Modified group divisible designs with block size 4 and $\lambda = 1$

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Abstract

In this paper it is shown that the MGD[4, 1, m, mn] exist when \((m - 1)(n - 1) \equiv 1 \pmod{3}\) and \(n, m \geq 4\) with some possible exceptions. © 1999 Elsevier Science B.V. All rights reserved

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1. Introduction

We assume that the reader is familiar with the basic concepts in design theory such as transversal design (TD), Latin square, resolvable design, etc. For general information and notations see [4]. Here we recall the definition of GDD.

A GDD is a triple \((X, G, B)\) which satisfies the following properties:

1. \(G\) is a partition of a set \(X\) (of points) into subsets called groups,
2. \(B\) is a family of subsets of \(X\) (called blocks) such that a group and a block contain at most one common point,
3. every pair of points from distinct groups occurs in exactly \(\lambda\) blocks.

The type of the GDD is the multiset \(\{ |G| : G \in G \}\). We shall use an 'exponential' notation to describe types: so type \(t_1^{u_1}t_2^{u_2}\cdots t_k^{u_k}\) denotes \(u_i\) occurrences of \(t_i\), \(1 \leq i \leq k\), in the multiset. We also use the notation \(GD(K, M; \lambda)\) to denote the GDD when its block sizes set is \(K\) and group sizes set is \(M\) or \(K\)-GDD when its group sizes is not specified.

If \(M = \{1\}\), then the GDD becomes a PBD. If \(K = \{k\}\), \(M = \{n\}\) and with the type \(n^k\), then the GDD becomes a TD[\(k, n]\). It is well known that the existence of a TD[\(k, n]\) is equivalent to the existence of \(k - 2\text{MOLS}(n)\).

Now, we give the definition of modified group divisible design which is first introduced in [1].

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Definition 1.1. Let $X$ be a set of $mn$ points where the points of $X$ are denoted as $(x_i, y_j)$, $0 \leq i \leq m - 1$, $0 \leq j \leq n - 1$. Let $\mathcal{B}$ be a collection of subsets of $X$ (called blocks), which satisfies the following conditions:
1. $|B| = k$ for every block $B \in \mathcal{B}$;
2. every pair of points $(x_{i_1}, y_{j_1})$ and $(x_{i_2}, y_{j_2})$ of $X$ are contained in exactly $\lambda$ blocks, where $i_1 \neq i_2$ and $j_1 \neq j_2$.
3. the pair of points $(x_{i_1}, y_{j_1})$ and $(x_{i_2}, y_{j_2})$ with $i_1 = i_2$ or $j_1 = j_2$ are not contained in any blocks.

Then we call $(X, \mathcal{B})$ a modified group divisible design and denote it by $\text{MGD}[k, \lambda, m, mn]$. The subsets $\{(x_i, y_j) | 0 \leq i \leq m - 1\}$, where $0 \leq j \leq n - 1$ are called groups and the subsets $\{(x_i, y_j) | 0 \leq j \leq n - 1\}$, where $0 \leq i \leq m - 1$ are called rows.

The modified group divisible designs are motivated by the existence problem of resolvable group divisible designs and other constructions of designs. An application can be found in [2]. It has also been generalized in [8].

By simple calculation, we can easily obtain the following necessary conditions for the existence of an MGD.

Lemma 1.2. The necessary conditions for the existence of an MGD$[k, \lambda, m, nm]$ are that $m \geq k$, $n \geq k$, $\lambda(nm + 1 - m - n) \equiv 0 \pmod{(k - 1)}$ and $\lambda nm(nm + 1 - m - n) \equiv 0 \pmod{(k(k - 1))}$.

In [1] it is proved that the necessary conditions are sufficient when $k = 3$. However, when $k = 4$, these conditions are not sufficient. A counterexample is that an MGD$[4, 1, 6, 24]$ does not exist because there do not exist two mutually orthogonal Latin squares of order 6.

The following simple but useful lemma comes from the definition of MGD.

Lemma 1.3. An MGD$[k, \lambda, m, nm]$ exists if and only if an MGD$[k, \lambda, n, mn]$ exists.

In this paper we consider the existence of MGD with block size 4. Let $E = \{(10, 8), (10, 15), (10, 18), (10, 23), (19, 11), (19, 12), (19, 14), (19, 15), (19, 18), (19, 23)\}$, we shall prove the following theorem.

Theorem 1.4. If $m, n \neq 6$, then an MGD$[4, 1, m, nm]$ exists if and only if $(n - 1)(m - 1) \equiv 0 \pmod{3}$ and $n, m \geq 4$ with possible exception of $(n, m) \in E$.

