the
equilibrium is given by

\[ w \int_{0}^{R} \log(1 + SINR(P_i, s, r)) \varsigma(r) dr. \]

We took the following parameters \( r_0 = 0.2, w = 20, \sigma = 0.2, \alpha = 3, \beta = 0.2 \) and \( R = 1 \). First, we show the impact of the density of nodes \( \lambda \) on the ESS and the average throughput. In figures 13-14, we depict the average throughput obtained at the equilibrium and the ESS, respectively, as a function of the density \( \lambda \). We recall that the interference for a mobile increases when \( \lambda \) increases. We observe that the mobiles become less aggressive when the density increases. In the figure 13, we observe that it is important to adapt the pricing as function of the density of nodes. Indeed, we observe that for low density of nodes, the lower pricing (\( \eta = 0.92 \)) gives better results than higher pricing (\( \eta = 0.97 \)). When the density of nodes increases, the better performance is obtained with higher pricing.

### 6.2.2 Dynamics

Now, we study the impact of the receiver distributions on the ESS. Figures 15 and 16 represent the fraction of population using the high power level for different initial states of the population: 0.99, 0.66, 0.25 and 0.03. We observe that the choice of the receiver distributions changes the ESS. For the
impact of the time delay on the convergence of replicator dynamics to ESS, we obtain the same behavior as in Slotted Aloha based wireless Networks.

7 Concluding remarks

In this paper we have adapted the theory of evolutionary games with a random number of players or nodes in wireless networks. This adaptation is needed in order to apply this theory for the study of access game and particularly in wireless networks. First, we have proposed different scenario based on the level of information for each player in the slotted Aloha model. In all cases, we have obtained the existence and uniqueness of the ESS and we have proposed optimization issues for the transmission probability of success. Finally, we have studied the impact of delays in the convergence to the ESS of the replicator dynamics. Second, we have applied also this evolutionary game framework with a random number of nodes in a context of W-CDMA network in which we have studied the level of aggressiveness of the population at equilibrium.
Figure 11: Impact of the time delay on the stability of the replicator dynamics (case 1).

References


Figure 12: Evolution of the fraction of transmitters varying the density parameter $\lambda$.


Figure 13: The average rate of a mobile at equilibrium as function of the density of nodes $\lambda$ for $\eta = 0.92, 0.97$.


Figure 14: ESS versus density of nodes $\lambda$ for $\eta = 0.92, 0.97$.


Figure 15: Convergence to the ESS in W-CDMA system: uniform distribution


Figure 16: Convergence to the ESS in W-CDMA system: quadratic distribution