2. Constructions

In this section, we first give some direct constructions of MGD. We need some knowledge of cyclotomic class and cyclotomic number. For more information, the readers are referred to [7].
Definition 2.1. Let $G$ be an additive group of a finite field $GF(q)$, where $q = ef + 1$. Suppose that $\zeta$ is a primitive root of $GF(q)$. Then the subsets $H^i_e$ of $G$ are called cyclotomic classes of $G$, where

$$H^i_e = \{ \zeta^{es+i} \mid s = 0, 1, \ldots, f - 1 \}, \quad i = 0, 1, \ldots, e - 1.$$

The number of the solutions of

$$1 + g_i = g_j, \quad g_i \in H^i_e, \quad g_j \in H^j_e$$

is called cyclotomic number and denoted by $(i, j)$.

Lemma 2.2 (Storer [7]). If $e = 2$, then the cyclotomic numbers are

1. $(0, 0) = (f - 2)/2, (0, 1) = (1, 0) = (1, 1) = f/2$, when $f$ is even; or
2. $(0, 0) = (1, 0) = (1, 1) = (f - 1)/2, (0, 1) = (f + 1)/2$, when $f$ is odd.

Lemma 2.3. Let $n \equiv 1 (\mod 4)$ be a prime power and $\zeta$ a primitive root of $GF(n)$. Then there exists an integer $k, k > 1$, such that $\zeta^{2(k-1)} - 1 \in H^1_2$, and $\zeta - 1$ and $\zeta^{2k} - 1$ are not in the same cyclotomic class.

Proof. Let $n = 4m + 1$. If $\zeta - 1 \in H^0_2$, since $(1, 1) = m$ by Lemma 2.2 we may assume that $\zeta^{2k_i} - 1, \zeta^{2k_i+1} - 1, \ldots, \zeta^{2k_{m}} - 1 \in H^i_2$, where $k_i > 1$. Then there must exist $k_i$ such that $\zeta^{2k_i} - 1 \in H^i_1$. Otherwise, it will contradict the fact $(0, 0) = m - 1$. If $\zeta - 1 \in H^1_2$, let $\zeta^{2t_i} - 1, \zeta^{2t_i+1} - 1, \ldots, \zeta^{2t_{m}} - 1 \in H^0_2$, where $t_i > 1$. Then we must have some $t_i$ such that $\zeta^{2t_i} - 1 \in H^i_1$, otherwise it will contradict the fact $(0, 0) = m - 1$. The conclusion follows. □

Lemma 2.4. If $n \equiv 1 (\mod 4)$, $n$ is a prime power, then there exists an MGD$[4, 1, 7, 7n]$.

Proof. Let the point set of the MGD be $GF(n) \times Z_7$ and a primitive root of $GF(n)$ be $\zeta$. Develop the following starter blocks modulo $(GF(n), 7)$:

$$\{(0, 0), (\zeta^{2i+1}, 1), (\zeta^{2i+2}, 3), (\zeta^{2i+2k}, 6)\}, \quad i = 0, 1, \ldots, 2m - 1,$$

where $k$ satisfies the conditions of Lemma 2.3 and $m = (n - 1)/4$. It is readily checked that these blocks form an MGD$[4, 1, 7, 7n]$ with groups $\{j\} \times Z_7, \quad j \in GF(n)$. □

Lemma 2.5. Let $n \equiv 3 (\mod 4)$, $n$ is a prime power. Let $\zeta$ be a primitive root of $GF(n)$. If there exists a positive integer $k$ such that $\zeta^{2k} - 1$ or $\zeta^{2k+1} - 1 \in H^i_2$, and $1 - \zeta$ and $1 - \zeta^{2k}$ are in the same cyclotomic class, then there exists an MGD$[4, 1, 7, 7n]$.

Proof. Let the point set of the MGD be $GF(n) \times Z_7$. Case 1, when $\zeta^{2k} - 1 \in H^i_2$ develop the following blocks modulo $(GF(n), 7)$:

$$\{(0, 0), (\zeta^{2i+1}, 1), (\zeta^{2i+2}, 3), (\zeta^{2i+2k+1}, 6)\} \quad i = 0, 1, \ldots, 2m.$$
Case 2, when $\zeta^{2k+1} - 1 \in H_2^1$ develop the following blocks modulo $(\text{GF}(n), 7)$:

$$\{(0,0), (\zeta^{2i+1}, 1), (\zeta^{2i}, 3), (\zeta^{2i+2k+1}, 6)\} \quad i = 0, 1, \ldots, 2m,$$

where $m = (n - 3)/4$. □

**Example 2.6.** There exist $\text{MGD}[4, 1, 7, 7n]$ for $n \in \{7, 11, 19, 23\}$.

**Proof.** In view of Lemma 2.5, we list $n, \zeta, k$ and case number as follows, where case 1 means $\zeta^{2k-1} - 1 \in H_2^1$, while case 2 means $\zeta^{2k+1} - 1 \in H_2^1$:

<table>
<thead>
<tr>
<th>$n$</th>
<th>7</th>
<th>11</th>
<th>19</th>
<th>23</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\zeta$</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>$k$</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>5</td>
</tr>
<tr>
<td>case</td>
<td>2</td>
<td>2</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Another method of constructing $\text{MGD}$ is by using base blocks. Then all blocks of the $\text{MGD}$ can be obtained by developing the base blocks. In the following examples, we use base blocks.

**Example 2.7.** There exists an $\text{MGD}[4, 1, 7, 7n]$, for $n = 6$ or $10$.

**Proof.** For $n = 6$, let the point set $X$ be $\mathbb{Z}_{42}$. Groups are \{i, i + 3, i + 6, i + 9, i + 12, i + 15, i + 18\} $\cup$ \{i + 21, i + 24, i + 27, i + 30, i + 33, i + 36, i + 39\}, $i = 0, 1, 2$. Rows are \{i, i + 7, i + 14, i + 21, i + 28, i + 35\}, $i = 0, \ldots, 6$. Furthermore, let $\alpha$ be the permutation $\alpha = (0, 1, 2, \ldots, 20)(21, 22, \ldots, 41)$. Then the required blocks are the distinct images of the following blocks under powers of $\alpha$. \{0, 1, 5, 23\}, \{0, 27, 38, 40\}, \{0, 2, 31, 36\}, \{0, 8, 32, 33\}, \{0, 10, 26, 30\}.

For $n = 10$, let $X = \mathbb{Z}_{70}$. Groups are \{i, i + 5, i + 10, \ldots, i + 30\} $\cup$ \{i + 35, \ldots, i + 65\}, $i = 0, 1, 2, 3, 4$. Rows are \{i, i + 7, i + 14, \ldots, i + 63\}, $i = 0, 1, \ldots, 6$. Let $\alpha$ be the permutation $\alpha = (0, 1, 2, \ldots, 34)(35, 36, \ldots, 69)$. The required blocks are the distinct images of the following blocks under powers of $\alpha$. \{0, 6, 17, 65\}, \{0, 43, 47, 60\}, \{0, 4, 12, 41\}, \{0, 38, 40, 67\}, \{0, 13, 16, 39\}, \{0, 54, 57, 66\}, \{0, 1, 45, 69\}, \{0, 2, 52, 53\}, \{0, 9, 36, 55\}. □

It is easy to see that an $\text{MGD}[4, 1, 4, 4n]$ is equivalent to two mutually orthogonal idempotent Latin squares of order $n$. So we have the following lemma from [6].

**Lemma 2.8.** An $\text{MGD}[4, 1, 4, 4n]$ exists when $n \geq 4$ and $n \neq 6$.

**Lemma 2.9.** There exists an $\text{MGD}[4, 1, 7, 7n]$ for $n = 4, 5, 6, 7, 9, 10, 11, 19, 23$.

**Proof.** The conclusion comes from Lemmas 2.4, 2.8 and Examples 2.6, 2.7. □

**Lemma 2.10.** There exists a $\text{MGD}[4, 1, 10, 10n]$ for $n = 4, 5, 6, 7, 9$. 

Proof. For $n = 4$ and $7$, the result comes from Lemmas 1.3, 2.8 and Example 2.7. For $n = 9$, let $X = Z_{90}$. The groups are $\{i, i+10, i+20, \ldots, i+80\}$, $i = 0, \ldots, 8$ and the rows are $\{j, j+9, j+18, \ldots, j+81\}$, $j = 0, \ldots, 8$. Then the blocks are the following under the action of group $Z_{90}$. $\{0, 1, 5, 17\}$, $\{0, 2, 8, 33\}$, $\{0, 3, 26, 41\}$, $\{0, 7, 29, 53\}$, $\{0, 11, 39, 58\}$, $\{0, 13, 34, 48\}$. For $n = 5$ and $6$ see appendix. $\square$

Lemma 2.11. There exists a MGD$[4, 1, 19, 19n]$ for $n = 4, 5, 6, 7, 9$.

Proof. For $n = 4$ and $7$, the result comes from Lemmas 1.3, 2.8 and Example 2.6. For $n = 5$, let $X = Z_{95}$. The groups are $\{i, i+5, i+10, \ldots, i+90\}$, $i = 0, \ldots, 4$ and the rows are $\{j, j+19, j+38, j+57, j+76\}$, $j = 0, \ldots, 18$. Then the blocks are the following blocks developed under the action of $Z_{95}$: $\{0, 1, 3, 7\}$, $\{0, 8, 21, 54\}$, $\{0, 9, 26, 73\}$, $\{0, 11, 29, 72\}$, $\{0, 12, 36, 68\}$, $\{0, 14, 42, 58\}$.

For $n = 6$, let $X = Z_{114}$. The groups are $\{i, i+3, i+6, \ldots, i+54\} \cup \{i+57, i+60, \ldots, i+111\}$, $i = 0, 1, 2$ and the rows are $\{j, j+19, j+38, j+57, j+76, j+95\}$, $j = 0, \ldots, 18$. Let $\alpha = (0, 1, 2, \ldots, 56)(57, 58, \ldots, 113)$. Then the blocks are the distinct images of the following blocks under powers of $\alpha$: $\{0, 8, 28, 59\}$, $\{0, 10, 26, 63\}$, $\{0, 58, 66, 92\}$, $\{0, 67, 72, 89\}$, $\{0, 11, 25, 61\}$, $\{0, 60, 62, 73\}$, $\{0, 1, 70, 113\}$, $\{0, 2, 79, 99\}$, $\{0, 4, 85, 86\}$, $\{0, 5, 101, 111\}$, $\{0, 7, 75, 91\}$, $\{0, 13, 78, 103\}$, $\{0, 17, 100, 104\}$, $\{0, 22, 74, 102\}$, $\{0, 23, 64, 71\}$.

For $n = 9$, let $X = Z_{171}$. The groups are $\{i, i+9, i+18, \ldots, i+162\}$, $i = 0, \ldots, 8$ and the rows are $\{j, j+19, j+38, \ldots, j+152\}$, $j = 0, \ldots, 18$. Then the blocks are the following blocks developed under the action of $Z_{171}$: $\{0, 1, 25, 85\}$, $\{0, 2, 30, 80\}$, $\{0, 7, 53, 75\}$, $\{0, 3, 8, 14\}$, $\{0, 4, 16, 33\}$, $\{0, 10, 31, 104\}$, $\{0, 13, 55, 120\}$, $\{0, 15, 47, 130\}$, $\{0, 20, 69, 112\}$, $\{0, 23, 58, 132\}$, $\{0, 26, 66, 127\}$, $\{0, 34, 71, 123\}$. $\square$

Next, we give some recursive constructions of MGD.

First, we state a recursive construction which comes from [1]. We use the notation $B[K, v]$ to denote the PBD with $\lambda = 1$, block size set $K$ and point set of order $v$. When $K = \{k\}$, we simply denote it by $B[k, v]$.

Lemma 2.12. If there exists a pairwise balanced design $B[K, n]$ and if for every $k \in K$ there exists an MGD$[r, 1, km]$, then there exists an MGD$[r, 1, m, km]$.

To use Lemma 2.12, we need some results about the existence of some kind of PBDs. The following lemma comes from [5, III.3, pp. 203–213].

Lemma 2.13. Let $v$ be an integer, $v \geq 4$ and $v \notin \{8, 10, 11, 12, 14, 15, 18, 19, 23\}$. Then there exists a $B[4, 5, 6, 7, 9, v]$.

Let $M = \{v: v \geq 4$ and $v \neq 8, 10, 11, 12, 14, 15, 18, 19, 23\}$. From Lemmas 2.12 and 2.13, we obtain the following theorem.

Theorem 2.14. There exist MGD$[4, 1, 7, 7n]$, MGD$[4, 1, 10, 10n]$ and MGD$[4, 1, 19, 19n]$ for $n \in M$. 

Proof. From Lemmas 2.9–2.11, there exists MGD[4, 1, m, mn] for m = 7, 10, 19 and n = 4, 5, 6, 7, 9. But for any n ∈ M, there exists B{4, 5, 6, 7, 9, n}. The conclusion follows from Lemma 2.12.

3. Main results

In this section we shall prove the main theorem of this paper. The following lemma comes from [4]. Combining this with the existence of MGD[4, 1, m, mn] for m = 4 and 7, we solve most cases of MGDs.

**Lemma 3.1.** A B{4, 7, m} exists when m ≡ 1 (mod 3) and m ≠ 10, 19.

For m = 4, we already have the complete result stated in Lemma 2.8. So we consider the case of m = 7 and n not in M.

**Lemma 3.2.** There exists an MGD[4, 1, 7, 7n] for n = 8, 10, 11, 12, 14, 15, 18, 19 or 23.

Proof. For n = 10, 11, 19 and 23, the results are from Lemma 2.9. The others are displayed in the appendix.

**Theorem 3.3.** Let n, m ≠ 6, 10 and 19. Then there exists an MGD[4, 1, m, mn] if and only if (n - 1)(m - 1) ≡ 0 (mod 3) and n, m ≥ 4.

Proof. When k = 4 and λ = 1, the necessary conditions of Lemma 1.2 become (m - 1)(n - 1) ≡ 0 (mod 3). By Lemma 1.3 we can always assume that m ≡ 1 (mod 3), so there exists a B{4, 7, m} by Lemma 3.1. Now an MGD[4, 1, 4, 4n] and an MGD[4, 1, 7, 7n] exist by Lemma 2.8, 3.2 and Theorem 2.14, so the conclusion follows from Lemma 3.1.

**Lemma 3.4.** There exists MGD[4, 1, 10, 10n] for n = 10, 11, 12, 14, and 19.

Proof. See the appendix for the details.

**Lemma 3.5.** There exists MGD[4, 1, 19, 19n] for n = 8, 10, and 19.

Proof. For n = 10, see Lemma 3.4. For the others, see the appendix.

Now, we are in a position to prove our main theorem. Recall that E = {(10, 8), (10, 15), (10, 18), (10, 23), (19, 11), (19, 12), (19, 14), (19, 15), (19, 18), (19, 23)}.

**Theorem 3.6.** Let n, m ≠ 6. An MGD[4, 1, m, mn] exists if and only if (n - 1)(m - 1) ≡ 0 (mod 3) and n, m ≥ 4 with possible exception of (n, m) ∈ E.
Proof. The conclusion follows from Theorems 2.14 and 3.3, and Lemmas 3.4 and 3.5. □

Appendix

In this section, we list some direct constructions of MGDs. For an MGD$[4, 1, n, nm]$, we display $(n, m)$, point set $X$, groups $G$, rows $R$, permutation or action group $\alpha$ and starter blocks.

$(n, m) = (7, 8),$
$X: Z_7 \times (Z_7 \cup \{\infty\}),$
$G: Z_7 \times \{i\}, \; i \in (Z_7 \cup \{\infty\}),$
$R: \{j\} \times (Z_7 \cup \{\infty\}), \; j \in Z_7,$
$\alpha: Z_7 \times Z_7,$
\[
\begin{align*}
\{(0, \infty), (1, 0), (4, 1), (2, 3)\}, & \quad \{(0, \infty), (6, 0), (3, 1), (5, 3)\}, \\
\{(0, 0), (2, 1), (1, 2), (4, 4)\}, & \quad \{(0, 0), (5, 1), (6, 2), (4, 4)\}.
\end{align*}
\]

$(n, m) = (7, 12),$
$X: Z_7 \times (Z_{11} \cup \{\infty\}),$
$G: Z_7 \times \{i\}, \; i \in (Z_{11} \cup \{\infty\}),$
$R: \{j\} \times (Z_{11} \cup \{\infty\}), \; j \in Z_7,$
$\alpha: Z_7 \times Z_{11},$
\[
\begin{align*}
\{(0, \infty), (1, 1), (2, 2), (4, 6)\}, & \quad \{(0, \infty), (6, 1), (5, 2), (3, 6)\}, \\
\{(2, 1), (0, 2), (4, 10), (1, 9)\}, & \quad \{(5, 1), (0, 2), (3, 10), (6, 9)\}, \\
\{(1, 2), (2, 4), (4, 9), (0, 7)\}, & \quad \{(6, 2), (5, 4), (3, 9), (0, 7)\}.
\end{align*}
\]

$(n, m) = (7, 14),$
$X: Z_{26} \cup H_7,$
$G: \{i, i + 13, i + 26, \ldots, i + 78\} \cup H_7, \; i = 0, \ldots, 12,$
$R: \{j, j + 7, j + 14, \ldots, j + 84, h_{\beta(j+1)}\}, \; j = 0, \ldots, 6, \text{ where } \beta(i) \text{ is the } i\text{th term in} (1, 7, 6, 5, 4, 3, 2),$  
$\alpha: (0, 1, 2, \ldots, 90)(h_7, h_6, h_5, h_4, h_3, h_2, h_1),$  
$\begin{align*}
\{0, 1, 3, 32\}, & \quad \{0, 4, 16, 22\}, \quad \{0, 5, 25, 55\}, \quad \{0, 8, 19, 46\}, \\
\{0, 9, 24, 57\}, & \quad \{0, 10, 47, h_7\}, \quad \{0, 51, 68, h_4\}.
\end{align*}$

$(n, m) = (7, 15),$
$X: Z_{105},$
$G: \{i, i + 7, i + 14, \ldots, i + 98\}, \; i = 0, \ldots, 6,$
$R: \{j, j + 15, j + 30, \ldots, j + 90\}, \; j = 0, \ldots, 14,$
$\alpha: Z_{105},$
$\begin{align*}
\{0, 1, 3, 9\}, & \quad \{0, 4, 16, 26\}, \quad \{0, 5, 37, 64\}, \quad \{0, 11, 36, 65\}, \\
\{0, 13, 47, 66\}, & \quad \{0, 17, 48, 72\}, \quad \{0, 18, 38, 61\}.
\end{align*}$


\((n, m) = (7, 18),\)
\(X: \mathbb{Z}_{126},\)
\(G: \{i, i+9, i+18, \ldots, i+54\} \cup \{i+63, i+72, \ldots, i+117\}, \; i = 0, \ldots, 8,\)
\(R: \{j, j+7, j+14, \ldots, j+119\}, \; j = 0, \ldots, 6,\)
\(\alpha: (0, 1, 2, \ldots, 62)(63, 64, \ldots, 125),\)

\(\{0, 17, 32, 125\}, \; \{0, 10, 30, 116\}, \; \{63, 66, 82, 92\}, \; \{0, 3, 26, 55\},\)
\(\{63, 65, 76, 96\}, \; \{0, 1, 65, 69\}, \; \{0, 2, 73, 74\}, \; \{0, 4, 82, 87\},\)
\(\{0, 5, 80, 95\}, \; \{0, 6, 85, 109\}, \; \{0, 12, 100, 122\}, \; \{0, 13, 89, 114\},\)
\(\{0, 16, 115, 123\}, \; \{0, 19, 67, 113\}, \; \{0, 22, 118, 124\}, \; \{0, 24, 81, 121\},\)
\(\{0, 25, 66, 117\}.\)

\((n, m) = (10, 5),\)
\(X: \mathbb{Z}_5 \times \mathbb{Z}_{10},\)
\(G: \mathbb{Z}_5 \times \{j\}, \; j \in \mathbb{Z}_{10},\)
\(R: \{i\} \times \mathbb{Z}_{10}, \; i \in \mathbb{Z}_5,\)
\(\alpha: \mathbb{Z}_{10},\)

\(\{(0, 0), (1, 1), (2, 2), (3, 3)\}, \; \{(0, 0), (1, 2), (2, 1), (3, 5)\},\)
\(\{(0, 0), (1, 3), (2, 6), (3, 1)\}, \; \{(0, 0), (1, 4), (2, 9), (4, 1)\},\)
\(\{(0, 0), (1, 6), (2, 8), (4, 7)\}, \; \{(0, 0), (1, 7), (2, 4), (4, 2)\},\)
\(\{(0, 0), (1, 5), (3, 6), (4, 9)\}, \; \{(0, 0), (1, 8), (3, 2), (4, 6)\},\)
\(\{(0, 0), (1, 9), (3, 8), (4, 5)\}, \; \{(0, 0), (2, 3), (3, 9), (4, 4)\},\)
\(\{(0, 0), (2, 5), (3, 7), (4, 8)\}, \; \{(0, 0), (2, 7), (3, 4), (4, 3)\},\)
\(\{(1, 0), (2, 8), (3, 6), (4, 2)\}, \; \{(1, 0), (2, 6), (3, 5), (4, 3)\},\)
\(\{(1, 0), (2, 4), (3, 7), (4, 9)\}.\)

\((n, m) = (10, 6),\)
\(X: \mathbb{Z}_{10} \times \mathbb{Z}_5 \cup H_{10},\)
\(G: \bigcup_{i \in \mathbb{Z}_5} \mathbb{Z}_{10} \times \{j\} \cup H_{10},\)
\(R: \{(i) \times \mathbb{Z}_5 \cup h_i, \; i \in \mathbb{Z}_{10},\)
\(\alpha: \mathbb{Z}_5,\)

\(\{(3, 0), (4, 1), (6, 2), (7, 3)\}, \; \{(4, 0), (5, 1), (7, 3), (8, 2)\}, \; \{(5, 0), (6, 1), (8, 2), (9, 3)\},\)
\(\{(0, 0), (6, 1), (7, 3), (9, 2)\}, \; \{(0, 0), (1, 1), (7, 2), (8, 3)\}, \; \{(1, 0), (2, 1), (8, 4), (9, 2)\},\)
\(\{(0, 0), (2, 1), (3, 2), (9, 4)\}, \; \{(1, 0), (4, 2), (6, 4), (9, 3)\}, \; \{(0, 0), (3, 1), (5, 3), (8, 2)\},\)
\(\{(2, 0), (4, 2), (7, 1), (9, 4)\}, \; \{(1, 0), (3, 3), (6, 2), (8, 1)\}, \; \{(0, 0), (2, 2), (5, 1), (7, 4)\},\)
\(\{(0, 0), (1, 3), (3, 4), (4, 1)\}, \; \{(2, 0), (3, 3), (5, 2), (6, 1)\}, \; \{(1, 0), (2, 3), (4, 4), (5, 1)\}.\)
\[(n, m) = (10, 10), \]
\[X: \mathbb{Z}_{10} \cup H_{10}, \]
\[G: \bigcup_{i=0}^{8} \{i, i + 9, i + 18, \ldots, i + 81\} \cup H_{10}, \]
\[R: \{j, j + 5, j + 10, \ldots, j + 40, h_{\beta(j+1)}\} \cup \{j + 45, j + 50, j + 55, \ldots, j + 85, h_{\beta(j+6)}\}, \]
\[j = 0, \ldots, 4, \text{ where } \beta(i) \text{ is the } i\text{th item in } (1, 5, 4, 3, 2, 6, 10, 9, 8, 7), \]
\[x: (0, 1, 2, \ldots, 44)(45, 46, \ldots, 89)(h_1, h_5, h_4, h_3, h_2)(h_{10}, h_9, h_8, h_7, h_6), \]
\[
\begin{array}{c}
0, 13, 21, h_{10}, \quad 4, 60, 74, h_{10}, \quad 2, 53, 82, h_{10}, \quad 45, 67, 84, h_1, \\
7, 24, 76, h_1, \quad 3, 26, 68, h_1, \quad 0, 1, 7, 47, \quad 0, 4, 16, 48, \\
0, 59, 71, 78, \quad 0, 62, 73, 75, \quad 0, 2, 68, 76, \quad 0, 3, 58, 82, \\
0, 11, 60, 61, \quad 0, 14, 53, 57, \quad 0, 19, 83, 86.
\end{array}
\]

\[(n, m) = (10, 11), \]
\[X: \mathbb{Z}_{11}, \]
\[G: \{i, i + 11, i + 22, \ldots, i + 99\}, \quad i = 0, \ldots, 10, \]
\[R: \{j, j + 5, j + 10, \ldots, j + 50\} \cup \{j + 55, j + 60, \ldots, j + 105\}, \quad j = 0, \ldots, 4, \]
\[x: (0, 1, 2, \ldots, 54)(55, 56, \ldots, 109), \]
\[
\begin{array}{c}
0, 3, 19, 31, \quad 55, 59, 73, 76, \quad 0, 58, 59, 65, \quad 0, 7, 21, 56, \\
0, 1, 62, 109, \quad 0, 2, 74, 87, \quad 0, 4, 71, 102, \quad 0, 6, 75, 101, \\
0, 8, 76, 92, \quad 0, 9, 60, 103, \quad 0, 13, 91, 93, \quad 0, 17, 81, 100, \\
0, 18, 70, 97, \quad 0, 23, 96, 105, \quad 0, 26, 57, 89.
\end{array}
\]

\[(n, m) = (10, 12), \]
\[X: \mathbb{Z}_{11} \cup H_{10}, \]
\[G: \bigcup_{i=0}^{10} \{i, i + 11, i + 22, \ldots, i + 99\} \cup H_{10}, \]
\[R: \{j, j + 10, j + 20, \ldots, j + 100, h_{\beta(j+1)}\}, \quad j = 0, \ldots, 9, \]
\[x: (0, 1, 2, \ldots, 109)(h_1, h_2, h_3, \ldots, h_{10}). \]
\{0, 21, 47, h_{10}\}, \{4, 45, 98, h_{10}\}, \{3, 32, 86, h_{10}\}, \{0, 1, 3, 9\},
\{0, 4, 17, 42\}, \{0, 5, 19, 78\}, \{0, 7, 31, 65\}, \{0, 12, 35, 74\},
\{0, 15, 43, 61\}.

\((n, m) = (10, 14),\)
\(X: \mathbb{Z}_{160} \cup H_{10},\)
\(G: \bigcup_{i=0}^{12} \{i, i + 13, i + 26, \ldots, i + 117\} \cup H_{10},\)
\(R: \{j, j + 5, j + 10, \ldots, j + 60, h_{\beta(j+1)}\} \cup \{j + 65, j + 70, j + 75, \ldots, j + 125, h_{\beta(j+1)}\},\)
j = 0, \ldots, 4, where \(\beta(i)\) is the \(i\)th item in \((1, 3, 5, 2, 4, 6, 8, 10, 7, 9),\)
\(x: (0, 1, 2, \ldots, 64)(65, 66, \ldots, 129)(h_1, h_3, h_5, h_2, h_4)(h_{10}, h_7, h_9, h_6, h_8),\)

\{0, 1, 80, h_{10}\}, \{13, 98, 126, h_{10}\}, \{39, 57, 129, h_{10}\}, \{13, 36, 47, h_1\},
\{65, 98, 112, h_1\}, \{4, 71, 114, h_1\}, \{0, 3, 17, 36\}, \{65, 69, 77, 96\},
\{0, 2, 66, 73\}, \{0, 12, 68, 69\}, \{0, 28, 70, 81\}, \{0, 4, 105, 128\},
\{0, 6, 102, 126\}, \{0, 9, 83, 112\}, \{0, 21, 109, 115\}, \{0, 7, 84, 93\},
\{0, 24, 106, 123\}, \{0, 27, 87, 89\}, \{0, 8, 100, 116\}, \{0, 22, 76, 97\},
\{0, 16, 111, 114\}.

\((n, m) = (10, 19),\)
\(X: \mathbb{Z}_{171} \cup H_{19},\)
\(G: \bigcup_{i=0}^{12} \{i, i + 9, i + 18, \ldots, i + 162\} \cup H_{19},\)
\(R: \{j, j + 19, j + 38, \ldots, j + 152, h_{\beta(j+1)}\}, j = 0, \ldots, 18, where \(\beta(i)\) is the \(i\)th item in\)
\((1, 18, 16, 14, \ldots, 2, 19, 17, 15, \ldots, 3),\)
\(x: (0, 1, 2, \ldots, 170)(h_{19}, h_{17}, h_{15}, \ldots, h_3, h_1, h_{18}, h_{16}, \ldots, h_4, h_2),\)

\{0, 28, 74, h_{19}\}, \{15, 41, 106, h_{19}\}, \{16, 64, 146, h_{19}\}, \{1, 88, 138, h_{19}\},
\{2, 4, 63, h_{19}\}, \{8, 18, 33, h_{19}\}, \{0, 1, 4, 12\}, \{0, 5, 21, 35\},
\{0, 6, 13, 83\}, \{0, 17, 40, 132\}, \{0, 20, 62, 113\}, \{0, 22, 71, 118\},
\{0, 24, 67, 127\}, \{0, 29, 66, 98\}, \{0, 31, 64, 116\}.

\((n, m) = (8, 19),\)
\(X: \mathbb{Z}_{133} \cup H_{19},\)
\(G: \bigcup_{i=0}^{12} \{i, i + 7, i + 14, \ldots, i + 126\} \cup H_{19},\)
\(R: \{j, j + 19, j + 38, \ldots, j + 114, h_{\beta(j+1)}\}, j = 0, \ldots, 18, where \(\beta(i)\) is the \(i\)th item in\)
\((1, 12, 4, 15, 7, 18, 10, 2, 13, 5, 16, 8, 19, 11, 3, 14, 6, 17, 9),\)
\(x: (0, 1, 2, \ldots, 132)(h_{19}, h_{11}, h_3, h_{14}, h_6, h_{17}, h_9, h_1, h_{12}, h_4, h_{15}, h_7, h_8, h_{10}, h_2, h_{13}, h_5, h_{16}, h_8),\)

\{0, 4, 30, h_{19}\}, \{7, 39, 100, h_{19}\}, \{2, 3, 67, h_{19}\}, \{14, 36, 94, h_{19}\},
\{9, 53, 92, h_{19}\}, \{6, 51, 103, h_{19}\}, \{0, 2, 5, 118\}, \{0, 6, 33, 43\},
\{0, 8, 54, 67\}, \{0, 9, 34, 82\}, \{0, 11, 23, 115\}, \{0, 16, 47, 71\}. 
\((n, m) = (19, 19)\),
\(X: \mathbb{Z}_{19} \times \mathbb{Z}_{19}\),
\(G: \mathbb{Z}_{19} \times \{j\}, \ j \in \mathbb{Z}_{19}\),
\(R: \{i\} \times \mathbb{Z}_{19}, \ i \in \mathbb{Z}_{19}\),
\(x: \mathbb{Z}_{19} \times \mathbb{Z}_{19}\),
\{(0,0),(1,1),(7,7),(11,11)\},  \{(0,0),(1,7),(7,11),(11,1)\},
\{(0,0),(1,1),(7,1),(11,7)\},  \{(0,0),(2,1),(14,7),(3,11)\},
\{(0,0),(2,7),(14,11),(3,1)\},  \{(0,0),(2,11),(14,1),(3,7)\},
\{(0,0),(4,1),(9,7),(6,11)\},  \{(0,0),(4,7),(9,11),(6,1)\},
\{(0,0),(4,11),(9,1),(6,7)\},  \{(0,0),(1,2),(7,14),(11,3)\},
\{(0,0),(1,14),(7,3),(11,2)\},  \{(0,0),(1,3),(7,2),(11,14)\},
\{(0,0),(17,2),(5,14),(16,3)\},  \{(0,0),(17,14),(5,3),(16,2)\},
\{(0,0),(17,3),(5,2),(16,14)\},  \{(0,0),(15,2),(10,14),(13,3)\},
\{(0,0),(15,14),(10,3),(13,2)\},  \{(0,0),(15,3),(10,2),(13,14)\},
\{(0,0),(11,4),(7,9),(1,6)\},  \{(0,0),(11,9),(7,6),(1,4)\},
\{(0,0),(11,6),(7,4),(1,9)\},  \{(0,0),(3,4),(14,9),(2,6)\},
\{(0,0),(3,9),(14,6),(2,4)\},  \{(0,0),(3,6),(14,4),(2,9)\},
\{(0,0),(13,4),(10,9),(15,6)\},  \{(0,0),(13,9),(10,6),(15,4)\},
\{(0,0),(13,6),(10,4),(15,9)\}.

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References